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# Semi-Groups of Analytic Maps

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**Abstract.** We consider a generalization, due to Baker and Rippon, of a version of the Denjoy-Wolff Theorem that is applicable to repeated compositions of different analytic self-maps of the unit disc $\mathbb{D}$. By placing the discussion in a topological context we show that a similar result holds for all hyperbolic subdomains of $\mathbb{C}$.

**Keywords.** Compositions of analytic maps, hyperbolic domains, Denjoy-Wolff Theorem.

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## 1. Introduction

Let $\mathcal{F}$ be a family of analytic self-maps of a domain $D$ in the complex plane $\mathbb{C}$. We say that a sequence of functions $g_1, g_2, \dots$ is a *composition sequence for* $\mathcal{F}$ if there are functions $f_1, f_2, \dots$ in $\mathcal{F}$ such that for $n \geq 1$, $g_n = f_1 \circ \cdots \circ f_n$. For brevity, we shall now write $f_1 \cdots f_n$ instead of $f_1 \circ \cdots \circ f_n$. This paper is concerned with assumptions on $D$ and $\mathcal{F}$ that imply that each composition sequence for $\mathcal{F}$ converges locally uniformly in $D$ to some constant. The situation in which the $f_n$ are Möbius transformations occurs in continued fraction theory, and the general situation arises naturally in complex dynamical systems. There has been much interest in this question in recent years, and we cite the papers [2], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13] and [15] as evidence of this.

The primordial result of this type is the Denjoy-Wolff Theorem which states that if $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, and if $f\colon \mathbb{D} \to \mathbb{D}$ is analytic but not a conformal automorphism of $\mathbb{D}$, then the $n$-th iterate $f^n$ of $f$ converges to a constant locally uniformly in $\mathbb{D}$. Here, we are seeking a generalization of this result in which the iterates $f^n$ are replaced by a general composition $f_1 \cdots f_n$, where the $f_j$ are chosed from the family $\mathcal{F}$, and where the unit disc $\mathbb{D}$ is replaced by a more general domain $D$. The results in this paper were motivated by a generalization, due to I. N. Baker and P. J. Rippon, of the Denjoy-Wolff Theorem to a result about a family of analytic self-maps of $\mathbb{D}$ ([2, Theorem 2.2]). Our generalization

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of their result is obtained by first placing the entire discussion in a topological context which we now describe.

Given a family $\mathcal{F}$ of analytic self-maps of $D$, the set of all finite compositions of functions chosen from $\mathcal{F}$ is a semi-group which we denote by $\mathcal{F}_0$. Now $\mathcal{F}_0$ is a semigroup lying in the semigroup $\mathcal{H}(D)$ of all analytic self-maps of $D$. The hypotheses in the Baker-Rippon Theorem are phrased in terms of limits of sequences of functions chosen from $\mathcal{F}_0$; thus we are led naturally into seeking some kind of topological closure of the semi-group $\mathcal{F}_0$. However, some functions in the closure of $\mathcal{F}_0$ will be constant functions with values in the boundary $\partial D$ of $D$, and as these limit functions are not in $\mathcal{H}(D)$, it becomes necessary to construct a space that is larger than $\mathcal{H}(D)$, and within which these limits can exist. The natural setting for these ideas is the space $\mathcal{H}(\mathbb{C})$ of all analytic maps $f\colon D \to \mathbb{C}$ endowed with the compact-open topology (which coincides with the topology of locally uniform convergence on $D$). Thus we shall be considering the closure of $\mathcal{F}_0$ in the larger space $\mathcal{H}(\mathbb{C})$. Let us now state the Baker-Rippon Theorem in the context of these new ideas.

**Theorem A.** *Let $\mathcal{F}$ be a family of analytic self-maps of $\mathbb{D}$, and suppose that the closure of $\mathcal{F}_0$ in $\mathcal{H}(\mathbb{C})$ does not contain the identity map, or any constant map whose value is in $\partial\mathbb{D}$. Then any composition sequence of $\mathcal{F}$ converges locally uniformly in $\mathbb{D}$ to a constant (whose value necessarily lies in $\mathbb{D}$).*

As a result of taking this topological point of view, we shall see quite clearly the different roles played by the assumptions concerning (i) the identity map, and (ii) the constant functions with values in $\partial D$. Briefly, we shall give a more or less complete analysis of the situation when the assumption (ii) is valid or, equivalently, when the $\mathcal{F}_0$-images of one (or every) point in $D$ do not accumulate on $\partial D$.

