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Constructive Theory of Analytic Functions on a Quasidisk

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Abstract. Let $G \subset \mathbb{C}$ be an arbitrary quasidisk, and f be analytic in G and continuous on $\overline{G}$. We prove two theorems (direct and inverse) establishing a connection between the rate of polynomial approximation of the function f on the boundary ∂G and its smoothness properties.

Keywords. Polynomial approximation, constructive description, quasidisk.

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1. Introduction

Let $K \subset \mathbb{C}$ be a compact set in the complex plane $\mathbb{C}$ with connected complement $\Omega := \overline{\mathbb{C}} \setminus K$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane. Denote by $\mathcal{A}(K)$ the class of all functions continuous on K and analytic in the interior of K. Let $\mathcal{P}_n$ be the class of complex polynomials of degree at most $n \in \mathbb{N} := \{1, 2, \ldots\}$. For $f \in \mathcal{A}(K)$ and $n \in \mathbb{N}$ denote by $\| \cdot \|_K$ the uniform norm on K, and by

$$E_n(f, K) := \inf\{\|f - p\|_K,\, p \in \mathcal{P}_n\},$$

denote best uniform approximations of f on K.

The rate of decrease of $E_n(f, K)$ as $n \to \infty$, the geometric structure of the boundary $L := \partial K$ of K, and the smoothness of f near the boundary interact in a complicated way. We mention the following two problems fundamental for the constructive theory of functions of a complex variable.

Problem A. *Describe the rate of polynomial approximation of all functions $f \in \mathcal{A}(K)$ with given smoothness properties (for example, satisfying the Hölder condition on K).*

Problem B. *Describe all functions $f \in \mathcal{A}(K)$ with given rate of decrease of $E_n(f, K)$ as $n \to \infty$ (for example, $E_n(f, K) = O(n^{-\alpha})$, $\alpha > 0$).*

In the years 1959–1963 Dzjadyk obtained a constructive characteristic of functions $f \in \mathcal{A}(\overline{G})$ satisfying the Hölder condition for a certain closed Jordan domains $K = \overline{G}$ with a piecewise smooth boundary (cf. Problem A). The distance $\rho_\delta(z)$, $\delta > 0$, from the boundary point $z \in L$ to the $(1 + \delta)$-th level line

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of the function $\Phi(z)$ which maps the domain $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ conformally, univalently and with standard normalization at infinity onto the exterior of the unit disk plays the central role in this description. Later, Dzjadyk, Lebedev, Shirokov, Tamrazov, Belyi, Mikljukov, Shevchuk, et al. extended these results to more general sets and more general classes of functions (defined by more delicate smoothness properties). We refer the reader to the monographs [17, 11, 13, 15, 4] and the many references therein for a comprehensive survey of this subject.

A survey of results concerning Problem B can be found in [12, 9, 11, 13]. The problem was solved in the general form in [2].

Approaches to the study of Problem A and Problem B are usually different. We know only two exceptions from this general trend. Dyn'kin [8] was the first who combined the direct parts of Problem A and Problem B into a single approximation problem. He introduced an auxiliary domain in order to define new smoothness properties of $f \in \mathcal{A}(\overline{G})$. In [3] this idea was adapted to the case of quasidisks (see [1, 14]) instead of domains bounded by a Radon curve without cusps as in [8].

In this paper we consider the case of functions on a quasidisk and connect the most general known solution of Problem A (see [18]) with the most general solution of Problem B (see [2]). The main idea of Theorems 1 and 2 below is that both problems can be treated in the same way.

Enlisting methods and concepts of the theory of quasiconformal mappings for solving problems of constructive theory of functions of a complex variable originated in Belyi's papers (see, for example, [7]).

2. Main definitions and results

In our considerations $K = \overline{G_1}$ will be a (closed) quasidisk, i.e., a domain bounded by a quasiconformal curve $L^1 = \partial G_1$ (see [1, 14]). A geometric characterization of quasiconformal curves can be stated as follows (see [14, p. 100]): L is a quasiconformal curve iff there exists a constant $c > 0$, depending only on L, such that for $z_1, z_2 \in L$,

$$\min\{\operatorname{diam} L', \operatorname{diam} L''\} \le c\,|z_1 - z_2|,$$

where L' and L'' denote the two arcs which $L \setminus \{z_1, z_2\}$ consists of and $\operatorname{diam} E$ is the diameter of a set $E \subset \mathbb{C}$. Thus we exclude regions with cusps on the boundary from our consideration.

