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The Wavelet Transform in Clifford Analysis

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Abstract. The upper half space $G = \{(x_0, \dots, x_n) : x_0 > 0\}$ can be considered as the group generated by dilations and translations on $\mathbb{R}^n$. This group has a natural unitary representation on $L_2(\mathbb{R}^n)$. Using the continuous wavelet transform, certain Banach and Hilbert spaces of functions monogenic (i.e. solutions of the Cauchy-Riemann operator) on the Poincaré half space are constructed. The Hilbert spaces are linked with the fractional calculus of the Dirac operator on $\mathbb{R}^n$.

Keywords. Clifford analysis, wavelet transform, weighted Bergman spaces, invariant Dirac operator.

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1. Introduction

Let G be a Lie group with a left invariant Haar measure μ. Then the representation α of G on $L_2(G, \mu)$ given by $\alpha(g)f(a) = f(ga)$ is obviously unitary. Suppose now that H is an arbitrary Hilbert space, with a representation β of G, and let f be an arbitrary non-zero vector of H. For each $h \in H$, we can define the function $\pi(h)$ on G by

$$\pi(h)(g) = (\beta(g)f, h),$$

where $(\cdot, \cdot)$ is the inner product on H. The circumstances under which π becomes a unitary mapping from H to (a subspace of) $L_2(G)$ are well known (see [7]). This technique is described under various forms and with various degrees of generality in the literature, and is very difficult to attribute to one author. Usually in the literature f is either called a vacuum vector or a wavelet. Since the application of this theory in this paper has similarities with certain wavelets (as described in e.g. [2]), we have adopted the name wavelet transform for this case. The group G we use here is generated by translations and dilations. Using suitable functions f, it is possible to make sure the image of $L_2(\mathbb{R}^n)$ consists of monogenic functions, i.e. solutions of some Cauchy-Riemann equation. One can consider either the Cauchy-Riemann equation for Euclidean or for Poincaré metrics (there is an

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elementary relation between the two), but it seems conceptually more logical to use the Poincaré metric.

Doing this we solve a question of Gilbert and Murray (see [6]) regarding the non-triviality of certain Banach spaces of monogenic functions. Apart from that we construct a series of Hilbert spaces of monogenic functions, which are linked with representations of the conformal group of $\mathbb{R}^n$, which encompasses G. The wavelets considered turn out to be, in a natural way, fractional Dirac derivatives of the Cauchy kernel, $\mathbf{D}^\beta E$, where $\mathbf{D}$ is the Dirac operator on $\mathbb{R}^n$ and E the Cauchy kernel in $\mathbb{R}^{n+1}$. Hence the boundary values of the wavelet transforms T_β are fractional Dirac derivatives of L_2 functions.

2. Preliminaries

2.1. Clifford algebra. Let $\mathbb{R}^n$ be n-dimensional Euclidean space, with the positive definite inner product $\langle \cdot, \cdot \rangle$. We can take an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ such that $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. We shall usually indicate vectors of $\mathbb{R}^n$ with lowercase boldface letters, i.e. use $\mathbf{x}$, $\mathbf{y}$, etc.

The Clifford algebra $\mathcal{C}\ell(\mathbb{R}, n)$ is the free algebra generated by $\mathbb{R}^n$ modulo the relation

$$\mathbf{x}^2 = -\langle \mathbf{x}, \mathbf{x} \rangle.$$

It can be considered as being the exterior algebra $\wedge \mathbb{R}^n$, where the extra information defining the inner product is included. From this point of view one has

$$\mathbf{x}\mathbf{y} = \mathbf{x} \wedge \mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle,$$

for arbitrary vectors. Clearly, for $n > 1$, $\mathcal{C}\ell(\mathbb{R}, n)$ is not commutative.

For practical reasons (e.g. use of the Fourier transform) it makes sense to complexify $\mathcal{C}\ell(\mathbb{R}, n)$ to $\mathcal{C}\ell(n)$, which then consists of elements of the form $a + \imath b$, where a and b are in $\mathcal{C}\ell(\mathbb{R}, n)$. Multiplication then assumes that $\imath$ commutes with all elements of the Clifford algebra. On $\mathcal{C}\ell(n)$ we can define an anti-automorphism by

$$\bar{\mathbf{x}} = -\mathbf{x}, \qquad \bar{\imath} = -\imath, \qquad \overline{ab} = \bar{b}\,\bar{a},$$

for arbitrary $\mathbf{x}$ in $\mathbb{R}^n$ and a and b in $\mathcal{C}\ell(n)$. There is a natural identification of $\mathbb{C}$ as a subspace of the finite-dimensional algebra $\mathcal{C}\ell(n)$, and so it makes sense to speak of the scalar part of a Clifford number, which will be denoted by $[a]_0$.

In this paper the so-called paravector formalism will be used. In this formalism, vectors in $\mathbb{R}^{n+1}$ are identified with elements of $\mathcal{C}\ell(n)$ by

$$x = x_0 + x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n = x_0 + \mathbf{x},$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^n$. Thus, instead of using the full Clifford algebra over $\mathbb{R}^{n+1}$, only the smaller algebra $\mathcal{C}\ell(n)$ is taken.

2.2. Monogenic functions. In Euclidean space $\mathbb{R}^n$ the Dirac operator $\mathbf{D}$ is defined by

$$\mathbf{D}f(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{e}_i \partial_i f(\mathbf{x}),$$

where ∂_i is the i-th partial derivative and f is a (sufficiently smooth) function which takes values in the Clifford algebra. Sometimes functions with values in a spinor space (i.e. an irreducible representation of the algebra, so that the multiplication with $\mathbf{e}_i$ makes sense) are used. This does not influence the theory.

Like in $\mathbb{R}^{n+1}$ there is a very similar operator, the Cauchy-Riemann operator

$$\mathcal{D} = \mathbf{D} + \partial_0.$$

Its properties are very similar to that of a Dirac operator, and there is an easy way to convert Dirac and Cauchy-Riemann operators. For a full overview of the theory of solutions of the equation $\mathcal{D}f = 0$, the so-called monogenic functions, we refer to [4].

Let Ω be a domain in $\mathbb{R}^{n+1}$, and let α be a strictly positive, sufficiently smooth (e.g. C^∞) function, the so-called conformal weight. Equipped with the metric ${ds_\alpha}^2 = {ds_E}^2/\alpha^2$, where ds_E is the Euclidean metric, Ω becomes a conformally flat manifold. The Cauchy-Riemann operator (sometimes called the spinor Dirac operator or Atiyah-Singer operator, as for this case other Dirac-like operators are in use) in this case is given by

$$\mathcal{D}_\alpha f(\mathbf{x}) = \alpha(\mathbf{x})^{(2+n)/2} \mathcal{D}\left(\alpha(\mathbf{x})^{-n/2} f(\mathbf{x})\right).$$

A function satisfying $\mathcal{D}_\alpha f = 0$ is called monogenic in Ω, or monogenic for α. Since any function f, monogenic for α, can be written in the form $\alpha^{(n-1)/2} g$, where g is Euclidean monogenic, the (local) theory of monogenic functions on conformally flat manifolds is equivalent with the (local) theory of monogenic functions on Euclidean space.

The specific case of interest here is where the domain is the half space $P = \{\mathbf{x} : x_0 > 0\}$ with the Poincaré metric ${ds_P}^2 = {ds_E}^2/{x_0}^2$. The Cauchy-Riemann operator $\mathcal{D}_P$ of this manifold is given by $\mathcal{D}_P = x_0^{(2+n)/2} \mathcal{D} x_0^{-n/2}$.