A glance at the proof of Theorem A in [2] shows that normal families play a key role in the argument. Now for any subdomain $D$ of $\mathbb{C}$, the set of analytic self-maps of $D$ is a normal family in $D$ if and only if $D$ is *hyperbolic*; that is if $D$ supports a hyperbolic metric (of constant negative curvature) or, equivalently, if the complement of $D$ in the extended complex plane $\mathbb{C}_\infty$ contains at least three points. Our primary aim in this paper is to show that the conclusion in Theorem A holds when the unit disc $\mathbb{D}$ is replaced by any hyperbolic subdomain of $\mathbb{C}$, and the comments above on normal families suggest that the restriction to hyperbolic domains here is natural. A formal statement of our first result now follows.

**Theorem 1.1.** *Let $\mathcal{F}$ be a family of analytic self-maps of a hyperbolic subdomain $D$ of $\mathbb{C}$, and let $\mathcal{F}_0$ be the semigroup of all finite compositions of elements of $\mathcal{F}$. Suppose that the closure of $\mathcal{F}_0$ in $\mathcal{H}(\mathbb{C})$ does not contain the identity map, or any constant function whose value lies in $\partial D$. Then any composition sequence of $\mathcal{F}$ converges locally uniformly in $D$ to a constant.*

Note that Theorem 1.1 contains the well-known result that if $K$ is a compact subset of a hyperbolic domain $D$, and if $f_1, f_2, \ldots$ are analytic in $D$ with $f_j(D) \subset K$, then $f_1 \cdots f_n$ converges locally uniformly on $D$ to a constant. For a deeper result of this type see [8].

Experience shows that for a given choice of $f_1, f_2, \ldots$ in $\mathcal{F}$, the main difficulties in this problem arise when the images $f_1 \cdots f_n(z_0)$, $n = 1, 2, \ldots$, of some point $z_0$ in $D$ accumulate somewhere on $\partial D$, and this difficulty is related to the hypotheses in Theorems A and 1.1 in the following way. Here (as in [2]) we start with a family $\mathcal{F}$ and then generate the semigroup $\mathcal{F}_0$ of analytic maps of $D$ into itself. We shall see later that the requirment that $\mathcal{F}_0$ contains no sequence converging locally uniformly to a value in $\partial D$ is equivalent to the assumption that for each $z$ in $D$, the set $\{f(z) : f \in \mathcal{F}_0\}$ does not accumulate on $\partial D$ (and so it lies in a compact subset of $D$). We shall also see that this last condition *is independent of the choice of $z$ in $D$*, and this fact depends in an essential way on the existence of the hyperbolic metric in $D$. It fails, for example, if we take $D = \mathbb{C}$, $\mathcal{F} = \{f_1, f_2, \ldots\}$, and $f_n(z) = nz$. A subdivision into the cases when the images of some $z$ do or do not accumulate on $\partial D$ is part of the folklore of this problem, and here we shall consider this division formally and in detail.

Section 2 contains a discussion of the topological material, and Section 3 contains our proof of Theorem 1.1. It will be clear that this proof owes much to the proof of Theorem A given in [2], but there are significant differences between the two proofs. In particular, we need to eliminate the use (in [2]) of the Denjoy-Wolff Theorem. Section 4 contains some closing remarks and extensions of these results.

## 2. The compact-open topology

For each compact subset $K$ of $D$, and each open subset $U$ of $\mathbb{C}$, let $[K, U]$ be the class of all analytic maps $f\colon D \to \mathbb{C}$ with $f(K) \subset U$. The *compact-open topology* $\mathcal{T}$ is the topology on $\mathcal{H}(\mathbb{C})$ generated by taking all sets of the form $[K, U]$ as a sub-base for the topology. To be more explicit, a subset of $\mathcal{H}(\mathbb{C})$ is open if and only if it is a union of sets of the form $[K_1, U_1] \cap \cdots \cap [K_q, U_q]$ (for more details, see [14], p.221). The compact-open topology is defined on the class of continuous maps from one topological space to another, but here we shall restrict ourselves to the class of holomorphic functions on $D$.

