Computational Methods and Function Theory
Volume 1 (2001), No. 1, 275–287

Zeros of Hypergeometric Functions

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Abstract. For certain ranges of parameters, it is shown that the hypergeometric function $F(a, b; b+1; z)$ has no zeros in a specified half-plane. It is also shown that the zeros of the hypergeometric polynomials

$$F(-n, kn+\ell+1; kn+\ell+2; z)$$

cluster on one loop of a specified lemniscate. Other results then follow from quadratic relations.

Keywords. Hypergeometric functions, hypergeometric polynomials, Jacobi polynomials, zeros, Euler integral, asymptotic curves.

2000 MSC. 33C05, 30C15, 33C45.

1. Introduction.

The standard Gauss hypergeometric function is

$$F(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k\,(b)_k}{k!\,(c)_k} z^k, \qquad |z| < 1,$$

where a, b, c are complex parameters and

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

is Pochhammer's symbol. It is clear that $F(a, b; c; z) = F(b, a; c; z)$. If a is a negative integer $-n$, the series terminates and reduces to a polynomial of degree n, called a *hypergeometric polynomial*. The classical orthogonal polynomials provide many examples.

If the parameters b and c are real, and if none of the numbers b, c, $c-b$ is a negative integer larger than $-n$, then the hypergeometric polynomial $F(-n, b; c; z)$ has precisely n simple zeros in the complex plane, symmetric with respect to the real axis. The zeros are simple because a hypergeometric function $w = F(a, b; c; z)$ satisfies the differential equation

$$z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0,$$

Received January 28, 2002.

ISSN 1617-9447/\$ 2.50

and so if $w(z_0) = w'(z_0) = 0$ at some point $z_0 \neq 0, 1$, then $w(z) \equiv 0$ by the uniqueness theorem. But $F(a, b; c; 0) = 1$, and $F(-n, b; c; 1) = (c-b)_n/(c)_n \neq 0$ by the Chu-Vandermonde identity ([1, p. 67]).

In recent papers [4, 5, 6, 7, 8, 9], the zeros of various classes of hypergeometric polynomials have been investigated. A general theorem of Borwein and Chen [3] was invoked in [4] to show that as $n \to \infty$, the zeros of $F(-n, kn+1; kn+2; z)$ cluster on the right-hand loop of the lemniscate

$$|z^k(z-1)| = \frac{k^k}{(k+1)^{k+1}}, \qquad \text{with } \operatorname{Re}\{z\} > \frac{k}{k+1}.$$

Here $k = 1, 2, \ldots$, and the link with the Borwein-Chen result was provided by the Euler integral formula

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt,$$

where $\operatorname{Re}\{c\} > \operatorname{Re}\{b\} > 0$. Subsequent work [9] gave an independent and detailed proof that applies to arbitrary $k > 0$, not necessarily an integer.

In the present paper, the analysis of [9] is adapted to obtain the same asymptotic result for zeros of the more general polynomials $F(-n, kn+\ell+1; kn+\ell+2; z)$, where $k > 0$ and $\ell \geq 0$. It is also shown that these polynomials have no zeros in the half-plane $\operatorname{Re}\{z\} \leq k/(k+1)$, a result conjectured in [9] for $\ell = 0$ on the basis of examples generated by Mathematica graphics. In fact, we are now able to establish more general statements about zero-free regions of hypergeometric functions, not necessarily polynomials. Specifically, we propose to prove two basic theorems, from which other results will follow.

Theorem 1. *For arbitrary real numbers $a < 0$, $k > 0$, and $b \geq 1 - ka$, the hypergeometric function*

$$F(a, b; b+1; z)$$

has no zeros in the half-plane

$$\operatorname{Re}\{z\} \leq \frac{k}{k+1}.$$

Theorem 2. *For arbitrary $k > 0$ and $\ell \geq 0$ the zeros of the hypergeometric polynomials*

$$F(-n, kn+\ell+1; kn+\ell+2; z)$$

cluster on the loop of the lemniscate

$$|z^k(z-1)| = \frac{k^k}{(k+1)^{k+1}} \qquad \textit{with} \qquad \operatorname{Re}\{z\} > \frac{k}{k+1}$$

as $n \to \infty$.