Next we introduce the concept of modulus of smoothness suitable for our problem. Let $G_2 \subset \mathbb{C}$ also be a quasidisk, i.e., a domain bounded by a quasiconformal curve $L^2 = \partial G_2$. Denote by $w = \Phi_j(z)$, $j = 1, 2$, the conformal mapping of $\Omega_j := \overline{\mathbb{C}} \setminus \overline{G_j}$ onto the exterior $\Delta := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ of the unit disk $\mathbb{D} := \{w : |w| < 1\}$ with the normalization $\Phi_j(\infty) = \infty$, $\Phi_j'(\infty) > 0$. As we know, Φ_j can be extended continuously to a homeomorphism $\Phi_j\colon \overline{\Omega_j} \to \overline{\Delta}$. For $\delta > 0$, $z \in \mathbb{C}$, $E \subset \mathbb{C}$, we set

$$\begin{aligned} d(z,E) = \operatorname{dist}(z,E) &:= \inf\{|z-\zeta|,\, \zeta \in E\}, \\ L^j_\delta &:= \{\zeta \in \Omega_j : |\Phi_j(\zeta)| = 1+\delta\}, \\ \rho^j_\delta(z) &:= d(z, L^j_\delta). \end{aligned}$$

Next we introduce the inverse conformal mapping $\Psi_j := \Phi_j^{-1}$ and composite conformal mappings $\Phi := \Psi_2 \circ \Phi_1$, $\Psi := \Phi^{-1} = \Psi_1 \circ \Phi_2$.

For a given function $f \in \mathcal{A}(\overline{G_1})$, $m \in \mathbb{N}$, $\delta > 0$, we put

$$\omega^*_m(\delta) = \omega^*_m(\delta, f, G_2) := \sup\{E_{m-1}(f, \Psi(J)), J \text{ is a subarc of } L^2, \operatorname{diam} J \le \delta\}.$$

In our consideration this function describes structure properties of f.

When $G_1 = G_2$, $\omega^*_m(\delta)$ is equivalent to the m-th modulus of continuity of the function f on L^1 (see [17, 9] for a more precise explanation). Also note that $\omega^*_1(\delta)$ is equivalent to the usual modulus of continuity of a function $f^*(w) := f[\Psi(w)]$, $w \in L^2$, on L^2.

We use α, β, c, c_1, … to denote positive constants (possibly different in different relations) that are either absolute or depend on parameters not essential for the argument; otherwise, such a dependence will be indicated.

Following [17], we call a function $\mu(\delta)$ a normal majorant if it is defined, finite, positive, nondecreasing for $\delta > 0$ and satisfies

$$\lim_{\delta\to 0+} \mu(\delta) = 0,$$

$$\mu(2\delta) \le c\mu(\delta) \qquad (\delta > 0). \tag{1}$$

Note that (1) is equivalent to the condition

$$\mu(t\delta) \le c_1 t^c \mu(\delta) \qquad (t \ge 1, \delta > 0). \tag{2}$$

For example, the function $\mu(\delta) = c_1\,\delta^c$ is a normal majorant.

Lemma 1. *Let $f \notin \mathcal{P}_{m-1}$. The function $\omega^*_m(\delta, f)$ is a normal majorant.*

The direct theorem of the constructive theory of analytic functions can be formulated as follows.

Theorem 1. *Let $f \in \mathcal{A}(\overline{G_1})$, $m \in \mathbb{N}$. For any $n \in \mathbb{N}$ there exists a polynomial $p_n \in \mathcal{P}_n$ such that the inequality*

$$|f(z) - p_n(z)| \le c\,\omega^*_m[\rho^2_{1/n}(\Phi(z))] \qquad (z \in L^1)$$

holds for some constant $c > 0$, independent of z and n.

Special cases of this theorem for $m = 1$ are proved in [8, 3].

The inverse theorem has the following form.

Theorem 2. *Let $f \in \mathcal{A}(\overline{G_1})$ and let $\mu(\delta)$ be a normal majorant. If for any $n \in \mathbb{N}$ there exists a polynomial $p_n \in \mathcal{P}_n$ such that the inequality*

$$|f(z) - p_n(z)| \le c\mu(\rho_{1/n}^2(\Phi(z))) \qquad (z \in L^1) \tag{3}$$

holds for some constant $c > 0$, independent of z and n, then there is $m \in \mathbb{N}$, independent of f, such that

$$\omega_m^*(\delta, f, G_2) \le c_1\mu(\delta) \qquad (\delta > 0) \tag{4}$$

for some constant $c_1 > 0$, independent of δ.