2.3. Hilbert modules. A (right) module $\mathcal{T}$ over $\mathcal{C}\ell(n)$ is a vector space such that for each a the map

$$\begin{array}{rcl} R(a)\colon \mathcal{T} & \to & \mathcal{T}, \\ f & \mapsto & R(a)f = fa, \end{array}$$

is a linear map which is well defined as a right multiplication, i.e. that $R(1) = Id$ and $R(ab) = R(b)R(a)$. We only consider modules here where elements of $\mathcal{T}$ are Clifford valued functions, and where multiplication $R(a)$ is got by multiplying pointwise from the right by a, that is,

$$R(a)f(\mathbf{x}) = f(\mathbf{x})a.$$

A linear map $\mathcal{T} \times \mathcal{T} \to \mathcal{C}\ell(n)$ satisfying

$$\begin{aligned}(f,g) &= \overline{(g,f)},\\ (f,g\lambda) &= (f,g)\lambda,\end{aligned}$$

for any f and g in $\mathcal{T}$ and λ in $\mathcal{C}\ell(n)$ is called a Clifford valued inner product. If the scalar inner product $[(\cdot,\cdot)]_0$ is positive definite, and if $\mathcal{T}$ is complete for the resulting norm, $\mathcal{T}$ is called a Hilbert module.

For a Hilbert module $\mathcal{T}$ of Clifford valued functions on a certain domain Ω, we say that $\mathcal{T}$ has a reproducing kernel K if the point evaluation is continuous in each point. The reproducing kernel K is a function on Ω such that for all $\mathbf{x} \in \Omega$

(i) $K(\mathbf{x},\cdot) \in \mathcal{T}$.
(ii) $f(\mathbf{x}) = \langle K(\mathbf{x},\cdot), f\rangle_{\mathcal{T}}$ for all f in $\mathcal{T}$.

It has been proved (see [3]) that such a reproducing kernel exists if and only if the point evaluation is continuous.

2.4. The group of motions and the wavelet transform.

We can turn $G = \mathbb{R}_0^+ \times \mathbb{R}^n$ into a group defining multiplication by

$$(a,\mathbf{u})(b,\mathbf{v}) = (ab, \mathbf{u} + a\mathbf{v}).$$

The geometric action of an element of G is given by

$$g(a,\mathbf{u})x = ax + \mathbf{u},$$

which is meaningful for $x \in \mathbb{R}^{n+1}$, so $g(a,\mathbf{u})$ can be considered as a similarity of $\mathbb{R}^{n+1}$ leaving $\mathbb{R}^n$ invariant. It is easily seen that g in fact defines a group action, given by, $g(a,\mathbf{u})g(b,\mathbf{v}) = g(ab, a\mathbf{u} + \mathbf{v})$.

Consider the space $L_2(\mathbb{R}^n)$ of $\mathcal{C}\ell(n)$ valued functions with the inner product $\langle\cdot,\cdot\rangle_{L_2}$ given by

$$\langle f,g\rangle_{L_2} = \int_{\mathbb{R}^n} \bar{f}g.$$

This inner product is $\mathcal{C}\ell(n)$ valued. The complex valued inner product $[\langle\cdot,\cdot\rangle_{L_2}]_0$ is positive definite, and turns $L_2(\mathbb{R}^n)$ into a Hilbert space with a norm, $\|\cdot\|_{L_2}$. However, to obtain reproducing properties, we shall need the full Clifford algebra valued inner product. This has to be treated with care because of the non-commutativity of the algebra. In general, it is not true, for example, that $\langle f,g\rangle_{L_2} = \langle \bar{g},\bar{f}\rangle_{L_2}$.

We now define the following unitary representation of G on $L_2(\mathbb{R}^n)$:

$$W(a,\mathbf{u})f(\mathbf{x}) = \frac{1}{a^{n/2}} f\left(\frac{\mathbf{x}-\mathbf{u}}{a}\right). \tag{1}$$

Its Fourier transform can be defined by $\widehat{W}\mathcal{F}f = \mathcal{F}(Wf)$. Explicitly it is given by

$$\widehat{W}(a,\mathbf{u})\hat{f}(\boldsymbol{\omega}) = a^{n/2} e^{\imath\langle\boldsymbol{\omega},\mathbf{u}\rangle} \hat{f}(a\boldsymbol{\omega}).$$

Notice that we use the Fourier transform in n dimensions normalised by

$$\begin{aligned}\mathcal{F}f(\boldsymbol{\omega}) &= \hat{f} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\imath\langle\boldsymbol{\omega},\mathbf{x}\rangle} f(\mathbf{x})\,d\mathbf{x},\\ \mathcal{F}^{-}f(\mathbf{x}) &= \check{f} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\imath\langle\boldsymbol{\omega},\mathbf{x}\rangle} f(\boldsymbol{\omega})\,d\boldsymbol{\omega},\end{aligned}$$

at least for L_1 functions. This way $\mathcal{F}$ and $\mathcal{F}^-$ are isometries on $L_2(\mathbb{R}^n)$, and they are the inverse of each other. Moreover we define an inner product for Clifford valued functions on G by

$$\langle f, g\rangle_G = \int_0^\infty \frac{da}{a^{n+1}} \int_{\mathbb{R}^n} \overline{f(a,\mathbf{u})}g(a,\mathbf{u})\,d\mathbf{u}.$$

It is not hard to check that this is a left invariant measure on G. Indeed, for an arbitrary fixed element $(b,\mathbf{v})$ of G, the mapping ϕ on G given by

$$\phi(a,\mathbf{u}) = (b,\mathbf{v})(a,\mathbf{u}) = (ba, b\mathbf{u}+\mathbf{v})$$

has Jacobian (with respect to Lebesgue measure $da\,d\mathbf{u}$) b^{n+1}. With this inner product we define the module $L_2(G)$ in the usual way, the relation between complex and Clifford valued inner products obtained in the same way as in $L_2(\mathbb{R}^n)$.

Let ψ be an arbitrary function in $L_2(\mathbb{R}^n)$. Then the wavelet transform with wavelet ψ is defined as the mapping

$$\begin{aligned} T\colon L_2(\mathbb{R}^n) &\to F(g),\\ f &\mapsto Tf, \qquad Tf(a,\mathbf{u}) = \langle W(a,\mathbf{u})\psi, f\rangle_{L_2}.\end{aligned}$$

Here $F(G)$ is the vector space of $\mathcal{C}\!\ell(n)$ valued functions on G.

2.5. Radially symmetric wavelets. We formulate the following theorem for scalar valued wavelets, because it is stated for scalar valued functions in the literature. Proofs given in this setting remain valid in the case of Clifford valued functions if the wavelet itself is scalar (if f is scalar, then $\langle f, g\rangle_{L_2} = \langle \bar{g}, \bar{f}\rangle_{L_2}$). We shall be using Clifford valued wavelets, and shall prove the necessary properties based on these for scalar wavelets.

Theorem 2.1. *Let ψ be a radially symmetric* scalar valued *function in $L_2(\mathbb{R}^n)$ satisfying the admissibility condition (notice that $\mathcal{F}\psi$ is also radially symmetric, and so it makes sense to consider $\mathcal{F}\psi$ as a function of one real variable, the radius)*

$$C_\psi = (2\pi)^n \int_0^\infty \frac{dt}{t}\,|\mathcal{F}\psi(t)|^2 < \infty.$$

Then

(i) $(1/\sqrt{C_\psi})T$ *is an isometry from $L_2(\mathbb{R}^n)$ to a submodule $\mathcal{T}$ of $L_2(G)$.*

(ii) *The inversion formula is given by*

$$f(\mathbf{x}) = \frac{1}{C_\psi} \int_0^\infty \overline{\langle W(a,\cdot)\psi(\mathbf{x}), Tf(a,\cdot)\rangle}_{L_2} \frac{da}{a^{n+1}}$$

in the weak sense.