2. Proofs of theorems

For $c = b + 1$, the Euler integral takes the form

$$F(a, b; b + 1; z) = b \int_0^1 t^{b-1}(1 - zt)^{-a}\, dt.$$

For $b = 1 - ka$, we can write

$$F(a, 1 - ka; 2 - ka; z) = (1 - ka) \int_0^1 f(t)^{-a}\, dt,$$

where $f(t) = t^k(1-zt)$. Observe now that $f(t)$, viewed as a function of a *complex* variable t, has zeros at 0 and $1/z$, while $f'(t)$ vanishes at $k/[(k + 1)z]$. Thus the surface of height $|f(t)|$ has a saddle point at $k/[(k + 1)z]$. We will need a simple lemma that was proved in [9].

Lemma 1. *If* $\mathrm{Re}\{z\} \leq k/(k+1)$*, the function* $|f(t)|$ *has a unique path of steepest ascent from* 0 *to* 1*. If* $\mathrm{Re}\{z\} > k/(k + 1)$*, it has a unique path of steepest ascent from* $1/z$ *to* 1*.*

Proof of Theorem 1. First suppose that $b = 1 - ka$. By the lemma, the function $|f(t)|$ has a path C of steepest ascent from 0 to 1. Since the integrand is analytic, we can deform the path of integration in the Euler integral and reduce the problem to showing that

$$\int_C [t^k(1 - zt)]^{-a}\, dt \neq 0 \qquad \text{for } \mathrm{Re}\{z\} \leq \frac{k}{k+1}.$$

But on the curve C the argument of $f(t)$ is constant. Thus it will suffice to show that the tangent vector dt lies always in a fixed half-plane $\mathrm{Re}\{e^{i\theta}dt\} > 0$, which clearly implies that the integral cannot vanish. However, the tangent direction to C is given by the gradient of $|f|$. For convenience we take $z = 1$ and consider the gradient of

$$g(u, v) = |t^k(1 - t)|^2 = (u^2 + v^2)^k[(1 - u)^2 + v^2],$$

where $t = u + iv$. But a simple calculation gives

$$\frac{\partial g}{\partial v} = 2v(u^2 + v^2)^{k-1}\{k[(1 - u)^2 + v^2] + u^2 + v^2\},$$

which is positive for $v = \mathrm{Im}\{t\} > 0$ and negative for $\mathrm{Im}\{t\} < 0$. Observe now that the path of steepest ascent of $|t^k(1-t)|$ from 0 to a point $t_0 = u_0 + iv_0$ with $u_0 < 1/2$ and $v_0 \neq 0$ cannot cross the real axis, so that the gradient vector lies always in the upper half-plane if $v_0 > 0$ and in the lower half-plane if $v_0 < 0$. If $v_0 = 0$, the path of steepest ascent is the linear segment from 0 to u_0, and its tangent direction is constant.

Finally, if $b = 1 - ka + \ell$ for some $\ell > 0$, we may apply the preceding analysis with k replaced by $k - \ell/a$ to conclude that $F(a, b; b+1; z) \neq 0$ in the half-plane

$$\mathrm{Re}\{z\} < \frac{k - \frac{\ell}{a}}{1 + k - \frac{\ell}{a}},$$

which contains the half-plane $\mathrm{Re}\{z\} < k/(k+1)$. ∎

Proof of Theorem 2. Let for convenience

$$F_n(z) = F(-n, kn + \ell + 1; kn + \ell + 2; z).$$

Then the Euler representation is

$$F_n(z) = (kn + \ell + 1) \int_0^1 t^\ell [f(t)]^n \, dt,$$

where $f(t) = t^k(1 - zt)$. By Theorem 1, the polynomial has no zeros in the half-plane $\mathrm{Re}\{z\} \leq k/(k+1)$. If $\mathrm{Re}\{z\} > k/(k+1)$, an appeal to the lemma allows us to deform the path of integration to write

$$\int_0^1 t^\ell [f(t)]^n \, dt = \int_0^{\frac{1}{z}} t^\ell [f(t)]^n \, dt + \int_{\frac{1}{z}}^1 t^\ell [f(t)]^n \, dt = I_1(z) + I_2(z),$$

following the linear path from 0 to $1/z$ and the path of steepest ascent from $1/z$ to 1. To compute the first integral, substitute $t = s/z$ for $0 \leq s \leq 1$, so that

$$z^{kn+\ell+1} I_1(z) = \frac{n!\,\Gamma(kn + \ell + 1)}{\Gamma((k+1)n + \ell + 2)} \sim \frac{\sqrt{2\pi}}{\sqrt{n}} \left(\frac{k^k}{(k+1)^{k+1}} \right)^n \frac{k^{\ell+1/2}}{(k+1)^{\ell+3/2}},$$

by Stirling's formula $\Gamma(x+1) \sim e^{-x} x^x \sqrt{2\pi x}$ and the fact that $(x+a)^{x+a} \sim e^a x^{x+a}$ as $x \to \infty$.