The example of function $f(z) = z^m$, $m \in \mathbb{N}$, and domains $G_2 = \mathbb{D}$,

$$G_1 = G_1(\alpha, m) = \left\{ z = re^{i\theta\pi} : 0 < r < 1, \frac{\alpha}{2m} < \theta < 2 \right\}, \qquad 0 < \alpha < 1,$$

shows that even the condition $E_n(f, \overline{G_1}) = 0$ for $n \ge m$ is not sufficient in order to assert that $\omega_m^*(\delta) = O(\delta^\alpha)$ as $\delta \to 0$. This fact, in particular, shows that (4) is not valid for all $m \in \mathbb{N}$, but only for sufficiently large values of m.

If $G_1 = G_2$, then statements of Theorem 1 and Theorem 2 essentially coincide with results in [18]. In case $G_2 = \mathbb{D}$ Theorem 1 and Theorem 2 correspond to results in [2].

We conclude this section with some definitions. Set for $z \in \mathbb{C}$ and $\delta > 0$,

$$D(z, \delta) := \{\zeta : |\zeta - z| < \delta\}, \qquad \overline{D}(z, \delta) := \{\zeta : |\zeta - z| \le \delta\}.$$

For $a > 0$ and $b > 0$ we use the expression $a \preceq b$ (order inequality) if $a \le cb$. The expression $a \asymp b$ denotes that $a \preceq b$ and $b \preceq a$ simultaneously.

3. Auxiliary results from function theory

In this section we recall some results from [1, 7, 2, 4] that are needed below.

The mappings Φ, Φ_j, Ψ and Ψ_j can be extended to quasiconformal homeomorphisms of the whole plane onto itself. Consequently, the following assertion (see [4, pp. 97–98]) is useful in the study of their local properties.

Lemma 2. *Suppose a function $w = F(z)$ is K-quasiconformal mapping of the plane onto itself, $F(\infty) = \infty$. Assume also that $z_j \in \mathbb{C}$, $w_j := F(z_j)$, $j = 1, 2, 3$.*

(i) *Then the conditions $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent.*

(ii) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{1/K} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K}.$$

In the following we use without proof some geometrical facts that can be proved by applying Lemma 2 to the specifically chosen triplets of points (see also [7, 2, 4]).

For $\delta > 0$ and $z \in \overline{\Omega_j}$, set

$$\tilde{z}^j_\delta := \Psi_j[(1+\delta)\Phi_j(z)].$$

Then

$$\rho^j_\delta(z) \asymp |z - \tilde{z}^j_\delta| \qquad (z \in L^j). \tag{5}$$

Let $u > v > 0$, $j = 1, 2$. Then for $z \in L^j$,

$$\left(\frac{u}{v}\right)^\alpha \preceq \frac{\rho^j_u(z)}{\rho^j_v(z)} \preceq \left(\frac{u}{v}\right)^\beta, \tag{6}$$

with some constants $\beta \geq \alpha > 0$.

Let $z_1, \zeta_1 \in L^1$, $z_2 := \Phi(z_1)$, $\zeta_2 := \Phi(\zeta_1)$. If $|z_1 - \zeta_1| \succeq \rho^1_\delta(z_1)$, then

$$\left|\frac{z_2 - \zeta_2}{z_2 - \tilde{z}^{\,2}_{2\delta}}\right| \preceq \left|\frac{z_1 - \zeta_1}{z_1 - \tilde{z}^{\,1}_{1\delta}}\right|^\alpha. \tag{7}$$

Let $z, \zeta \in L^j$, $|z - \zeta| \preceq \rho^j_\delta(z)$. Then

$$\rho^j_\delta(z) \asymp \rho^j_\delta(\zeta). \tag{8}$$

For any $z, \zeta \in \overline{G}_j$ there exists an arc $\gamma \subset \overline{G}_j$, joining z and ζ, whose length $|\gamma|$ satisfies the condition (see [7], [4, p. 24])

$$|\gamma| \leq c|z - \zeta|. \tag{9}$$

Proof of Lemma 1. It is obvious only condition (1) is nontrivial. Let $0 < 2\delta < \operatorname{diam} L^2$, and let $J \subset L^2$ be an arc for which

$$E_{m-1}(f, \Psi(J)) = \omega^*_m(2\delta), \qquad \operatorname{diam} J \leq 2\delta.$$

Without loss of generality we can assume that $\operatorname{diam} J > \delta$. Denote by $J_1, \dots, J_k$ the system of subarcs of J with the following properties:

(i) $J = \cup_{j=1}^k J_j$,
(ii) $c_1\delta \leq \operatorname{diam} J_j \leq \delta$, $j = 1, \dots, k$,
(iii) $\operatorname{diam} J_j \cap J_{j+1} \geq c_2\delta$, $j = 1, \dots, k-1$,
(iv) $k \leq c_3$.