(iii) $\mathcal{T}$ *has a reproducing kernel given by*

$$K(a_1, \mathbf{u}_1; a_2\mathbf{u}_2) = \frac{1}{C_\psi} \langle W(a_1, \mathbf{u}_1)\psi, W(a_2, \mathbf{u}_2)\psi\rangle_{L_2}.$$

Proof. The proof for $n = 1$ is given explicitly in [2]. As indicated there, the proof for $n > 1$ is quite similar. ■

3. Generalised Cauchy wavelets

3.1. The Cauchy transform as a wavelet transform. An important rôle in this paper will be played by a generalisation of the Cauchy kernel

$$E(\mathbf{y} + y_0, \mathbf{x} + x_0) = \frac{1}{s_{n+1}} \frac{\mathbf{y} + y_0 - \mathbf{x} - \mathbf{x}_0}{|\mathbf{y} + y_0 - \mathbf{x} - \mathbf{x}_0|^{n+1}}.$$

Here s_{n+1} is the surface area of the unit sphere in $\mathbb{R}^{n+1}$, $s_k = 2\pi^{k/2}\Gamma(k/2)$. With it we associate the Cauchy transform of a function in $L_2(\mathbb{R}^n)$ by

$$Cf(\mathbf{y} + y_0) = \langle E(\mathbf{y} + y_0, \cdot), f\rangle_{L_2}.$$

This is a monogenic function in $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$. Notice that we only need the function $E(\mathbf{y} + y_0, \mathbf{x} + x_0)$ restricted to the hyperplane $x_0 = 0$ for the Cauchy transform.

The Cauchy transform is closely related to a wavelet transform. Indeed, if we take

$$\psi_0(\mathbf{x}) = E(1, \mathbf{x}),$$

we immediately get, for $a > 0$, that

$$W(a, \mathbf{u})\psi(\mathbf{x}) = \frac{a^{n/2}}{s_{n+1}} \frac{\mathbf{x} - \mathbf{u}}{|1 - (\mathbf{x} - \mathbf{u})|^{n+1}} = a^{n/2} E(\mathbf{u} + a, \mathbf{x}).$$

Writing the wavelet transform for this wavelet as T_0 we have

$$T_0 f(\mathbf{u} + a) = a^{n/2} Cf(\mathbf{u} + a).$$

To be able to define a suitable generalisation of this Cauchy transform we first need to calculate the n-dimensional Fourier transform of E. Fix $\mathbf{y} + y_0$, with $y_0 > 0$. Then the Fourier transform over $\mathbb{R}^n$ of $E(\mathbf{y} + y_0, \cdot)$ is given by

$$\mathcal{F}E(\mathbf{y} + y_0, \boldsymbol{\omega}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle \boldsymbol{\omega}, \mathbf{x}\rangle} E(\mathbf{y} + y_0, \mathbf{x})\, d\mathbf{x}.$$

In order to calculate this integral we consider the following functions, monogenic in $\mathbf{x}$ for the whole of $\mathbb{R}^{n+1}$

$$\begin{aligned}\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{x}) &= \frac{1}{2}\left(1+\imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)e^{\imath\langle\boldsymbol{\omega},\mathbf{x}-x_0\rangle|\boldsymbol{\omega}|},\\ \mathcal{P}^{-}_{\boldsymbol{\omega}}(\mathbf{x}) &= \frac{1}{2}\left(1-\imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)e^{\imath\langle\boldsymbol{\omega},\mathbf{x}+x_0\rangle|\boldsymbol{\omega}|}.\end{aligned}$$

It is fairly easy to prove monogenicity. We have

$$\mathcal{D}_{\mathbf{x}}e^{\imath\langle\boldsymbol{\omega},\mathbf{x}-x_0\rangle|\boldsymbol{\omega}|} = (\imath\boldsymbol{\omega}-|\boldsymbol{\omega}|)e^{\imath\langle\boldsymbol{\omega},\mathbf{x}-x_0\rangle|\boldsymbol{\omega}|},$$

and monogenicity follows from the fact that $(|\boldsymbol{\omega}|-\imath\boldsymbol{\omega})(|\boldsymbol{\omega}|+\imath\boldsymbol{\omega})=0$ (recall that $\boldsymbol{\omega}^2=-|\boldsymbol{\omega}|^2$). The function $\mathcal{P}^{-}_{\boldsymbol{\omega}}$ can be treated in a similar way. On the hyperplane $x_0=0$, $\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{x})+\mathcal{P}^{-}_{\boldsymbol{\omega}}(\mathbf{x})=e^{\imath\langle\boldsymbol{\omega},\mathbf{x}\rangle}$, and so

$$\mathcal{F}E(\mathbf{y}+y_0,\boldsymbol{\omega}) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\overline{\overline{E(\mathbf{y}+y_0,\mathbf{x})}}\left(\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{x})+\mathcal{P}^{-}_{\boldsymbol{\omega}}(\mathbf{x})\right)\,d\mathbf{x}.$$

For both terms we apply Cauchy's representation formula (this is the reason why we wrote $\overline{\overline{E}}$ instead of E: $\overline{E}$ is a monogenic function). For $\mathcal{P}^{+}_{\boldsymbol{\omega}}$ we use the domain $C^{+}_{r}=\{\mathbf{x}:\ x_0>0 \text{ and } |\mathbf{x}|<r\}$. For $r\to+\infty$ we obtain, using Jordan's Lemma,

$$\int_{\mathbb{R}^n}E(\mathbf{y}+y_0,\mathbf{x})\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{x})\,d\mathbf{x} = \lim_{r\to+\infty}\int_{\partial C^{+}_{r}}E(\mathbf{y}+y_0,\mathbf{x})\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{x})\,dS_{\mathbf{x}}.$$

But, according to Cauchy's representation formula, we have

$$\int_{\partial C^{+}_{r}}E(\mathbf{y}+y_0,\mathbf{x})\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{x})\,dS_{\mathbf{x}} = \mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{y}+y_0)$$

whenever $r>|\mathbf{y}|$, and therefore

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}E(\mathbf{y}+y_0,\mathbf{x})\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{x})\,d\mathbf{x} = \frac{1}{(2\pi)^{n/2}}\mathcal{P}^{+}_{\boldsymbol{\omega}}(\mathbf{y}+y_0).$$

In a similar way C^{-}_{r} is defined as $C^{-}_{r}=\{\mathbf{x}:\ x_0<0 \text{ and } |\mathbf{x}|<r\}$. Notice that $\mathbf{y}+y_0\notin C^{-}_{r}$ because $y_0>0$. So one obtains

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}E(\mathbf{y}+y_0,\mathbf{x})\mathcal{P}^{-}_{\boldsymbol{\omega}}(\mathbf{x})\,d\mathbf{x}=0.$$

Therefore

$$\mathcal{F}E(\mathbf{y}+y_0,\boldsymbol{\omega}) = \frac{1}{2(2\pi)^{n/2}}\left(1+\imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)e^{\imath\langle\boldsymbol{\omega},\mathbf{y}-y_0\rangle|\boldsymbol{\omega}|},\qquad y_0>0.$$

In a quite similar way we obtain for the lower half-space

$$\mathcal{F}E(\mathbf{y}-y_0,\boldsymbol{\omega}) = \frac{1}{2(2\pi)^{n/2}}\left(-1+\imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)e^{\imath\langle\boldsymbol{\omega},\mathbf{y}-y_0\rangle|\boldsymbol{\omega}|},\qquad y_0>0.$$