The second integral can be computed as follows. Along the path of steepest ascent from $1/z$ to 1, the function $f(t)$ will have constant argument, with $f(1/z) = 0$ and $f(1) = 1 - z$. Hence we can parametrize the path by letting

$$f(t) = t^k(1 - zt) = r(1 - z), \qquad 0 \leq r \leq 1 \tag{1}$$

so that

$$t^{k-1}[k - (k+1)zt] \, dt = (1 - z) \, dr$$

and

$$I_2(z) = (1 - z)^n \int_0^1 \frac{(1 - zt)\, t^{\ell+1} \, r^n}{k - (k+1)zt} \, dr = (1 - z)^n \, J(z),$$

say. But the zeros of F_n satisfy $I_1(z) + I_2(z) = 0$, so the above expressions show that

$$|z^k(1 - z)| \cdot |J(z)|^{1/n} \to \frac{k^k}{(k+1)^{k+1}} \tag{2}$$

as $n \to \infty$, uniformly for all zeros $z = z_{nj}$ of F_n, $j = 1, 2, \ldots, n$.

It now remains to show that $|J(z)|^{1/n} \to 1$ as $n \to \infty$, uniformly for all zeros $z = z_{nj}$. We observe first that $|z-1| \geq \varepsilon$ for some $\varepsilon > 0$ and all zeros $z = z_{nj}$. Indeed, if some sequence of zeros were to converge to the point 1, then $J(z)$ would be bounded and the left-hand side of (2) would tend to zero, a contradiction. We may also restrict attention to zeros that satisfy $|z - k/(k+1)| \geq \varepsilon$, since the point $k/(k+1)$ lies on the alleged asymptotic curve. Then the integrand in $J(z)$ is uniformly bounded, and

$$\limsup_{n\to\infty} |J(z)|^{1/n} \leq 1,$$

uniformly for points z with $\mathrm{Re}\{z\} > k/(k+1)$ and $|z-k/(k+1)| \geq \varepsilon$. Note that $\mathrm{Re}\{z\} > k/(k+1)$ implies that $|1/z| < (k+1)/k$, so that the points t on the path of steepest ascent from $1/z$ to 1 are uniformly bounded for $\mathrm{Re}\{z\} > k/(k+1)$.

To obtain a lower bound for $|J(z)|$, we argue as follows. The assumptions that $|z-1| \geq \varepsilon$ and $|z - k/(k+1)| \geq \varepsilon$ imply that

$$0 < A \leq \left| \frac{1-z}{k-(k+1)z} \right| \leq B < \infty$$

for some constants A and B depending only on ε. Let

$$\varphi = \varphi(z) = \arg\left\{ \frac{1-z}{k-(k+1)z} \right\}.$$

Then for some $\delta > 0$ sufficiently small, we see that

$$\mathrm{Re}\left\{ e^{-i\varphi} \frac{(1-zt)t^{\ell+1}}{k-(k+1)zt} \right\} > \frac{A}{2}$$

whenever $|t-1| < \delta$. But the function $t = t(r)$, as determined implicitly by (1) with $t(1) = 1$, has a derivative

$$t'(1) = \frac{1-z}{k-(k+1)z},$$

and so $|t'(1)| \leq B$. Hence it is possible to choose a number ρ with $0 < \rho < 1$, independent of z and sufficiently close to 1 that $0 < 1 - t(r) < \delta$ whenever $\rho \leq r \leq 1$. We can then write

$$J(z) = \int_0^\rho \frac{(1-zt)\, t^{\ell+1}\, r^n}{k-(k+1)zt}\, dr + \int_\rho^1 \frac{(1-zt)\, t^{\ell+1}\, r^n}{k-(k+1)zt}\, dr = J_1(z) + J_2(z).$$