As $D$ admits a countable exhaustion by compact sets (whose interiors cover $D$), we can construct a metric $\sigma$ on $\mathcal{H}(\mathbb{C})$ such that $\sigma(f_n, f) \to 0$ if and only if $f_n \to f$ locally uniformly on $D$ (see [1, p. 220]). Moreover, it is well known (and easy to prove) that the topology on $\mathcal{H}(\mathbb{C})$ generated by the metric $\sigma$ coincides with the compact-open topology (for a more general discussion of these ideas, see Theorem 13.2.3 in [18]). The importance of this remark is that the closure $\overline{E}$ of any subset $E$ of $\mathcal{H}(\mathbb{C})$ with respect to the compact-open topology can now be defined in in terms of convergent sequences; this will be used frequently in what follows without explicit mention.

We turn now to consider the interior, and the closure, of the subset $\mathcal{H}(D)$ of $\mathcal{H}(\mathbb{C})$. First, it is of interest (even though we shall not use it here) to see that in many cases the set $\mathcal{H}(D)$ has empty interior. Indeed, suppose that $D$ is simply connected, and that there is a disc $\{z : |z - \zeta| < r\}$ lying outside $D$. Take any $f$ in $\mathcal{H}(D)$ and any neighbourhood $\mathcal{N}$ of $f$. Thus there are sets $K_i$ and $U_i$ as above such that $f \in [K_1, U_1] \cap \cdots \cap [K_q, U_q] \subset \mathcal{N}$. As $D$ is simply connected we can construct a compact subset $K$ of $D$ that contains each $K_i$ and that is homeomorphic to a closed disc. Let $d_j$ be the minimum distance between $f(K_j)$ and $\partial U_j$, and let $d$ be the minimum of the $d_j$. Next, choose a point $z_0$ in $D \backslash K$. Then $K \cup \{z_0\}$ is a compact subset of $D$ that does not separate $\mathbb{C}$ and so, by Runge's Theorem ([16, p. 257]), there is a polynomial $p$ such that $|p - f| < d$ on $K$, and $|p - \zeta| < r$ on $\{z_0\}$. Clearly, $p \in \mathcal{H}(\mathbb{C})$, and as $p(K_j)$ is within a distance $d$ of $f(K_j)$, we see that $p(K_j) \subset U_j$; thus $p \in \mathcal{N}$. On the other hand, $p(z_0) \notin D$, so that $p \notin \mathcal{H}(D)$. We conclude that $f$ is not an interior point of $\mathcal{H}(D)$, and hence that $\mathcal{H}(D)$ has empty interior in $\mathcal{H}(\mathbb{C})$.

Our primary interest is in the closure $\overline{\mathcal{H}(D)}$ of $\mathcal{H}(D)$ in $\mathcal{H}(\mathbb{C})$. It is convenient to denote the constant function on $D$ with value $c$ by $\chi_c$, and the set of constant functions $\chi_c$, where $c \in E$, by $\chi(E)$. We now use this notation to describe the closure of $\mathcal{H}(D)$.

**Lemma 2.1.** *For any domain $D$, $\overline{\mathcal{H}(D)} = \mathcal{H}(D) \cup \chi(\partial D)$.*

**Proof.** If $c \in \partial D$, there are points $z_n$ in $D$ converging to $c$, and then $\chi_{z_n} \to \chi_c$ uniformly in $D$. This shows that $\chi(\partial D) \subset \overline{\mathcal{H}(D)}$. As $\mathcal{H}(D) \subset \overline{\mathcal{H}(D)}$, we have only to show that $\overline{\mathcal{H}(D)} \subset \mathcal{H}(D) \cup \chi(\partial D)$.

Suppose now that $g \in \overline{\mathcal{H}(D)}$, that $g$ is nonconstant, and that $g(D) \cap \partial D \neq \emptyset$. First, there is a sequence of analytic maps $f_n \colon D \to D$ such that $f_n \to g$ locally uniformly in $D$. Next, there is some $w$ in $D$ and some $\zeta$ in $\partial D$ such that $g(w) = \zeta$. Now draw a small circle $C$ around $w$ such that $C$ and its interior $\Delta$ lies in $D$. As $g$ is nonconstant we may assume that $C$ is chosen so that $g \neq \zeta$ on $C$. Then, for all sufficiently large $n$,

$$|f_n(z) - g(z)| < \inf_{z \in C} |g(z) - \zeta|,$$

so that (by Rouché's Theorem), $f_n(z) = \zeta$ at some point $z$ in $\Delta$. This, however, contradicts the assumption that $f_n(D) \subset D$, and we conclude that if $g \in \overline{\mathcal{H}(D)}$, then either $g$ is constant or $g(D) \subset D$. Clearly, if $g$ is constant its value lies in the closure $\overline{D}$ of $D$, and if $g$ is constant with value in $D$, then $g(D) \subset D$. This shows that if $g \in \overline{\mathcal{H}(D)}$ then $g \in \mathcal{H}(D)$ or $g = \chi_c$ for some $c$ in $\partial D$. The proof is complete. ■