The existence of such a covering of J follows easily from Lemma 2 written for a quasiconformal extension of Φ_2.

We put $l := \Psi(J)$, $l_j := \Psi(J_j)$, $j = 1, \dots, k$. Choose j and let $P_j \in \mathcal{P}_{m-1}$ be a polynomial such that

$$\|f - P_j\|_{l_j} = E_{m-1}(f, l_j).$$

Consider the polynomial $Q_j(z) := P_{j+1}(z) - P_j(z)$. For $z \in l_j \cap l_{j+1}$ we have

$$|Q_j(z)| \le |P_j(z) - f(z)| + |f(z) - P_{j+1}(z)| \le 2\,\omega_m^*(\delta). \tag{10}$$

On the arc $J_j \cap J_{j+1}$ one can construct a system of points $w_1, \dots, w_m$ according to the following rule:

(i) If $m = 1$, then $w_1 \in J_j \cap J_{j+1}$ is an arbitrary point.

(ii) If $m > 1$, then w_1 and w_m are end points of the arc $J_j \cap J_{j+1}$ and the other points are defined by the condition

$$|w_k - w_1| = \frac{k-1}{m-1}|w_m - w_1|, \qquad k = 1, \dots, m.$$

By Lemma 2 applied to a quasiconformal extension of Φ the set of points $z_i := \Psi(w_i)$ satisfies

$$|z_i - z_s| \asymp \operatorname{diam} l_j \asymp \operatorname{diam} l_{j+1} \asymp \operatorname{diam} l$$

for all $i, s = 1, \dots, m$, $i \ne s$. By virtue of inequality (10) and Lagrange's interpolation formula

$$Q_j(z) = \sum_{i=1}^{m} Q_j(z_i)\frac{\pi_i(z)}{\pi_i(z_i)}, \qquad \pi_i(z) := \prod_{\substack{s=1\\ s\ne i}}^{m}(z - z_s),$$

we obtain

$$|Q_j(z)| \preceq \omega_m^*(\delta) \qquad (z \in l).$$

Therefore, if $z \in l_j$, then

$$|f(z) - P_1(z)| \le |f(z) - P_j(z)| + \sum_{i=1}^{j-1}|Q_i(z)| \preceq \omega_m^*(\delta). \tag{11}$$

Consequently,

$$\omega_m^*(2\delta) = E_{m-1}(f, l) \le \|f - P_1\|_l \preceq \omega_m^*(\delta).$$

■

Now let $r(z,h)$, $z \in L^1$, $h > 0$, be a function defined by the identity

$$\rho^1_{r(z,h)}(z) = h.$$

Let $z \in L^1$ be an arbitrary point, $w := \Phi(z)$, and let $J \subset L^2$ be an arc such that $w \in J$, $\operatorname{diam} J = \rho^2_{r(z,h)}(w) := \delta$. Denote by $P_0 \in \mathcal{P}_{m-1}$ the polynomial for which

$$\|f - P_0\|_{\Psi(J)} = E_{m-1}(f, \Psi(J)).$$

Reasoning as in the proof of inequality (11) for $\zeta \in L^1$ one can obtain

$$|f(\zeta) - P_0(\zeta)| \preceq \begin{cases} \omega_m^*(\delta), & |\zeta - z| \le h; \\ \omega_m^*(\delta)\left|\dfrac{\Phi(\zeta) - w}{\delta}\right|^c, & |\zeta - z| > h. \end{cases} \tag{12}$$

In the case $|\zeta - z| > h$, using (5), (7) and the estimate

$$\left|\frac{\Phi(\zeta)-w}{\delta}\right| \asymp \left|\frac{\Phi(\zeta)-w}{\tilde{w}^2_{r(z,h)}-w)}\right| \preceq \left|\frac{\zeta-z}{h}\right|^{\alpha},$$

we can write (12) in the form

$$|f(\zeta)-P_0(\zeta)| \preceq \omega^*_m(\delta)\left(1+\left|\frac{\zeta-z}{h}\right|^{c\alpha}\right) \qquad (\zeta \in L^1).$$

Due to the result of Tamrazov (see, for example, [4, p. 186]) for $h > 0$, we have

$$\begin{aligned} \omega_{f,m,z,\overline{G_1}}(h) &:= E_{m-1}(f, \overline{G_1} \cap \overline{D}(z,h)) \\ &\leq \|f-P_0\|_{\overline{G_1}\cap\overline{D}(z,h)} \preceq \omega^*_m[\rho^2_{r(z,h)}(\Phi(z))]. \end{aligned} \tag{13}$$

4. Polynomial approximation on a quasidisk

In our definition of polynomials, approximating a function $f \in \mathcal{A}(\overline{G_1})$, we use some basic ideas from [7, 18, 5] and follow the scheme of [6].