Now

$$n|J_1(z)| \leq Cn \int_0^\rho r^n\, dr \leq C\rho^{n+1} \to 0$$

as $n \to \infty$. On the other hand,

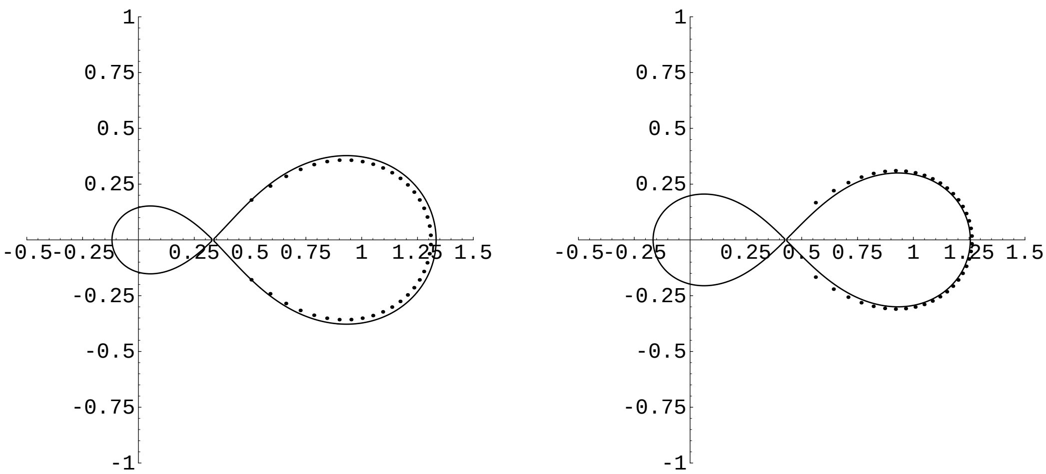
$$\begin{aligned} n\,\mathrm{Re}\Big\{e^{-i\varphi}J_2(z)\Big\} &= n\int_\rho^1 \mathrm{Re}\Big\{e^{-i\varphi}\frac{(1-zt)\,t^{\ell+1}}{k-(k+1)zt}\Big\}r^n\,dr \\ &> \frac{An}{2}\int_\rho^1 r^n\,dr \to \frac{A}{2} \end{aligned}$$

as $n \to \infty$. Thus $n|J_2(z)| \ge A/3$ for all $n \ge N$, and it follows that

$$\liminf_{n\to\infty} |J(z)|^{1/n} = \liminf_{n\to\infty} \{n|J_1(z)+J_2(z)|\}^{1/n} \ge 1,$$

uniformly for points z with $|z-1| \ge \varepsilon$ and $|z-k/(k+1)| \ge \varepsilon$. Combining this with the upper estimate, we conclude that $|J(z)|^{1/n} \to 1$ as $n \to \infty$, uniformly for all zeros $z = z_{nj}$ with $|z-k/(k+1)| \ge \varepsilon$. In view of (2), this completes the proof. ∎

Figure 1 illustrates both theorems with the help of Mathematica graphics. It displays the zeros of hypergeometric polynomials $F(-n, kn+\ell+1; kn+\ell+2; z)$ and lemniscates $|z^k(1-z)| = k^k/(k+1)^{k+1}$ for $n = 40$ and various values of k and ℓ.



$n = 40,\ k = 0.5,\ \ell = 5$ $n = 40,\ k = 0.75,\ \ell = 2$

Figure 1. Zeros of $F(-n, kn+\ell+1; kn+\ell+2; z)$, curve $|z|^k|1-z| = k^k/(k+1)^{k+1}$.

In connection with Theorem 1, it should be noted that Weir [12, 13] has studied the zeros of $F(a, b; b+1; z)$ for $a < -1$ and $b > 0$. For applications to Bergman space theory, it is important to determine the range of parameters a and b for which these functions have no zeros in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Weir's work [12] with Mathematica graphics suggests that this is true at least for $-4 < a < -1$ and $0 < b < \infty$. In [13] she proves that $F(a, b; b+1; z)$ has no zeros in $\mathbb{D}$ when $-2 < a < -1$ and $1/2 \leq b < \infty$. Remarkably, her proof rests entirely on the theory of Bergman spaces.

3. Consequences

Appealing to standard transformations of hypergeometric functions, we can apply Theorems 1 and 2 to obtain other results of similar type. It is also possible to derive information about the zeros of Jacobi polynomials and other special functions expressible in terms of hypergeometric functions.