We turn now to examine the consequences of the assumption that the closure of $\mathcal{F}_0$ does not contain any constant function whose value lies in $\partial D$. This assumption is

$$\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset, \tag{2.1}$$

and we do not need to assume yet that $D$ is hyperbolic. Now $\mathcal{H}(D)$ is closed under composition but its closure is not (for $f \circ \chi_c$ is not defined when $c \in \partial D$ and $f$ is defined only on $D$). Likewise, the family $\mathcal{F}_0$ is closed under composition but its closure is not. We shall now show that an important consequence of the assumption (2.1) is that $\overline{\mathcal{F}_0}$ is indeed closed under composition.

**Lemma 2.2.** *Suppose that $\mathcal{F} \subset \mathcal{H}(D)$ and $\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset$. Then $\overline{\mathcal{F}_0} \subset \mathcal{H}(D)$. Further, $\overline{\mathcal{F}_0}$ is closed under composition, and the map $(f, g) \mapsto f \circ g$ of $\overline{\mathcal{F}_0} \times \overline{\mathcal{F}_0}$ to $\overline{\mathcal{F}_0}$ is continuous.*

**Proof.** The first assertion follows from Lemma 2.1 for $\mathcal{F}_0 \subset \mathcal{H}(D)$ so that

$$\overline{\mathcal{F}_0} = \overline{\mathcal{F}_0} \cap \overline{\mathcal{H}(D)} = \overline{\mathcal{F}_0} \cap \Big[\mathcal{H}(D) \cup \chi(\partial D)\Big] = \overline{\mathcal{F}_0} \cap \mathcal{H}(D) \subset \mathcal{H}(D).$$

This shows that $f \circ g$ is defined, and in $\mathcal{H}(D)$, whenever $f$ and $g$ are in $\overline{\mathcal{F}_0}$; the second assertion in Lemma 2.2 asserts that this composition is actually in $\overline{\mathcal{F}_0}$. To verify the last sentence in Lemma 2.1 it suffices to show that if $f$ and $g$ are in $\overline{\mathcal{F}_0}$, and if $f_n \to f$ and $g_n \to g$, then $f_n \circ g_n \to f \circ g$ (all in the compact-open topology), for this shows first that $f \circ g \in \overline{\mathcal{F}_0}$, and second, that the given map is continuous (we recall that because the compact-open topology is metrizable, we can work with sequences here).

Suppose, then, that $f$ and $g$ are in $\overline{\mathcal{F}_0}$, and hence also in $\mathcal{H}(D)$, and that $f_n \to f$ and $g_n \to g$. It is sufficient to show that for any compact subset $K$ of $D$, and for any open any open subset $U$ of $\mathbb{C}$, if $f(g(K)) \subset U$ then $f_n(g_n(K)) \subset U$ for all sufficiently large $n$. As $f\colon D \to \mathbb{C}$ is continuous, $f^{-1}(U)$ is an open subset of $D$ that contains the compact subset $g(K)$ of $D$. Thus we can find an open subset $V$ of $D$ such that $g(K) \subset V \subset \overline{V} \subset f^{-1}(U)$, where the closure $\overline{V}$ is a compact subset of $D$. It follows that $g \in [K, V]$ and $f \in [\overline{V}, U]$. As $g_n \to g$ and $f_n \to f$ in the compact-open topology, we see that for all sufficiently large $n$, $g_n \in [K, V]$ and $f_n \in [\overline{V}, U]$. Thus, for these $n$,

$$f_n(g_n(K)) \subset f_n(V) \subset f_n(\overline{V}) \subset U,$$

and hence $f_n \circ g_n \in [K, U]$. This proves that $f_n \circ g_n \to f \circ g$ in $\mathcal{H}(\mathbb{C})$ and so completes the proof of Lemma 2.2. ■