We fix a point $z_0 \in G_1$ and consider an antiderivative of f

$$F(\zeta) := \int\limits_{\gamma(z_0,\zeta)} f(\xi)\,d\xi \qquad (\zeta \in \overline{G_1}),$$

where $\gamma(z_0,\zeta) \subset \overline{G_1}$ is an arbitrary rectifiable arc joining z_0 and ζ. The structure properties of F are described by its m-th local modulus of continuity

$$\omega_{F,m,z,\overline{G_1}}(\delta) := E_{m-1}(F, \overline{G_1} \cap \overline{D}(z,\delta)),$$

where $m \in \mathbb{N}$, $z \in L^1$, $\delta > 0$.

We will need the continuous extension of F into the complex plane which preserves its smoothness properties. The corresponding construction proposed by Dyn'kin [9], [10] is based on the Whitney unity partition (see [16]) and properties of the local m-th modulus of continuity of F. Slight modifications of the reasoning in [9], [10] and [16] as well as an application of (9) give the following result (cf. [4, pp. 13-15]).

Lemma 3. *The function F can be continuously extended to the complex plane (we preserve the notation F for the extension) such that:*

(i) *$F(z) = 0$ for z with $d(z, \overline{G_1}) \geq 3$, i.e., F has compact support;*
(ii) *for $z \in \mathbb{C} \setminus \overline{G_1}$,*

$$\left|\frac{\partial F(z)}{\partial \overline{z}}\right| \preceq \frac{\omega_{F,m,z^*,\overline{G_1}}(23d(z,\overline{G_1}))}{d(z,\overline{G_1})},$$

where $z^* \in L^1$ is an arbitrary point among the ones that are closest to z;

(iii) *if* $\zeta \in L^1$, $z \in \mathbb{C}$, $|z-\zeta| < \delta$, $0 < 2\delta < \operatorname{diam} \overline{G_1}$, *then*

$$|F(z) - P_{F,m,\zeta,\overline{G_1},\delta}(z)| \preceq \omega_{F,m,\zeta,\overline{G_1}}(25\,\delta),$$

where $P_{F,m,\zeta,\overline{G_1},\delta}(z) \in \mathcal{P}_{m-1}$ *is the polynomial such that*

$$\|F - P_{F,m,\zeta,\overline{G_1},\delta}\|_{\overline{G_1}\cap\overline{D}(\zeta,\delta)} = \omega_{F,m,\zeta,\overline{G_1}}(\delta);$$

(iv) F *satisfies a Lipschitz condition in* $\mathbb{C}$, *i.e.*,

$$|F(z) - F(\zeta)| \preceq |z-\zeta| \qquad (z,\zeta \in \mathbb{C}).$$

In order to approximate the Cauchy kernel $1/(\zeta - z)$, $z \in \overline{G_1}$, $\zeta \in \overline{\Omega_1}$, by polynomial kernels of the form

$$K_n(\zeta, z) = \sum_{j=0}^{n} a_j(\zeta)\, z^j, \tag{14}$$

we use the functions $K_{r,m,k,n}(\zeta, z)$ introduced by Dzjadyk (see [11, Chapter 9] or [4, Chapter 3]). Taking them as a basis for our discussion we can state the following result.

Lemma 4. *Let* $k \in \mathbb{N}$. *Then for any* $n \in \mathbb{N}$ *there exists a polynomial kernel of the form (14) such that the following relations hold for* $l = 0, 1$, $z \in L^1$, *and* $\zeta \in \overline{\Omega_1}$ *with* $d(\zeta, \overline{G_1}) \le 3$:

$$\begin{aligned} \left|\frac{\partial^l}{\partial z^l}\left(\frac{1}{\zeta - z} - K_n(\zeta, z)\right)\right| &\preceq \frac{1}{|\zeta - z|^{l+1}}\left(\frac{\rho^1_{1/n}(z)}{|\zeta - z| + \rho^1_{1/n}(z)}\right)^k, \\ \left|\frac{\partial^l}{\partial z^l} K_n(\zeta, z)\right| &\preceq \frac{1}{(|\zeta - z| + \rho^1_{1/n}(z))^{l+1}}. \end{aligned}$$