Two standard transformations are relevant. Their simple proofs may be found in [1, p. 68].

$$\begin{aligned} \text{Euler's transformation:} \quad F(a,b;c;z) &= (1-z)^{c-a-b}\, F(c-a,c-b;c;z). \\ \text{Pfaff's transformation:} \quad F(a,b;c;z) &= (1-z)^{-a}\, F\big(a,c-b;c;\tfrac{z}{z-1}\big) \\ &= (1-z)^{-b}\, F\big(c-a,b;c;\tfrac{z}{z-1}\big). \end{aligned}$$

When specialized to $c = b+1$, these transformations become

$$F(a,b;b+1;z) = (1-z)^{1-a}\, F(b-a+1,1;b+1;z), \tag{3}$$

$$F(a,b;b+1;z) = (1-z)^{-a} F\big(a,1;b+1;\tfrac{z}{z-1}\big), \tag{4}$$

$$F(a,b;b+1;z) = (1-z)^{-b} F\big(b-a+1,b;b+1;\tfrac{z}{z-1}\big). \tag{5}$$

We shall also use the basic quadratic transformation (cf. [1, p. 125], [2, p. 111, no. 2])

$$F(2a,2b;a+b+\tfrac{1}{2};z) = F(a,b;a+b+\tfrac{1}{2};4z(1-z)).$$

When combined with Euler's transformation, it leads to the formula ([2, p. 112, no. 22], [10, p. 459, no. 84])

$$F(a,1-a;c;z) = (1-z)^{c-1} F\big(\tfrac{c-a}{2},\tfrac{c+a-1}{2};c;4z(1-z)\big). \tag{6}$$

With $c = 2-a$, this reduces to

$$F(a,1-a;2-a;z) = (1-z)^{1-a} F(\tfrac{1}{2},1-a;2-a;4z(1-z)). \tag{7}$$

An application of Euler's transformation to (6) yields the similar formula ([2, p. 112, no. 23], [10, p. 459, no. 85])

$$F(a,1-a;c;z) = (1-z)^{c-1}(1-2z) F\big(\tfrac{a+c}{2},\tfrac{c-a+1}{2};c;4z(1-z)\big),$$

which gives in particular

$$F(a,1-a;2-a;z) = (1-z)^{1-a}(1-2z) F(1,\tfrac{3}{2}-a;2-a;4z(1-z)). \tag{8}$$

An application of Pfaff's transformation to (6) yields the additional formula ([2, p. 112, no. 24]; [10, p. 459, no. 86])

$$F(a,1-a;c;z) = (1-z)^{c-1}(1-2z)^{a-c} F\big(\tfrac{c-a}{2},\tfrac{c-a+1}{2};c;\tfrac{4z(z-1)}{(1-2z)^2}\big),$$

which with $c = 2 - a$ reduces to

(9) $F(a, 1-a; 2-a; z) = (1-z)^{1-a}(1-2z)^{2(a-1)}F\big(\frac{3}{2} - a, 1-a; 2-a; \frac{4z(z-1)}{(1-2z)^2}\big).$

Combining Theorems 1 and 2 with the special transformations (3), (4), and (5), we arrive at the following results.

Corollary 1. *For $a < 0$, $k > 0$, and $b \geq 1 - ka$, the hypergeometric functions*

$$F(a, 1; b+1; z) \qquad \text{and} \qquad F(b-a+1, b; b+1; z)$$

have no zeros in the disk $\big|z - (1-k)/2\big| \leq (k+1)/2$; while the function

$$F(b-a+1, 1; b+1; z)$$

has no zeros in the half-plane $\mathrm{Re}\{z\} \leq k/(k+1)$.

Corollary 2. *For fixed $k > 0$ and $\ell \geq 1$, the hypergeometric function*

$$F((k+1)n + \ell + 1, 1; kn + \ell + 1; z)$$

has exactly n simple zeros, which cluster on the right-hand loop of the lemniscate $|z^k(z-1)| = k^k/(k+1)^{k+1}$ as $n \to \infty$. The polynomial

$$F(-n, 1; kn + \ell + 1; z)$$

and the function

$$F((k+1)n + \ell + 1, kn + \ell; kn + \ell + 1; z)$$

have identical zero-sets, which cluster on the outer loop of the curve

$$(k+1)^{k+1}|z|^k = k^k|z-1|^{k+1}$$

as $n \to \infty$.