We end this section with a discussion of the constant functions $\chi_c$, where $c \in \partial D$, and their relation to the hyperbolicity of the domain $D$. First, we note that if a subdomain $D$ of $\mathbb{C}_\infty$ is not hyperbolic then $D$ is conformally equivalent to one of the domains $\mathbb{C}$, $\mathbb{C}_\infty$ or $\mathbb{C}\backslash\{0\}$. Consider the case when $D = \mathbb{C}$ and $\mathcal{F} = \{f_1, f_2, \ldots\}$, $f_n(z) = nz$. In this case $\mathcal{F} = \mathcal{F}_0$, and the images under $\mathcal{F}_0$ of 0 and 1 accumulate at and only at 0 and $\infty$, respectively. The same is true if we regard the $f_n$ as maps from $\mathbb{C}_\infty$ onto itself. If $D = \mathbb{C}\backslash\{0\}$, we can use the maps $f_n(z) = e^{-nz}$ and consider the images under $\mathcal{F}_0$ of the points 1 and $2\pi i$; these accumulate at and only at the points 0 and 1, respectively. We shall now show that this type of behaviour cannot occur for hyperbolic domains, and this is a crucial part of our argument.

For each $z$ in $D$, and each subfamily $\mathcal{F}$ of $\mathcal{H}(D)$, we define the *grand orbit* of $z$ to be the subset $\mathcal{F}_0(z) = \{f(z) : f \in \mathcal{F}_0\}$ of $D$. We are interested in the set

$$\Lambda(z) = \partial D \cap \overline{\mathcal{F}_0(z)} \tag{2.2}$$

and we shall now show that this is independent of $z$.

**Lemma 2.3.** *Let $D$ be a hyperbolic domain, and suppose that $\mathcal{F} \subset \mathcal{H}(D)$. Then the set $\Lambda(z)$ in* (2.2) *is independent of $z$ in $D$.*

**Proof.** Suppose that $\zeta \in \Lambda(z)$. Then there is a sequence of functions $f_n$ in $\mathcal{F}_0$ such that $f_n(z) \to \zeta$. Now let $\rho$ be the hyperbolic metric of $D$, and let $z'$ be any point in $D$. Then, by the generalized Schwarz-Pick Lemma,

$$\rho\big(f_n(z), f_n(z')\big) \le \rho(z, z'), \tag{2.3}$$

so that the sequence $\rho\big(f_n(z), f_n(z')\big)$, $n = 1, 2, \ldots$, is bounded. As $f_n(z)$ converges to the boundary point $\zeta$ of $D$, this implies that $|f_n(z) - f_n(z')| \to 0$ and so $\zeta \in \Lambda(z')$. Thus $\Lambda(z) \subset \Lambda(z')$ and, by symmetry, $\Lambda(z) = \Lambda(z')$. ■

Essentially the same idea allows us to reformulate the basic assumption (2.1) that there is no sequence of elements of $\mathcal{F}_0$ that converges to some constant function $\chi_c$, where $c \in \partial D$.

**Lemma 2.4.** *Suppose that $\mathcal{F} \subset \mathcal{H}(D)$, and that $w \in \partial D$. Then $\chi_w \in \overline{\mathcal{F}_0}$ if and only if one (or every) grand orbit accumulates at $w$. In particular, $\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset$ if and only if one (or every) grand orbit lies in a compact subset of $D$.*

Lemma 2.4 implies that $\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset$ if each grand orbit $\mathcal{F}_0(z)$ lies in a compact subset of $D$. This does *not* mean that there is a compact subset that contains every grand orbit; it simply means that no grand orbit accumulates anywhere on $\partial D$.

**Proof of Lemma 2.4.** Suppose that $\chi_w \in \overline{\mathcal{F}_0}$, where $w \in \partial D$. Let $z_0$ be any point of $D$, and put $K = \{z_0\}$. Then there are functions $f_n$ in $\overline{\mathcal{F}_0}$ such that $f_n \to \chi_w$ (uniformly) on $K$, and this means that $f_n(z_0) \to w$.

Conversely, suppose that for a given $z_0$, there are functions $f_n$ in $\overline{\mathcal{F}_0}$, and a point $w$ in $\partial D$, such that $f_n(z_0) \to w$. We claim that $f_n \to \chi_w$ locally uniformly on $D$. To see this, take any compact subset $K$ of $D$. Then $K \cup \{z_0\}$ is a compact subset of $D$, and this set has a bounded diameter, say $d$, in the hyperbolic metric $\rho$ of $D$. Each element $f$ in $\mathcal{H}(D)$ contracts, or preserves, hyperbolic distances in $D$, and as $f_n \in \mathcal{H}(D)$, for each $n$ the set $f_1 \cdots f_n(K \cup \{z_0\})$ has hyperbolic diameter at most $d$. It is known that this, together with the fact that $f_n(z_0)$ converges to a boundary point of $D$, implies that the Euclidean diameter of $f_1 \cdots f_n(K \cup \{z_0\})$ converges to zero. Thus $f_n \to \chi_w$ uniformly on $K$ and this completes the proof. ∎