Further, we consider the polynomial

$$p_n(z) := -\frac{1}{\pi}\int_{\overline{\Omega_1}} \frac{\partial F(\zeta)}{\partial \overline{\zeta}}\,\frac{\partial}{\partial z} K_n(\zeta, z)\, dm(\zeta) \qquad (z \in \overline{G_1}),$$

where $dm(\zeta)$ denotes integration with respect to the two-dimensional Lebesgue measure (area). Let $z \in L^1$, $\rho := \rho^1_{1/n}(z)$, $\overline{D} := \overline{D}(z, \rho)$, $\sigma := \partial \overline{D}$. According to assertion (iv) of Lemma 3, F is an ACL-function (absolutely continuous on lines parallel to the coordinate axes) in $\mathbb{C}$. Hence, the Green formula can be applied (see [14]) to obtain

$$
\begin{aligned}
f(z)-p_n(z) &= \frac{1}{\pi}\int\limits_{\overline{\Omega_1}\setminus\overline{D}} \frac{\partial F(\zeta)}{\partial\overline{\zeta}}\frac{\partial}{\partial z}\left(K_n(\zeta,z)-\frac{1}{(\zeta-z)}\right)dm(\zeta)\\
&\quad+\frac{1}{\pi}\int\limits_{\overline{D}} \frac{\partial F(\zeta)}{\partial\overline{\zeta}}\frac{\partial}{\partial z}K_n(\zeta,z)\,dm(\zeta)\\
&\quad+f(z)-\frac{1}{2\pi i}\int\limits_{\sigma}\frac{F(\zeta)}{(\zeta-z)^2}\,d\zeta\\
&:= U_1(z)+U_2(z)+U_3(z).
\end{aligned}\tag{15}
$$

5. Proof of Theorem 1

Using polynomials from the previous section, we only need to show that for any $z\in L^1$,

$$|U_j(z)|\preceq\omega_m^*[\rho^2_{1/n}(\Phi(z))]=:B_n(z)\qquad(j=1,2,3).\tag{16}$$

Since for $z\in L^1$, $\zeta\in\overline{G_1}$ with $|\zeta-z|\le\delta$,

$$
\begin{aligned}
F(\zeta) &= F(z)+\int\limits_{\gamma(z,\zeta)} f(\xi)\,d\xi\\
&= t_\delta(\zeta,z)+\int\limits_{\gamma(z,\zeta)}\left(f(\xi)-P_{f,m,z,\overline{G_1},c\delta}(\xi)\right)d\xi,
\end{aligned}
$$

where $c\ge1$ is the constant from (9), we obtain by (13)

$$\omega_{F,m+1,z,\overline{G_1}}(\delta)\le\|F-t_\delta(\cdot,z)\|_{\overline{G_1}\cap\overline{D}(z,\delta)}\preceq\delta\omega_m^*[\rho^2_{r(z,\delta)}(\Phi(z))].$$

Hence, by the estimate in assertion (ii) of Lemma 3 we can write

$$\left|\frac{\partial F(\zeta)}{\partial\overline{\zeta}}\right|\preceq\omega_m^*[\rho^2_{r(\zeta^*,|\zeta-\zeta^*|)}(\Phi(\zeta^*))]\qquad(\zeta\in\Omega_1^*),\tag{17}$$

where $\Omega_1^*:=\{\zeta\in\Omega_1:\,d(\zeta,L^1)\le3\}$ and $\zeta^*\in L^1$ is an arbitrary point among the ones that are closest to ζ. Moreover, for $z\in L^1$, $\zeta\in\mathbb{C}$ with $|z-\zeta|\le\delta<1/2\,\mathrm{diam}\,\overline{G_1}$, we have

$$|F(\zeta)-t_\delta(\zeta,z)|\preceq\delta\omega_m^*[\rho^2_{r(z,\delta)}(\Phi(z))].\tag{18}$$

Indeed, since for $\zeta\in\overline{G_1}\cap\overline{D}(z,\delta)$,

$$
\begin{aligned}
&|t_\delta(\zeta,z)-P_{F,m+1,z,\overline{G_1},\delta}(\zeta)|\\
&\quad\le|F(\zeta)-t_\delta(\zeta,z)|+|F(\zeta)-P_{F,m+1,z,\overline{G_1},\delta}(\zeta)|\preceq\delta\omega_m^*[\rho^2_{r(z,\delta)}(\Phi(z))],
\end{aligned}
$$

by the Bernstein-Walsh lemma [19, p. 77] we have

$$\|t_\delta(\cdot,z)-P_{F,m+1,z,\overline{G_1},\delta}\|_{\overline{D}(z,\delta)}\preceq\delta\omega_m^*[\rho^2_{r(z,\delta)}(\Phi(z))].$$