This also follows directly from the fact that

$$E(\mathbf{y}-y_0,\mathbf{x}) = E(\mathbf{y}+y_0,\mathbf{x})-2[E(\mathbf{y}+\mathbf{y}_0,\mathbf{x})]_0.$$

If we now define the radially symmetrical wavelet

$$\psi_0(\mathbf{x}) = 2\left[E(1,\mathbf{x})\right]_0 = \frac{2}{s_{n+1}|1-\mathbf{x}|^n},$$

we see that

$$\mathcal{F}\psi_0(\boldsymbol{\omega}) = 2[\mathcal{F}E(1,\boldsymbol{\omega})]_0 = \frac{1}{(2\pi)^{n/2}}e^{-|\boldsymbol{\omega}|}.$$

Therefore ψ_0 does not satisfy the admissibility condition of Theorem 2.1 and we do not have an inversion formula of the form given there. It is known however, that the Cauchy transform has an inversion formula of the form $f(\mathbf{x}) = \lim_{x_0\downarrow 0}(Cf(\mathbf{x}+x_0) - Cf(\mathbf{x}-x_0))$. In terms of the wavelet transform T_0 defined by ψ_0 this becomes

$$f(\mathbf{x}) = \lim_{a\downarrow 0} a^{-n/2}T_0 f(\mathbf{x}+a).$$

We now define two subspaces of $L_2 = L_2(\mathbb{R}^n)$ as follows:

Definition 3.1. *The spaces of boundary values from above and from below are defined as*

$$\begin{aligned} L_2^+ &= \left\{f : f = \lim_{x_0\downarrow 0}(Cf(\cdot + x_0)\right\}, \\ L_2^- &= \left\{f : f = \lim_{x_0\downarrow 0}(-Cf(\cdot - x_0)\right\}. \end{aligned}$$

An important characterisation uses the functions $(1/2)(1 \pm \imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|})$ which (for $\boldsymbol{\omega} \neq 0$) are self-adjoint idempotents whose product is zero. These idempotents were already used by F. Sommen in [9], and by A. McIntosh in [8].

Theorem 3.2. *Elements of $L_2^\pm$ are characterised as follows: $f \in L_2^\pm$ if and only if*

$$\mathcal{F}f(\boldsymbol{\omega}) = \frac{1}{2}\left(1 \pm \imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)\mathcal{F}f(\boldsymbol{\omega}).$$

The spaces $L_2^\pm$ are mutually orthogonal.

Proof. The two spaces $A^\pm = \{f : \mathcal{F}f = (1/2)(1 \pm \imath(\boldsymbol{\omega}/|\boldsymbol{\omega}|))\mathcal{F}f\}$ are orthogonal for the inner product of $L_2(\mathbb{R}^n)$. Indeed, in the integrand defining $(\mathcal{F}f, \mathcal{F}g)$, with $f \in A^-$ and $g \in A^+$ we have the factor $(1-\imath(\boldsymbol{\omega}/|\boldsymbol{\omega}|))(1+\imath(\boldsymbol{\omega}/|\boldsymbol{\omega}|))$ which equals 0. Moreover, for $f \in L_2(\mathbb{R}^n)$ we have that $f = f^+ + f^-$, where

$$\mathcal{F}f^\pm = \frac{1}{2}\left(1 \pm \imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)\mathcal{F}f.$$

Because the factors in front are idempotents $f^\pm \in A^\pm$, and $L_2(\mathbb{R}^n)$ is the direct orthogonal sum of A^+ and A^-. Now, for $x_0 > 0$ and $\mathbf{x}$ arbitrary we have that $E(\mathbf{x} - x_0) \in A^-$. Therefore, if $f \in A^+$, $Cf(\mathbf{x} - x_0) = \langle E(\mathbf{x} - x_0, \cdot), f\rangle_{L_2} = 0$

and $f \in L_2^+$. Likewise $A^- \subset L_2^-$. But because A^+ and A^- span $L_2(\mathbb{R}^n)$, we must have equality of the spaces, $A^\pm = L_2^\pm$. ■

Corollary 3.3. *Both spaces $L_2^\pm$ are invariant under the action W of the motion group G.*

Proof. This follows immediately from the Fourier transform $\widehat{W}$ of W. ■

With the wavelet ψ_0 we can now associate the two wavelets ψ_0^+ and ψ_0^- which are the projections of ψ_0 onto L_2^+ and L_2^-, resulting in the wavelet transforms $T_0^\pm$. We immediately have that $T_0 = T^+ + T^-$, while $T_0^+ f = T_0 f$ if $f \in L_2^+$, and similar relations. However, if we look at the explicit form of the Fourier transforms of $\psi_0^\pm$ we immediately see that

$$\begin{aligned} \psi_0^+(\mathbf{x}) &= E(1, \mathbf{x}), \\ \psi_0^-(\mathbf{x}) &= -E(-1, \mathbf{x}). \end{aligned}$$

In other words, the Cauchy transform can be considered as being the split over the decomposition $L_2^+ \oplus L_2^-$ of a wavelet transform with radially symmetric wavelet.

3.2. The generalised Cauchy transform. Consider, for real β with $\beta > 0$ the wavelet given by

$$\widehat{\psi}_\beta(\boldsymbol{\omega}) = (2\pi)^{-n/2} |\boldsymbol{\omega}|^\beta e^{-|\boldsymbol{\omega}|}.$$

It is easy to check that the normalising constant for this wavelet equals

$$C_\beta = C_{\psi_\beta} = 2^{-2\beta} \Gamma(2\beta).$$

It should be remarked that it is possible to take β complex, with $\operatorname{Re}\beta > 0$. The normalising constant then equals

$$C_\beta = C_{\psi_\beta} = 2^{-2\operatorname{Re}\beta} \Gamma(2\operatorname{Re}\beta).$$

For ease of notation we have chosen to work with real β only; the reader will have no difficulty in taking the holomorphic extension of the results below, which are valid for complex values of β.

We can immediately calculate the reproducing kernel associated with this wavelet transform in terms of an inner product, as described in Theorem 2.1. For future reference however, we first calculate a slightly more general inner product.

$$\begin{aligned} Q_\gamma^\beta(a_2,\mathbf{u}_2;a_1,\mathbf{u}_1) &= \langle W(a_2,\mathbf{u}_2)\psi_\gamma, W(a_1,\mathbf{u}_1)\psi_\beta\rangle_{L_2} \\ &= \frac{a_1^{n/2+\beta}a_2^{n/2+\gamma}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\imath\langle\boldsymbol{\omega},\mathbf{u}_1-\mathbf{u}_2\rangle}|\boldsymbol{\omega}|^{\beta+\gamma}e^{-(a_1+a_2)|\boldsymbol{\omega}|}\,d\boldsymbol{\omega} \\ &= \frac{a_1^{n/2+\beta}a_2^{n/2+\gamma}}{(2\pi)^n(a_1+a_2)^{\beta+\gamma+n}}\int_{\mathbb{R}^n} e^{\imath\langle\boldsymbol{\omega},\frac{\mathbf{u}_1-\mathbf{u}_2}{a_1+a_2}\rangle}|\boldsymbol{\omega}|^{\beta+\gamma}e^{-|\boldsymbol{\omega}|}\,d\boldsymbol{\omega} \\ &= \frac{a_1^{n/2+\beta}a_2^{n/2+\gamma}}{(a_1+a_2)^{\beta+\gamma+n}}\psi_{\beta+\gamma}\left(\frac{\mathbf{u}_1-\mathbf{u}_2}{a_1+a_2}\right) \qquad (2) \\ &= \frac{a_1^{n/2+\beta}a_2^{n/2+\gamma}}{(a_1+a_2)^{\beta+\gamma+n/2}}W(a_1+a_2,\mathbf{u}_2)\psi_{\beta+\gamma}(\mathbf{u}_1). \qquad (3) \end{aligned}$$

The reproducing kernel of the image of T_{ψ_β} is given putting γ equal to β, and dividing by C_β:

$$K_\beta = (1/C_\beta){Q_\beta}^\beta.$$

To generalise the Cauchy transform we use the projections of ψ_β onto the spaces $L_2^\pm$, creating the wavelets

$$\widehat{\psi_\beta^\pm}(\boldsymbol{\omega}) = \frac{1}{2}\left(1 \pm \imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)\widehat{\psi_\beta}.$$

To simplify notation, we shall concentrate here on the upper half space, and consider mostly the submodule L_2^+, together with the function ψ_β^+. The case of L_2^- will be quite analogous; whenever the switch is straightforward we shall not mention it.