Note that the curve indicated in Corollary 2 is the preimage of the lemniscate $|z^k(z-1)| = k^k/(k+1)^{k+1}$ under the Möbius transformation $z \mapsto z/(z-1)$.

Taking $k = 1$ in Corollary 1, we see that the functions $F(a, 1; b+1; z)$ and $F(b-a+1, b; b+1; z)$ have no zeros in the unit disk $\mathbb{D}$ when $a < 0$ and $b \geq 1 - a$. Thus $F(a, b; b+1; z)$ has no zeros in $\mathbb{D}$ when $b + 1 < a \leq 2b$, so that $b > 1$.

Figure 2 shows the zeros of $F(a, 1; b+1; z)$ for $a = -40$ and $b = 41$. According to Corollary 1, these points lie outside the unit circle ($k = 1$); and by Corollary 2 (with $n = 40$, $k = 1$, and $\ell = 1$), they cluster on the outer loop of the curve $4|z| = |z-1|^2$ as $n \to \infty$.

Next we appeal to the quadratic relations (7), (8), and (9) to derive some further consequences of Theorems 1 and 2. The following statements appeared in [4], but the proofs were incomplete in several respects because Theorem 1 was not then available.

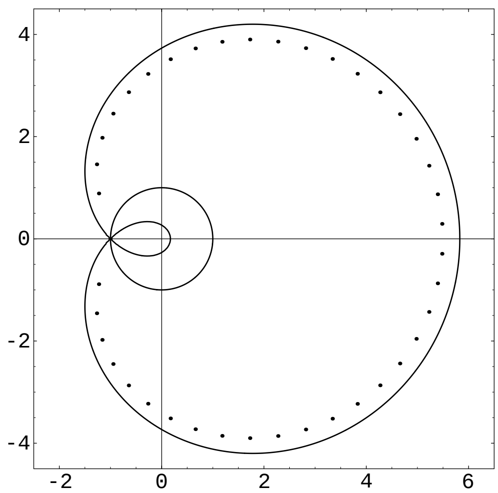


FIGURE 2. Zeros of $F(-40, 1; 42; z)$ with unit circle and curve $4|z| = |z-1|^2$.

Corollary 3. *The hypergeometric functions*

$$F(\tfrac{1}{2}, n+1; n+2; z) \qquad \text{and} \qquad F(1, n+\tfrac{3}{2}; n+2; z)$$

have identical zero-sets, consisting of precisely n points, which tend uniformly to the unit circle as $n \to \infty$. Similarly, the function $F(n+3/2, n+1; n+2; z)$ has n simple zeros, and they cluster on the vertical line $\mathrm{Re}\{z\} = 1/2$ as $n \to \infty$.

Proof. By (7) and (8),

$$\begin{aligned} F(-n, n+1; n+2; z) &= (1-z)^{n+1} F(\tfrac{1}{2}, n+1; n+2; 4z(1-z)) \\ &= (1-z)^{n+1}(1-2z) F(1, n+\tfrac{3}{2}; n+2; 4z(1-z)). \end{aligned}$$

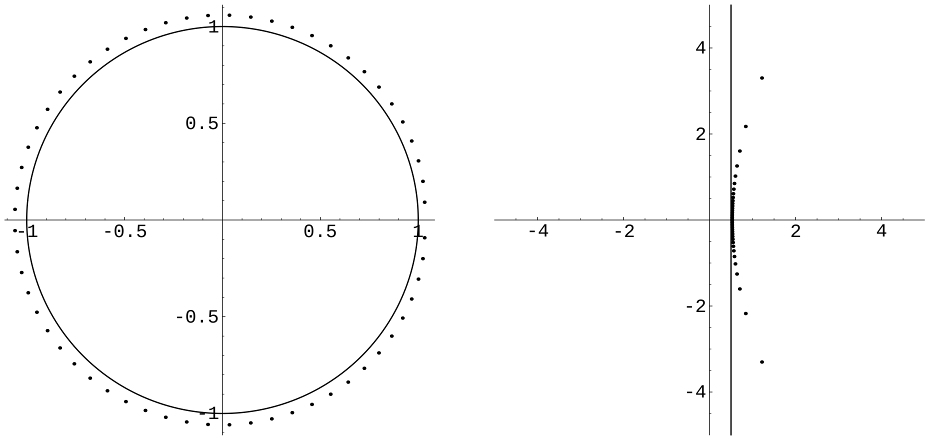
Now take $k = 1$ and $\ell = 0$, and apply Theorems 1 and 2 to conclude that $F(-n, n+1; n+2; z) \neq 0$ in $\mathrm{Re}\{z\} \leq 1/2$, and that its n zeros cluster on the lemniscate $|4z(1-z)| = 1$ as $n \to \infty$. Thus the hypergeometric functions $F(1/2, n+1; n+2; w)$ and $F(1, n+3/2; n+2; w)$ both have exactly n simple zeros, images of the zeros of $F(-n, n+1; n+2; z)$ under the mapping $w = 4z(1-z)$, which is *one-to-one* for $\mathrm{Re}\{z\} > 1/2$. In particular, the zero-sets of these two hypergeometric functions coincide, and they cluster on the circle $|w| = 1$ as $n \to \infty$.