Hence (18) follows from the above inequality and assertion (iii) of Lemma 3. Since by Lemma 1, (2) and (6) for $\zeta \in \Omega_1^* \setminus \overline{D}$,

$$\begin{aligned}\omega_m^*[\rho^2_{r(\zeta^*,|\zeta-\zeta^*|)}(\Phi(\zeta^*))] &\preceq \omega_m^*[\rho^2_{r(z,|\zeta-z|)}(\Phi(z))] \\ &\preceq B_n(z)\left(\frac{\rho^2_{r(z,|\zeta-z|)}(\Phi(z))}{\rho^2_{1/n}(\Phi(z))}\right)^c \\ &\preceq B_n(z)\left(\frac{r(z,|\zeta-z|)}{1/n}\right)^\alpha \\ &\preceq B_n(z)\left|\frac{\zeta-z}{\rho^1_{1/n}(z)}\right|^\beta,\end{aligned}$$

the first integral in (15) can be estimated in an appropriate way by passing to polar coordinates and using Lemma 4 (with k large enough):

$$|U_1(z)| \preceq B_n(z)\int_\rho^c \frac{\rho^k}{r^{k+1}}\left(\frac{r}{\rho}\right)^\beta dr \preceq B_n(z). \tag{19}$$

In the same way we obtain the estimate

$$|U_2(z)| \preceq B_n(z)\int_0^\rho \frac{r\,dr}{\rho^2} \preceq B_n(z). \tag{20}$$

In order to estimate the third term in (15) we note that

$$|f(z) - (t_\rho)'_\zeta(z,z)| = |f(z) - P_{f,m,z,\overline{G_1},c\rho}(z)| \preceq B_n(z),$$

so that by (18):

$$|U_3(z)| \le |f(z) - (t_\rho)'_\zeta(z,z)| + \frac{1}{2\pi}\left|\int_\sigma \frac{F(\zeta)-t_\rho(\zeta,z)}{(\zeta-z)^2}\,d\zeta\right| \preceq B_n(z). \tag{21}$$

Comparing (19)–(21) we obtain (16).

6. Proof of the inverse theorem

In the proof of Theorem 2 we follow the standard scheme (cf. [11, 17, 4]) adapting it to the new understanding of the structure properties of a function $f \in \mathcal{A}(\overline{G_1})$. The following analogue of the Markov-Bernstein theorem plays a central role.

Lemma 5. *Suppose that the polynomial $p_n \in \mathcal{P}_n$ satisfies the inequality*

$$|p_n(z)| \le \mu[\rho^2_{1/n}(\Phi(z)] \qquad (z \in L^1),$$

where μ is a normal majorant. Then for any $s \in \mathbb{N}$, $z \in L^1$, and any point $\zeta \in \overline{D}(z, \rho^1_{1/n}(z)/2)$, the inequality

$$|p_n^{(s)}(\zeta)| \le c\frac{\mu[\rho^2_{1/n}(\Phi(z)]}{[\rho^1_{1/n}(z)]^s}$$

holds with a constant $c > 0$ which is independent of z and n.

Proof. Let $z, \zeta \in L^1$, $w := \Phi(z)$, $u := \Phi(\zeta)$. Using (2), (5) and (7) we obtain

$$\begin{aligned} |p_n(\zeta)| &\leq \mu[\rho^2_{1/n}(u)] \\ &\leq \mu(\rho^2_{1/n}(w) + |w-u|) \\ &\preceq \mu[\rho^2_{1/n}(w)] \left(1 + \left|\frac{w-u}{w-\tilde{w}^2_{1/n}}\right|^{\alpha}\right) \\ &\preceq \mu[\rho^2_{1/n}(w)] \left(1 + \left(\frac{|z-\zeta|}{\rho^1_{1/n}(z)}\right)^{\beta}\right). \end{aligned}$$

According to Tamrazov (see [4, p. 185])

$$|p_n(\zeta)| \preceq \mu[\rho^2_{1/n}(w)] \qquad (\zeta \in \overline{D}(z, \rho^1_{1/n}(z))).$$

Therefore, for $\zeta \in \overline{D}(z, \frac{1}{2}\rho^1_{1/n}(z))$,

$$\begin{aligned} |p_n^{(s)}(\zeta)| &= \frac{s!}{2\pi}\left|\int_{|\xi-z|=\rho^1_{1/n}(z)} \frac{p_n(\xi)}{(\xi-\zeta)^{s+1}}\, d\xi\right| \\ &\preceq \frac{\mu[\rho^2_{1/n}(w)]}{[\rho^1_{1/n}(z)]^s}. \end{aligned}$$