## 3. The proof of Theorem 1.1

We begin with a sequence $f_1 \cdots f_n$, $n = 1, 2, \ldots$, where $f_j \in \mathcal{F}$, and we say that $\varphi$ is a *limit function* of this sequence if there is a subsequence of it that converges locally uniformly in $D$ to $\varphi$.

**Lemma 3.1.** *Let $D$ be a hyperbolic subdomain of $\mathbb{C}$. Suppose that $\mathcal{F} \subset \mathcal{H}(D)$, and that $\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset$, and let $f_j$ be a given sequence of elements of $\mathcal{F}$. Then for every pair of limit functions $\varphi$ and $\psi$ of $f_1 \cdots f_n$, there is a function $h$ in $\overline{\mathcal{F}_0}$ such that $\psi = \varphi \circ h$ in $D$.*

**Proof.** We write $F_{m,n} = f_m \cdots f_n$, and note that

$$F_{1,n} = F_{1,m} \circ F_{m+1,n}.$$

By assumption, $F_{1,n} \to \psi$ as $n \to \infty$ through some sequence, say $n_j$, and $F_{1,n} \to \varphi$ as $n \to \infty$ through some other sequence, say $m_j$, where we may assume (by passing to subsequences) that $n_1 < m_1 < n_2 < m_2 < \ldots$. Now consider the sequence of maps $F_{n_j+1,m_j}$. As $\mathcal{H}(D)$ is a normal family in $D$, there is some subsequence of this that converges to some map $h$ analytic in $D$. By letting $j \to \infty$ in a suitable manner in the relation

$$F_{1,m_j} = F_{1,n_j} \circ F_{n_j+1,m_j},$$

and by using the continuity described in Lemma 2.2, we find that $\psi = \varphi \circ h$ as required. ∎

Lemma 3.1 has a striking corollary. Suppose that the sequence $f_1 \cdots f_n$ has a constant limit function, say $\chi_c$. Then for any limit function $\psi$ there is some function $h$ such that $\psi = \chi_c \circ h$. As $\chi_c \circ h = \chi_c$, this proves the following result.

**Lemma 3.2.** *Suppose that in the situation described in Lemma 3.1, the sequence $f_1 \cdots f_n$ has a constant limit function $\chi_c$. Then $f_1 \cdots f_n \to \chi_c$ locally uniformly in $D$.*

We now examine the situation in which the sequence $f_1 \cdots f_n$ has only nonconstant limit functions, and in this case we prove the following result.

**Theorem 3.3.** *Suppose that $D$ is a hyperbolic subdomain of $\mathbb{C}$, $\mathcal{F} \subset \mathcal{H}(D)$, and that $\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset$. Suppose further that the sequence $f_1 \cdots f_n$ has a nonconstant limit function $\varphi$. Then every limit function of the sequence $f_1 \cdots f_n$ is of the form $\varphi \circ h$, where $h$ is an analytic bijection of $D$ onto itself.*

**Proof.** Given that a non-constant limit function $\varphi$ exists, Lemma 3.2 implies that every limit function is non-constant. Suppose, then, that $\varphi$ and $\psi$ are two nonconstant limit functions of the sequence $f_1 \cdots f_n$. Then, by Lemma 3.1, there are functions $h_1$ and $h_2$ such that $\varphi = \psi \circ h_1$ and $\psi = \varphi \circ h_2$, and these imply that

$$\varphi = \varphi \circ (h_2 \circ h_1), \quad \psi = \psi \circ (h_1 \circ h_2). \tag{3.1}$$