Similarly, the relation (9) gives

$$F(-n, n+1; n+2; z) = \left[\frac{1-z}{(1-2z)^2}\right]^{n+1} F\left(n+\frac{3}{2}, n+1; n+2; \frac{-4z(1-z)}{1-4z(1-z)}\right).$$

Again the polynomial $F(-n, n+1; n+2; z)$ has n simple zeros, all in the half-plane $\mathrm{Re}\{z\} > 0$, and they cluster on the lemniscate $|4z(1-z)| = 1$. But the mapping $\zeta = 4z(1-z)$ sends the right-hand loop of the lemniscate one-to-one onto the unit circle $|\zeta| = 1$. Furthermore, the zeros of $F(n+3/2, n+1; n+2; w)$ are in one-to-one correspondence with points $\zeta = 4z(1-z)$ under the Möbius transformation $w = -\zeta/(1-\zeta)$, which carries the circle $|\zeta| = 1$ to the vertical line $\mathrm{Re}\{w\} = 1/2$. ∎

Figure 3 illustrates Corollary 3 by showing the zeros of $F(1, n+3/2; n+2; z)$ for $n = 60$ with the unit circle (where they cluster as $n \to \infty$); and the zeros of $F(n+3/2, n+1; n+2; z)$ for $n = 40$ with their asymptotic line $\mathrm{Re}\{z\} = 1/2$.



$F(1, n+3/2; n+2; z)$, $n = 60$ $F(n+3/2, n+1; n+2; z)$, $n = 40$

FIGURE 3. Zeros of hypergeometric functions and their asymptotic curves.

Finally, Theorem 2 gives asymptotic information about the zeros of Jacobi polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal over the interval $[-1, 1]$ with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$, provided that $\alpha > -1$ and $\beta > -1$, so that w is integrable. Then all n zeros of $P_n^{(\alpha,\beta)}(z)$ lie in the real interval $(-1, 1)$. However, the zeros behave in a complicated way when either of the parameters α or β drops below the critical value -1. Classical formulas of Hilbert and Klein (cf. [11, p. 144]) then specify the number of real zeros in each of the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. The actual trajectories of the zeros in the ultraspherical case, where $\alpha = \beta < -1$, were described in [7].

The Jacobi polynomials have the representation ([11, p. 62])

$$P_n^{(\alpha,\beta)}(z) = \binom{n+\alpha}{n} F\left(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1-z}{2}\right). \tag{10}$$

Thus our theorems can be applied to give the following results.

Corollary 4. *(i) For $k > 0$ and $\ell \geq 1$, the Jacobi polynomials*

$$P_n^{(kn+\ell, -n-1)}(z)$$

have no zeros in the half-plane $\mathrm{Re}\{z\} \geq (1-k)/(1+k)$*. As* $n \to \infty$*, their zeros cluster on the loop of the lemniscate*

$$|(z-1)^k(z+1)| = \left(\frac{2}{k+1}\right)^{k+1} k^k \qquad \textit{with} \qquad \mathrm{Re}\{z\} < \frac{1-k}{1+k}.$$