For this function $\widehat{W}(a,\mathbf{u})\widehat{\psi_\beta^+}(\boldsymbol{\omega})$ is explicitly given by

$$\widehat{W}(a,\mathbf{u})\widehat{\psi_\beta^+}(\boldsymbol{\omega}) = \frac{1}{2(2\pi)^{n/2}}a^{\beta+n/2}e^{\imath\langle\boldsymbol{\omega},\mathbf{u}\rangle}\left(1+i\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)|\boldsymbol{\omega}|^\beta e^{-a|\boldsymbol{\omega}|}.$$

Notice that $a^{-\beta-n/2}\widehat{W}(a,\mathbf{u})\widehat{\psi_\beta^+}(\boldsymbol{\omega})$, considered as a function of the $n+1$-dimensional variable $\mathbf{y} = \mathbf{u} + a$, is monogenic. Indeed, if one takes the part which depends on a and $\mathbf{u}$, $g = e^{\imath\langle\boldsymbol{\omega},\mathbf{u}\rangle}e^{-a|\boldsymbol{\omega}|}$, (which is scalar) one obtains

$$\begin{aligned} \partial_a g &= -|\boldsymbol{\omega}|g, \\ \mathbf{D}_\mathbf{u} g &= \imath\boldsymbol{\omega}g, \\ \mathcal{D}g &= (\imath\boldsymbol{\omega} - |\boldsymbol{\omega}|)g, \end{aligned}$$

where $\mathbf{D}_\mathbf{u}$ stands for the $n-1$-dimensional Dirac operator in the variable $\mathbf{u}$. Multiplication with the remaining factor gives

$$\mathcal{D}g\left(1+\imath\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right) = 0.$$

The *monogenic wavelet transform* of a function f for the wavelet ψ_β is now defined by

$$\begin{aligned} C_\beta^+ f(a,\mathbf{u}) &= a^{-\beta-n/2}\int_{\mathbb{R}^n} \overline{\widehat{W}(a,\mathbf{u})\widehat{\psi}_\beta(\boldsymbol{\omega})}\hat{f}(\boldsymbol{\omega})d\boldsymbol{\omega} \\ &= a^{-\beta-n/2}\langle \widehat{W}(a,\mathbf{u})\widehat{\psi}_\beta, \hat{f}\rangle_{L_2} \\ &= a^{-\beta-n/2}\langle W(a,\mathbf{u})\psi_\beta, f\rangle_{L_2}. \end{aligned}$$

The symbol C_β^+ is used because it is a generalisation of the Cauchy transformation, as will be seen later. The Cauchy transform itself will be denoted by C_0, as it is in some sense the limit for $\beta \to 0$ of the monogenic wavelet transform C_β^+. So we define

$$\begin{aligned} C_0^+ f(a,\mathbf{u}) &= Cf(a+\mathbf{u}), \\ C_0^- f(a,\mathbf{u}) &= Cf(-a+\mathbf{u}), \qquad a>0. \end{aligned}$$

3.3. Explicit expression for the Cauchy wavelets. Let us now calculate the function $a^{-\beta-n/2}W(a,\mathbf{u})\psi_\beta^+$ itself, rather than its Fourier transform. As a central rôle is played by monogenicity, we shall look at this function, rather than $W(a,\mathbf{u})\psi_\beta^+$. We have that

$$a^{-\beta-n/2}W(a,\mathbf{u})\psi_\beta^+(\mathbf{x}) = \frac{c_n}{2}\int_{\mathbb{R}^n} e^{\imath\langle\boldsymbol{\omega},\mathbf{u}-\mathbf{x}\rangle}\left(1+i\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right)|\boldsymbol{\omega}|^\beta e^{-a|\boldsymbol{\omega}|}\,d\boldsymbol{\omega}, \tag{4}$$

where we have written c_n for $(2\pi)^{-n}$. It is seen immediately that this function is a function of a and $\mathbf{u}-\mathbf{x}$ only, say $\phi_\beta(a,\mathbf{u}-\mathbf{x})$, that $\phi_\beta(a,\mathbf{y})$ is monogenic in the variable $a+\mathbf{y}$, that ϕ_β is homogeneous of degree $-\beta-n$ again in the variable $a+\mathbf{y}$, and is invariant under rotations in $\mathbb{R}^n$. Therefore it must have the form (we put $\mathbf{y}=r\xi$, with $\xi=\sum_i \mathbf{e}_i\xi_i$ a unit vector)

$$\phi_\beta(a,r\xi) = a^{-\beta-n}A\left(\frac{r}{a}\right) + a^{-\beta-n-1}r\xi B\left(\frac{r}{a}\right).$$

The function ϕ_β is monogenic with $\mathcal{D}\phi_\beta=0$. To express this in terms of the functions A and B, we need the following formulae for the Dirac operator: if f is a scalar, radially symmetric function (i.e. f is a function of r only) then it follows from the definition of the Dirac operator that

$$\mathbf{D}f(r\xi) = \sum_{i=1}^n \mathbf{e}_i\xi_i f'(r) = \xi f'(r),$$

while taking the derivatives termwise gives

$$\mathbf{D}\mathbf{x}f(r\xi) = \mathbf{D}(f(r)\mathbf{x}) = (\mathbf{D}f(r))\mathbf{x} + f(r)\mathbf{D}\mathbf{x} = -rf'(r) - nf(r).$$

Thus it follows that

$$\begin{aligned} 0 = \mathcal{D}\phi_\beta &= (\partial_a + \mathbf{D})\left(a^{-\beta-n}A\left(\frac{r}{a}\right) + a^{-\beta-n-1}r\xi B\left(\frac{r}{a}\right)\right) \\ &= a^{-\beta-n}\left(\frac{-\beta-n}{a}A - \frac{r}{a^2}A' + \frac{r\xi}{ra}A' \right. \\ &\qquad \left. + \frac{-\beta-n-1}{a^2}r\xi B - \frac{r}{a^3}r\xi B' - \frac{n}{a}B - \frac{r}{a^2}B'\right). \end{aligned}$$

Both scalar and vector part must be zero, which leads to the system

$$\begin{cases} \dfrac{-\beta-n}{a}A - \dfrac{r}{a^2}A' - \dfrac{n}{a}B - \dfrac{r}{a^2}B' = 0, \\ \dfrac{-\beta-n-1}{a^2}B - \dfrac{r}{a^3}B' + \dfrac{1}{ra}A' = 0. \end{cases}$$

Substituting $t = r/a$, and multiplying the first equation with a and the second with a^2, results in

$$\begin{cases} (-\beta-n)A - tA' - nB - tB' = 0, \\ (-\beta-n-1)B - tB' + (1/t)A' = 0. \end{cases}$$