■

Proof of Theorem 2. Let $0 < \delta < \frac{1}{2} \operatorname{diam} L^2$, and let $J \subset L^2$ be an arbitrary subarc with $\operatorname{diam} J \leq \delta$. Let $w \in J$ be one of the end points of J. Set $z := \Psi(w)$, $l := \Psi(J)$, $h := \operatorname{diam} l$. Suppose that the number $N \in \mathbb{N}$ is such that

$$\rho^1_{2^{-N-1}}(z) < 2h \leq \rho^1_{2^{-N}}(z).$$

Hence, by (5) and Lemma 2, we have

$$\rho^2_{2^{-N}}(w) \asymp \delta.$$

Denote by $P_{m,n,z}(\zeta) \in \mathcal{P}_{m-1}$ polynomials that interpolate $p_{2^n}(\zeta)$ and their derivatives up to the order $m-1$ at the point z. For $\zeta \in l$,

$$\begin{aligned} f(\zeta) - P_{m,N,z}(\zeta) &= [f(\zeta) - p_{2^N}(\zeta)] + [p_1(\zeta) - P_{m,0,z}(\zeta)] \\ &\quad + \sum_{j=0}^{N-1}[(p_{2^{j+1}}(\zeta) - p_{2^j}(\zeta)) \\ &\qquad -(P_{m,j+1,z}(\zeta) - P_{m,j,z}(\zeta))]. \end{aligned} \tag{22}$$

Let $z(t) = z + (\zeta - z)t$, $0 \leq t \leq 1$, be a parametrization of the segment $[z, \zeta]$, and let

$$\phi_j(t) := (p_{2^{j+1}}(z(t)) - p_{2^j}(z(t))) - (P_{m,j+1,z}(z(t)) - P_{m,j,z}(z(t))).$$

By the choice of $P_{m,j,z}$ we have $\phi_j^{(r)}(0)=0$, $r=0,\dots,m-1$, and due to the Taylor formula

$$\begin{aligned}\phi_j(1) &= \phi_j(0)+\frac{\phi_j'(0)}{1!}+\dots+\frac{\phi_j^{(m-1)}(0)}{(m-1)!}+\frac{1}{m!}\int_0^1(1-t)^{m-1}\phi_j^{(m)}(t)\,dt\\ &= \frac{1}{m!}\int_0^1(1-t)^{m-1}\phi_j^{(m)}(t)\,dt.\end{aligned}$$

It follows from the assumption (3) and inequalities (2) and (6) that for $\zeta\in L^1$, $u:=\Phi(\zeta)$ and $j=0,\dots,N-1$,

$$\begin{aligned}|p_{2^{j+1}}(\zeta)-p_{2^j}(\zeta)| &\le |p_{2^{j+1}}(\zeta)-f(\zeta)|+|f(\zeta)-p_{2^j}(\zeta)|\\ &\preceq \mu(\rho^2_{2^{-j}}(u))\\ &\asymp \mu(\rho^2_{2^{-j-1}}(u)).\end{aligned}$$

By Lemma 5, this yields the estimate

$$|\phi_j^{(m)}(t)|\preceq\frac{\mu(\rho^2_{2^{-j-1}}(w))}{[\rho^1_{2^{-j-1}}(z)]^m}|z-\zeta|^m\qquad(0\le t\le 1)$$

and, therefore, the same estimate from above for $|\phi_j(1)|$, i.e.,

$$|\phi_j(1)|\preceq\frac{\mu(\rho^2_{2^{-j-1}}(w))}{[\rho^1_{2^{-j-1}}(z)]^m}|z-\zeta|^m.$$

Consequently, (2) and (6) imply that

$$\begin{aligned}\sum_{j=0}^{N-1}|\phi_j(1)| &\preceq h^m\sum_{j=0}^{N-1}\frac{\mu(\rho^2_{2^{-j-1}}(w))}{[\rho^1_{2^{-j-1}}(z)]^m}\\ &\preceq \mu(\delta)\sum_{j=0}^{N-1}\left(\frac{2^{-j}}{2^{-N}}\right)^{\beta}\left(\frac{2^{-N}}{2^{-j}}\right)^{m\,c}.\end{aligned}$$

Hence, taking $m>(1+\beta)/c$, we have

$$\sum_{j=0}^{N-1}|\phi_j(1)|\preceq\mu(\delta).\tag{23}$$

Note that by (8)

$$|f(\zeta)-p_{2^{N+1}}(\zeta)|\preceq\mu(\rho^2_{2^{-N-1}}(u))\preceq\mu(\delta),\tag{24}$$

and for $m\ge 2$,

$$p_1(\zeta)=P_{m,0,z}(\zeta).\tag{25}$$

Thus, taking into account (22)–(25), we have

$$E_{m-1}(f,l)\le\|f-P_{m,2^N,z}\|_l\preceq\mu(\delta).$$

Since J is arbitrary, we have (4) (for sufficiently large m). ∎

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