Now let $h = h_2 \circ h_1$. As each $h_j$ is in $\overline{\mathcal{F}_0}$, so is $h$, and hence $h \in \mathcal{H}(D)$. This means that $h$ maps $D$ into itself and so for any point $z_0$ in $D$ we can consider the sequence $h(z_0), h^2(z_0), h^3(z_0), \ldots$, where $h^m$ denotes the $m$-th iterate of $h$. Now this sequence cannot accumulate at $\partial D$ since if it does then, by Lemma 2.4, $\overline{\mathcal{F}_0}$ will contain some function of the form $\chi_c$, where $c \in \partial D$. It follows that the sequence $h(z_0), h^2(z_0), h^3(z_0), \ldots$ lies in a compact subset of $D$. If the terms in this sequence form an infinite set then, from (3.1) we find that $\varphi$ is constant contrary to our assumption (that it is not constant). Thus for each $z_0$, the sequence $h^m(z_0)$ only contains a finite number of distinct terms. This in turn implies that for every $z_0$ in $D$ there is some integer $m$ such that $h^m$ fixes $z_0$. Let $D_m$ be the set of $z$ in $D$ such that $h^m$ is the earliest iterate of $h$ that fixes $z$. Then $D$ is the countable union of the $D_m$, and so by Baire's Category Theorem, some $D_m$ is dense in some open subset of $D$. We deduce that for this $m$, $h^m = I$, and as this implies that $h\colon D \to D$ has a common left and right inverse, namely $h^{m-1}$, we see that $h$ is a conformal map of $D$ onto itself.

We recall that $h = h_2 \circ h_1$, and that we are trying to show that $h_1$ is a conformal map of $D$. Because $h = h_2 \circ h_1$, and $h$ is injective, then so is $h_1$. By symmetry (of $\varphi$ and $\psi$), so too is $h_2$. Finally, as

$$D = h^m(D) \subset h(D) = h_2\big(h_1(D)\big) \subset h_2(D) \subset D,$$

we see that $h_2\colon D \to D$ is surjective and so too (by symmetry) is $h_1$. This shows that each $h_j$ is a conformal map of $D$ onto itself, and the result follows as $\varphi = \psi \circ h_1$. ■

Notice that this proof also proves the following fact.

**Lemma 3.4.** *Suppose that $\varphi$ is a nonconstant limit function of the sequence $f_1 \cdots f_n$, and that $\varphi \circ h = \varphi$, where $h$ is in $\overline{\mathcal{F}_0}$. Then $h^m = I$ for some integer $m$.*

We can now give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Take any sequence $f_j$ from the family $\mathcal{F}$, form the sequence $f_1 \cdots f_n$, $n = 1, 2, \ldots$, and consider all limit functions of this sequence. As $\mathcal{H}(D)$ is a normal family in $D$, such limit functions do exist. Lemma 3.2 implies that it is sufficient to show that the sequence $f_1 \cdots f_n$ does not have a nonconstant limit function. Suppose that it does have such a function, say $\varphi$. Then, by Lemma 3.4, $\overline{\mathcal{F}_0}$ contains an element $h$ of finite order, say with $h^m = I$.

As $\overline{\mathcal{F}_0}$ is closed under composition (Lemma 2.2) this means that $I \in \overline{\mathcal{F}_0}$, and this contradicts the assumption in Theorem 1.1. This completes the proof of Theorem 1.1; notice that the assumption concerning the identity map is only used in the last line of the proof. ■

## 4. Further results

The analysis given in Section 3 also provides a proof of the following result that is closely related to Theorem 1.1.

**Theorem 4.1.** *Suppose that $\mathcal{F} \subset \mathcal{H}(D)$, where $D$ is a hyperbolic domain, and $\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset$. Let $f_1, f_2, \ldots$ be elements of $\mathcal{F}$. Then either*

(1) *$f_1 \cdots f_n \to \chi_c$ locally uniformly in $D$, for some $c$ in $D$, or*
(2) *$f_1 \cdots f_n$ has a nonconstant limit function $\varphi$, and every limit function of the sequence is of the form $\varphi \circ h$, where $h$ is a conformal automorphism of $D$.*

If $D$ is a domain of finite connectivity greater than two, then the automorphism group (of conformal self-maps) of $D$ is finite (see [10] or [17, p. 427]). It follows immediately that for such domains, the sequences $f_1 \cdots f_n$ under consideration can only have a finite number of nonconstant limit functions. However, we can say even more than this. The generic domain of this type has only a trivial group of conformal automorphisms, and in this sense we have the following result.