The function ϕ_β being a monogenic function is analytic, and it is permissible to expand A and B into series, $A(t) = \sum_{i=0}^{+\infty} A_i t^i$ and $B(t) = \sum_{i=0}^{+\infty} B_i t^i$. Because of analyticity, both A and B are even functions of their argument. This results in the system

$$\begin{cases} (-\beta-n-k)A_k - (n+k)B_k = 0, \\ (-\beta-n-1-k)B_k + (k+2)A_{k+2} = 0, \end{cases}$$

which can be written as

$$\begin{cases} -A_{k+2} = \dfrac{(\beta+n+k)(\beta+n+k+1)}{(k+2)(k+n)}A_k, \\ -B_{k+2} = \dfrac{(\beta+n+k+1)(\beta+k+n+2)}{(k+2)(k+n+2)}B_k. \end{cases}$$

So, with the classical notation $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, and $F = {}_2F_1$ denoting the hypergeometric function, we get

$$\begin{aligned} A\Big(\frac{r}{a}\Big) &= A_0 \sum_{k=0}^{\infty} \frac{((\beta+n)/2)_k((\beta+n+1)/2)_k}{k!(n/2)_k}\left(-\frac{r^2}{a^2}\right)^k \\ &= A_0F\left(\frac{\beta+n}{2},\frac{\beta+n+1}{2},\frac{n}{2},-\frac{r^2}{a^2}\right), \end{aligned} \tag{5}$$

$$\begin{aligned} B\Big(\frac{r}{a}\Big) &= B_0 \sum_{k=0}^{\infty} \frac{((\beta+n+1)/2)_k((\beta+n+2)/2)_k}{k!((n+1)/2)_k}\left(-\frac{r^2}{a^2}\right)^k \\ &= -\frac{\beta+n}{n}A_0F\left(\frac{\beta+n+1}{2},\frac{\beta+n+2}{2},\frac{n+2}{2},-\frac{r^2}{a^2}\right). \end{aligned} \tag{6}$$

The constant A_0 can be determined from the defining integral putting $\mathbf{u}-\mathbf{x}=0$ and $a=1$. We then get

$$\begin{aligned} A_0 &= \frac{c_n}{2}\int_{\mathbb{R}^n} |\boldsymbol{\omega}|^{\beta} e^{-|\boldsymbol{\omega}|}\,d\boldsymbol{\omega} = \frac{c_n s_n}{2}\int_0^{\infty} r^{\beta+n-1}e^{-r}\,dr \\ &= c_n s_n \Gamma(\beta+n) = \frac{1}{2^n\pi^{n/2}}\frac{\Gamma(\beta+n)}{\Gamma(n/2)}, \end{aligned}$$

where $s_n = 2\pi^{n/2}\Gamma(n/2)$ is the surface area of the unit sphere. It was permissible to disregard the term in $\boldsymbol{\omega}/|\boldsymbol{\omega}|$, because the integrand for this term is antisymmetric in $\boldsymbol{\omega}$, and therefore this part of the integral equals 0.

The series expansion is only valid in the domain $|\mathbf{u}-\mathbf{x}| < a$. The integral however defines a monogenic function in the domain $a > 0$. To determine the functions in the domain $|\mathbf{u}-\mathbf{x}| \geq a > 0$ we must look at the inner and outer power functions, introduced by F. Sommen in [10]. These are monogenic functions in the variable $\mathbf{y}+a$, defined for a complex parameter s by

$$\begin{aligned} p_s(\mathbf{y}+a) &= r^sF\left(1-\frac{n+s}{2},\frac{-s}{2},\frac{1}{2},-\frac{a^2}{r^2}\right) \\ &\quad -sar^{s-1}\xi F\left(1-\frac{n+s}{2},\frac{-s}{2},\frac{3}{2},-\frac{a^2}{r^2}\right), \\ q_s(\mathbf{y}+a) &= (s+n-1)ar^{s-1}F\left(\frac{1-s}{2},\frac{3-s-n}{2},\frac{3}{2},-\frac{a^2}{r^2}\right) \\ &\quad +r^s\xi F\left(\frac{1-s}{2},\frac{1-s-n}{2},\frac{1}{2},-\frac{a^2}{r^2}\right). \end{aligned}$$

These functions are homogeneous of degree s, and are monogenic in the domain $|\mathbf{y}| < 0$. Notice that the series definition of the hypergeometric function converges absolutely for the argument $-a^2/r^2 = -1$, and so these functions are continuous up to each boundary point where $r = a \neq 0$.

On the cone $r = a$, both of these functions are determined by two real constants: we have for s fixed and $a > 0$ arbitrary

$$\begin{aligned} p_s(a(1+\xi)) &= a^s(u + v\xi), \\ q_s(a(1+\xi)) &= a^s(c + d\xi), \end{aligned}$$

for some scalars u, v, c and d. Indeed, this is the only possible form for a function homogeneous of degree s which is axially invariant on this cone. Moreover, it is easily seen from the properties of the hypergeometric series that $ud - vc \neq 0$. As a consequence we have:

Lemma 3.4. *Let f be a function on the cone $r = a$ of the form*

$$f(a(1+\xi)) = a^s(c_1 + c_2\xi),$$

then f has a monogenic extension into the domain $|\mathbf{y}| < |a|$, which is a linear combination of p_s and q_s.

Now, our function $\phi_\beta(a, r\xi)$, when restricted to the cone satisfies the conditions of Lemma 3.4. Therefore the series expansions (5) and (6) have two monogenic extensions into the domain $|\mathbf{y}| < a$: one given by ϕ_β, the other one given by Lemma 3.4. By the unique continuation property of the Cauchy-Riemann operator, both extensions are equal in this domain.

One can take the derivative with respect to a under the integral sign and obtains immediately

$$\partial_a \phi_\beta(a, \mathbf{y}) = \phi_{\beta+1}(a, \mathbf{y}). \tag{7}$$

Notice that this equation is also valid for $\beta = 0$. But since ϕ_β is monogenic, we have $\mathbf{D}\phi_\beta = -\partial_a \phi_\beta$, and so $-\mathbf{D}\phi_\beta(a, \mathbf{y}) = \phi_{\beta+1}(a, \mathbf{y})$.

We can summarise the results of this subsection:

Theorem 3.5. *The Cauchy wavelets $\phi_\beta(a, \mathbf{y})$ defined by the integral* (4) *can be extended to monogenic functions in the domain $a > -|\mathbf{y}|$. In the domain $a > |\mathbf{y}|$ they are defined using the hypergeometric series by* (5) *and* (6)*. In the region $|a| \leq |\mathbf{y}| > 0$ they are defined by the monogenic power functions $p_{-\beta-n}$ and $q_{-\beta-n}$. Finally, if β is an integer, repeated application of* (7) *shows that ϕ_β is a β-th derivative of the Cauchy kernel, and can therefore be extended to the domain $\mathbb{R}^{n+1} \setminus \{0\}$.*

It was proved in [10] that, if β is not an integer, one cannot construct a monogenic function of degree $-\beta - n$ in the entire domain $\mathbb{R}^{n+1} \setminus \{0\}$. Therefore, in this case, a similar extension for ϕ_β is impossible.

4. Spaces of monogenic functions

The wavelet transform T_β, associated to the real valued radially symmetric wavelet ψ_β, maps $L_2(\mathbb{R}^n)$ (up to a constant) isometrically to a closed subspace of $L_2(G)$. Thus it is natural to assume that the image of the generalised Cauchy transform C^+_β will have the form of a weighted Bergman space $B_{2\beta-1}$. We shall presently show that this is indeed the case. But the wavelet transform provides a link between these spaces and a kind of Hardy spaces introduced by Gilbert and Murray ([6, p. 288]). They ask the question whether these Hardy spaces are trivial. Using the wavelet transform we can give a complete answer to this question, because the image of C^+_β is contained in these Hardy spaces.