(ii) For $k > 0$ *and* $\ell \geq 1$*, the Jacobi polynomials*

$$P_n^{(kn+\ell,-(k+1)n-\ell)}(z)$$

have no zeros in the disk $|z-k| \leq k+1$*. As* $n \to \infty$*, their zeros cluster on the outer loop of the curve*

$$2(k+1)^{k+1}|z-1|^k = k^k|z+1|^{k+1}.$$

Proof. (i). In view of (10),

$$P_n^{(kn+\ell,-n-1)}(z) = \binom{(k+1)n+\ell}{n} F\big(-n, kn+\ell; kn+\ell+1; \tfrac{1-z}{z}\big).$$

But Theorem 1 says that all zeros of $F(-n, kn+\ell; kn+\ell+1; w)$ lie in the half-plane $\mathrm{Re}\{w\} > k/(k+1)$ if $\ell \geq 1$, and Theorem 2 says that they cluster on the lemniscate $|w^k(w-1)| = k^k/(k+1)^{k+1}$. Thus all zeros of the Jacobi polynomial satisfy $\mathrm{Re}\{(1-z)/2\} > k/(k+1)$, or $\mathrm{Re}\{z\} < (1-k)/(1+k)$; and they cluster on the lemniscate

$$\left|\left(\frac{1-z}{2}\right)^k\left(\frac{1-z}{2}-1\right)\right| = \frac{k^k}{(k+1)^{k+1}},$$

which reduces to the given form.

(ii). Now apply (10) to see that

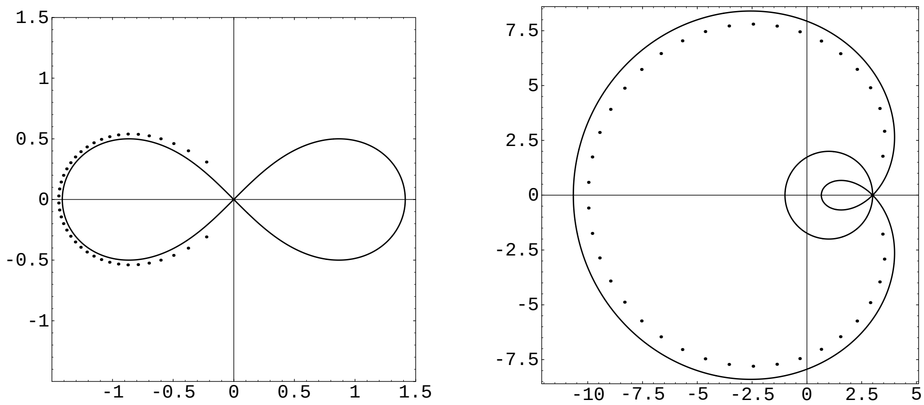
$$P_n^{(kn+\ell,-(k+1)n-\ell)}(z) = \binom{(k+1)n+\ell}{n} F\big(-n, 1; kn+\ell+1; \tfrac{1-z}{z}\big).$$

By Corollary 1, all zeros of the polynomial $F(-n, 1; kn+\ell+1; w)$ lie outside the circle $|w-(1-k)/2| = (k+1)/2$ when $\ell \geq 1$. This says that all zeros of the corresponding Jacobi polynomial lie outside the circle $|z-k| = k+1$. By Corollary 2, the zeros of $F(-n, 1; kn+\ell+1; w)$ cluster on the outer loop of the curve

$$(k+1)^{k+1}|w|^k = k^k|w-1|^{k+1},$$

which transforms to the given curve under the mapping $w = (1-z)/2$. ∎

A special case of the asymptotic statement in Part (i) of Corollary 4 was given (with a slight error) in [4]. Part (ii) seems to be new. Figure 4 illustrates both results. It shows the zeros of $P_n^{(kn+\ell,-n-1)}(z)$ for $n = 40$, $k = 1$, $\ell = 1$, with their asymptotic lemniscate $|z^2-1| = 1$; and the zeros of $P_n^{(kn+\ell,-(k+1)n-\ell)}(z)$ for $n = 40$, $k = 1$, $\ell = 1$, with their asymptotic curve $8|z-1| = |z+1|^2$. These zeros lie respectively in the left half-plane $\mathrm{Re}\{z\} < 0$, and outside the circle $|z-1| = 2$, as Corollary 4 asserts.



$P_{40}^{(41,-41)}(z)$ $P_{40}^{(41,-81)}(z)$

FIGURE 4. Zeros of Jacobi polynomials and their asymptotic curves.

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