**Theorem 4.2.** *Let $D$ be a generic hyperbolic subdomain of $\mathbb{C}$ whose connectivity is finite and at least two. Suppose that $\mathcal{F} \subset \mathcal{H}(D)$, and $\overline{\mathcal{F}_0} \cap \chi(\partial D) = \emptyset$. Then for any sequence $f_j$ in $\mathcal{F}$, the sequence $f_1 \cdots f_n$ converges locally uniformly in $D$ to some analytic self-map of $D$.*

Finally, we recall Lemma 2.4 and the discussion following Corollary 1.2. First, we can replace the assumption that $\overline{\mathcal{F}_0} \cap \partial D = \emptyset$ by the assumption that each grand orbit lies in a compact subset of $D$, and this is a weaker assumption than $f(D) \subset K$ for a compact set K and all $f$ in $\mathcal{F}$. This leads to our final result.

**Theorem 4.3.** *Let $D$ be a hyperbolic domain, and let $\Delta$ be a closed disc in $D$. Suppose also that the maps $f_j$ are analytic maps of $D$ into $D \backslash \Delta$, and that for some $z_0$ in $D$, the points $f_1 \cdots f_n(z_0)$, $n = 1, 2, \ldots$, lie in a compact subset of $D$. Then the sequence $f_1 \cdots f_n$ converges locally uniformly on $D$ to a constant.*

Briefly, the existence of $\Delta$ means that $g(D)$ is disjoint from $\Delta$ whenever $g \in \mathcal{F}_0$, and this implies that no sequence of elements from $\mathcal{F}_0$ can converge to $I$. The existence of $z_0$ rules out any constant limits with value in $\partial D$, and so Theorem 1.1 is applicable and the result follows.

The authors of [2] comment (on page 186) that the hypothesis that the $f_j$ map $\mathbb{D}$ into a compact set $K$ if $\mathbb{D}$ "seems stringent when compared with the original Denjoy-Wolff Theorem". They then add that "it is tempting to hope that some weaker condition on $K$ will yield the conclusion of the corollary", and follow this with the comment that "the illusory nature of this hope ... is shown by the following example" ([2, p. 186, Example 2.6]). In fact, Theorem 4.3 shows that some progress can be made in weakening the assumption that $f_j(\mathbb{D}) \subset K$ for some compact $K$ and all $j$.

## References


1. L. V. Ahlfors, *Complex Analysis*, Third Ed., McGraw-Hill, 1979.
2. I. N. Baker and P. J. Rippon, Towers of exponents and other composite maps, *Complex Variables* **12** (1989), 181–200.
3. I. N. Baker and P. J. Rippon, On compositions of analytic self-mappings of a convex domain, *Arch. Math.* **55** (1990), 380–386.
4. A. F. Beardon, Iteration of contractions and analytic maps, *J. London Math. Soc.* **41** 2 (1990), 141–150.
5. A. F. Beardon, Iteration of analytic Euclidean contractions, *Computational methods and function theory*, ed. by R. M. Ali, St. Ruscheweyh, and E. B. Saff, World Scientific Pub. Co., 1995, 57–74.
6. A. F. Beardon, Analytic contractions of the unit disc, *Bull. Hong Kong Math. Soc.* **1** (1997), 209–218.
7. A. F. Beardon, The dynamics of contractions, *Ergod. Th. & Dynam. Sys.* **17** (1997), 1257–1266.
8. A. F. Beardon, T. W. Carne, D. Minda, and T. W. Ng, Random iteration of analytic maps, *Ergodic Theory and Dynamical Systems*, to be published.
9. Gill, J., Compositions of analytic functions of the form $F_n(z) = F_{n-1}(f_n(z))$, $f_n(z) \to f(z)$, *J. Comput. Appl. Math.* **23** 2 (1988), 179–184.
10. M. Heins, On the number of 1-1 directly conformal maps which a multiply-connected plane region of finite connectivity ($p > 2$) admits onto itself, *Bull. Amer. Math. Soc.* **52** (1946), 454–457.
11. L. Lorentzen, Compositions of contractions, *J. Comput. Appl. Math.* **32** (1990), 169–178.
12. L. Lorentzen, A convergence question inspired by Stieltjes and by value sets in continued fraction theory, *J. Comput. Appl. Math.* **65** (1995), 233–251.
13. L. Lorentzen, Convergence of compositions of self-mappings, *Ann. Univ. Mariae Curie-Sklodowska* A **53** 13 (1999), 121-145.
14. J. L. Kelley, *General Topology*, Van Nostrand, 1955.
15. P. J. Rippon, A generalization of a theorem of Beardon on analytic contraction mappings, *J. Math. Anal. and Appl.* **199** (1996), 157–161.
16. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966.
17. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.
18. A. Wilansky, *Topology for Analysis*, Ginn & Co., 1970.



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