First we define the relevant Bergman spaces (actually they are Hilbert modules over $\mathcal{C}\ell(n)$) B_μ to be the spaces of functions which are monogenic in the upper half space and which have finite norm for the inner product

$$[f,g]_\mu = \int_0^\infty a^\mu \langle f(a+\cdot), g(a+\cdot)\rangle_{L_2}\, da.$$

The norm will be denoted by $[[f]]_\mu$.

Consider for any real μ and an arbitrary Clifford valued function on the half space f the norm (which may be infinite)

$$\|f\|_\mu = \sup_{a>0} \frac{1}{a^\mu} \|f(a+\cdot)\|_{L_2}.$$

We construct, for μ fixed, two function spaces with this norm:

- the space H_μ of functions satisfying $\mathcal{D}_P f = 0$ with finite norm $\|f\|_\mu$;
- the space M_μ of monogenic functions with finite norm $\|f\|_\mu$.

The Hardy spaces considered in [6] are the spaces H_μ. Since any solution of $\mathcal{D}_P f = 0$ can be written as $f = a^{(n-1)/2} g$ where g is monogenic, H_μ is isomorphic to (and isometric with) the space $M_{\mu+(1-n)/2}$.

It is fairly elementary to prove that the space M_μ is trivial for $\mu > 0$. Indeed, for such μ, we have that $\lim_{a\downarrow 0} \|f(a+\cdot)\|_{L_2} = 0$, which implies that f is in the Szegö space of the half space, with boundary value zero: hence f is zero everywhere.

For $\mu \leq 0$ we shall show that the image of C^+_β, where $\beta = -\mu$, is contained in the space M_μ. In other words, that

$$B_{2\beta-1} \subset M_{-\beta}.$$

Theorem 4.1. *Let $\beta > 0$. Then $C^+_\beta \colon L_2(\mathbb{R}^n) \to B_{2\beta-1}$ is, up to a constant, an isometry.*

Proof. Let T_β be the wavelet transform associated with the radially symmetric wavelet ψ_β, and let $S_\beta f(a,\mathbf{u}) = a^{-\beta-n/2} T_\beta f(a,\mathbf{u})$. It is known, see [2], that S_β is, up to a constant, an isometry from $L_2(\mathbb{R}^n)$ to $\mathcal{T}_\beta$, the Bergman space of harmonic functions with inner product $[[\cdot,\cdot]]_{2\beta-1}$. Clearly for any $f \in L_2^+$ we have $C_\beta^+ f = S_\beta f$, because ψ_β^+ is the orthogonal projection of ψ_β onto L_2^+. We have already proved that $C_\beta^+ f$ is monogenic, and so C_β^+ is, up to a constant, an isometry from L_2^- into $M_{2\beta-1}$. We now prove that C_β^+ is onto. Assume therefore that there exists $F \in M_{2\beta-1}$ such that F is orthogonal to the image of C_β^+. Since S_β is up to a constant an isometry, $g = S_\beta^{-1}F$ exists and sits in L_2^-. But an explicit calculation shows that $S_\beta g$ is monogenic for $g \in L_2^-$ if and only if $g = 0$. This proves that $F = 0$, and C_β^+ is onto. ■

Corollary 4.2. *The reproducing kernel of $B_{2\beta-1}$ is given by*

$$\frac{1}{C_\beta(a_1+a_2)^{2\beta+n}}\,\psi_{2\beta}^+\left(\frac{\mathbf{u}_1-\mathbf{u}_2}{a_1+a_2}\right) = \frac{1}{C_\beta}\phi_{2\beta}(a_1+a_2,\mathbf{u}_1-\mathbf{u}_2),$$

where $\phi_{2\beta}$ is a generalised Cauchy wavelet.

Proof. We start with the reproducing kernel (2) for the image space of the wavelet transform T_β. First of all we must correct it, multiplying it with a factor $(a_1a_2)^{-n/2-\beta}$, to take into account the difference between wavelet and Cauchy transform. Then we must project it onto the space of monogenic functions in the second variable (because we only use the image of L_2^+), and the conjugates of monogenic functions in the first variable (because the result of applying the reproducing kernel must be a monogenic function). This gives the result. ■

Since each of the spaces $B_{2\beta-1}$ is isomorphic with L_2^+, they are mutually isomorphic. We can therefore define the isomorphism

$$\begin{aligned} \mathcal{P}_\gamma^\beta\colon B_{2\beta-1} &\to B_{2\gamma-1},\\ f &\to \mathcal{P}_\gamma^\beta f = C_\gamma^+(C_\beta^+)^{-1}. \end{aligned}$$

This is obviously well defined. We now prove

Theorem 4.3. *$\mathcal{P}_\gamma^\beta$ is an integral transform with kernel P_γ^β given by*

$$P_\gamma^\beta(a_2,\mathbf{u}_2;a_1,\mathbf{u}_1) = C_\beta^{-1}\phi_{\beta+\gamma}(a_1+a_2,\mathbf{u}_1-\mathbf{u}_2).$$

In other words, $\mathcal{P}_\gamma^\beta G(a_2,\mathbf{u}_2) = [P_\gamma^\beta(a_2,\mathbf{u}_2;\cdot),G]_{2\beta-1}$.

Proof. We start with the similar map $\mathcal{Q}_\gamma^\beta\colon \mathcal{T}_\beta \to \mathcal{T}_\gamma$. Let F be a function in $\mathcal{T}_\beta$. According to the inversion formula of Theorem 2.1 it is the image under T_β of f, given by

$$f(\mathbf{x}) = \frac{1}{C_\beta}\int_0^\infty \frac{da_1}{a_1^{n+1}}\int_{\mathbb{R}^n} d\mathbf{u}_1\,W(a_1,\mathbf{u}_1)\psi_\beta(\mathbf{x})F(a_1,\mathbf{u}_1).$$

Taking $\mathcal{Q}_\gamma^\beta F = T_\gamma f$ results in

$$\begin{aligned}
\mathcal{Q}_\gamma^\beta F(a_2, \mathbf{u}_2) &= \int_{\mathbb{R}^n} d\mathbf{x} W(a_2, \mathbf{u}_2)\psi_\gamma(\mathbf{x}) \, \frac{1}{C_\beta} \int_0^\infty \frac{da_1}{a_1^{n+1}} \\
&\qquad \int_{\mathbb{R}^n} d\mathbf{u}_1 W(a_1, \mathbf{u}_1)\psi_\beta(\mathbf{x}) F(a_1, \mathbf{u}_1) \\
&= \frac{1}{C_\beta} \int_0^\infty \frac{da_1}{a_1^{n+1}} \\
&\qquad \int_{\mathbb{R}^n} d\mathbf{u}_1 \left(\int_{\mathbb{R}^n} d\mathbf{x} \, W(a_2, \mathbf{u}_2)\psi_\gamma(\mathbf{x}) W(a_1, \mathbf{u}_1)\psi_\beta(\mathbf{x}) \right) F(a_1, \mathbf{u}_1) \\
&= \frac{1}{C_\beta} \int_0^\infty \frac{da_1}{a_1^{n+1}} \int_{\mathbb{R}^n} d\mathbf{u}_1 \, Q_\gamma^\beta(a_2, \mathbf{u}_2; a_1, \mathbf{u}_1) F(a_1, \mathbf{u}_1) \\
&= \frac{1}{C_\beta} \int_0^\infty \frac{da_1}{a_1^{n+1}} \int_{\mathbb{R}^n} d\mathbf{u}_1 \, Q_\gamma^\beta(a_2, \mathbf{u}_2; a_1, \mathbf{u}_1) F(a_1, \mathbf{u}_1) \\
&= \frac{1}{C_\beta} \langle Q_\gamma^\beta(a_2, \mathbf{u}_2; \cdot), F \rangle_G .
\end{aligned}$$

In other words, $\mathcal{Q}_\gamma^\beta$ is an integral transform with kernel $(1/C_\beta)Q_\gamma^\beta$.

As in the case of the reproducing kernel, we only have to adjust the exponent in a_1 to account for the difference of the inner products $\langle \cdot, \cdot \rangle_G$ and $[\cdot, \cdot]_{2\beta-1}$, and project onto the space of monogenic functions. ■

Theorem 4.4. *Let $\beta \geq 0$. Then C_β^+ defines a continuous mapping from L_2^+ into $M_{-\beta}$ with empty kernel.*

Proof. Keep a fixed. We shall calculate an upper bound for $\|C_\beta^+ f(a, \cdot)\|_{L_2}$ starting from $\|f\|_{L_2}$. First notice that

$$\begin{aligned}
\widehat{W}(a, \mathbf{u})\widehat{\psi}_\beta(\boldsymbol{\omega}) &= \frac{1}{(2\pi)^{n/2}} a^{\beta+n/2} e^{\imath\langle \boldsymbol{\omega}, \mathbf{u} \rangle} |\boldsymbol{\omega}|^\beta e^{-a|\boldsymbol{\omega}|} \\
&= e^{\imath\langle \boldsymbol{\omega}, \mathbf{u} \rangle} \widehat{W}(a, 0)\widehat{\psi}_\beta(\boldsymbol{\omega}),
\end{aligned}$$

which implies for $f \in L_2^+$, for which we can use ψ_β or ψ_β^+ without changing the wavelet transform, the equation

$$\begin{aligned}
C_\beta^+ f(a, \mathbf{u}) &= a^{-\beta-n/2} \int_{\mathbb{R}^n} e^{-\imath\langle \boldsymbol{\omega}, \mathbf{u} \rangle} \overline{\widehat{W}(a, 0)\widehat{\psi}_\beta(\boldsymbol{\omega})} f(\boldsymbol{\omega}) \, d\boldsymbol{\omega} \\
&= a^{-\beta-n/2} (2\pi)^{n/2} \mathcal{F}^- \left(\overline{\widehat{W}(a, 0)\widehat{\psi}_\beta} \, \hat{f} \right) (\mathbf{u}).
\end{aligned}$$

Calculating the L_2 norm of $C_\beta^+ f(a,\cdot)$ gives

$$\begin{aligned}
\|C_\beta^+ f(a,\cdot)\|_{L_2}^2 &= \left[\langle C_\beta^+ f(a,\cdot), C_\beta^+ f(a,\cdot)\rangle_{L_2}\right]_0 \\
&= (2\pi)^n \left[\langle \mathcal{F} C_\beta^+ f(a,\cdot), \mathcal{F} C_\beta^+ f(a,\cdot)\rangle_{L_2}\right]_0 \\
&= (2\pi)^n a^{-2\beta-n} \left[\langle \overline{\widehat{W}(a,0)\widehat{\psi}_\beta}\,\hat{f}, \overline{\widehat{W}(a,0)\widehat{\psi}_\beta}\,\hat{f}\rangle_{L_2}\right]_0 \\
&= (2\pi)^n a^{-2\beta-n} \left[\int_{\mathbb{R}^n} \overline{\hat{f}}\,\widehat{W}(a,0)\widehat{\psi}_\beta \overline{\widehat{W}(a,0)\widehat{\psi}_\beta}\,\hat{f} d\boldsymbol{\omega}\right]_0 \\
&= (2\pi)^n a^{-2\beta-n} \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\omega})|^2 |\widehat{W}(a,0)\widehat{\psi}_\beta(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\omega})|^2 |\boldsymbol{\omega}|^{2\beta} e^{-2a|\boldsymbol{\omega}|} d\boldsymbol{\omega}
\end{aligned}$$

The last but one equality is valid because $\widehat{W}(a,0)\widehat{\psi}_\beta\overline{\widehat{W}(a,0)\widehat{\psi}_\beta}$ is real. The function $|\boldsymbol{\omega}|^{2\beta}e^{-2a|\boldsymbol{\omega}|}$ is bounded. It reaches its maximum at $|\boldsymbol{\omega}| = \beta/a$, and the maximum is $a^{-2\beta}C^2$, where $C = (\beta/e)^\beta$. Replacing $|\boldsymbol{\omega}|^{2\beta}e^{-2a|\boldsymbol{\omega}|}$ by this bound in the integral gives

$$\|C_\beta^+ f(a,\cdot)\|_{L_2} \le a^{-\beta} C \|f\|_{L_2}.$$

This proves that C_β^+ is a continuous mapping from $L_2(\mathbb{R}^n)$ into $M_{-\beta}$ with norm at most C.

Moreover it is easily seen, since $|\boldsymbol{\omega}|^{2\beta}e^{-2a|\boldsymbol{\omega}|}$ is different from zero almost everywhere, that $\|C_\beta^+ f(a,\cdot)\|_{L_2} = 0$ if and only if $f = 0$. ■

Corollary 4.5. *H_μ is not empty if $\mu \le -1+n$.*

5. Fractional calculus of the Dirac operator

The symbol of the Dirac operator maps $\boldsymbol{\omega}$ to $\imath\boldsymbol{\omega}$, in other words

$$\mathcal{F}(\mathbf{D}f)(\boldsymbol{\omega}) = -\imath\boldsymbol{\omega}\mathcal{F}f(\boldsymbol{\omega}),$$

assuming, of course, that f is in the suitable Sobolev space. As a consequence we immediately have

$$\mathbf{D}\psi_\beta^\pm = \mp\psi_{\beta+1}^\pm.$$

Formally, the adjoint of $\mathbf{D}$ in $L_2(\mathbb{R}^n)$ equals $\mathbf{D}$. On the other hand, it follows directly from the representation W (see (1)) that

$$\mathbf{D}_{\mathbf{u}} W(a,\mathbf{u})\psi(\mathbf{x}) = -\mathbf{D}_{\mathbf{x}} W(a,\mathbf{u})\psi(\mathbf{x}).$$

for any wavelet ψ. So we obtain, assuming $\mathbf{D}f \in L_2(\mathbb{R}^n)$, that

$$\begin{aligned}
C_\beta^\pm \mathbf{D}f(a,\mathbf{u}) &= a^{-\beta-n/2}\langle W(a,\mathbf{u})\psi_\beta^\pm, \mathbf{D}f\rangle_{L_2} \\
&= a^{-\beta-n/2}\langle \mathbf{D}_\mathbf{x} W(a,\mathbf{u})\psi_\beta^\pm, f\rangle_{L_2} \\
&= a^{-\beta-n/2}\langle a^{-1} W(a,\mathbf{u})(\mathbf{D}\psi_\beta^\pm), f\rangle_{L_2} \\
&= a^{-\beta-n/2-1}\langle a^{-1} W(a,\mathbf{u})(\mp\psi_{\beta+1}^\pm), f\rangle_{L_2} \\
&= \mp C_{\beta+1}^\pm f(a,\mathbf{u}).
\end{aligned}$$

Let $\mathcal{S}^\beta$ be the Sobolev space of functions in $L_2(\mathbb{R}^n)$ for which the Sobolev norm

$$\|f\|_{S,\beta} = \|(1+|\boldsymbol{\omega}|)^\beta \mathcal{F}f(\boldsymbol{\omega})\|_{L_2}$$

is finite.

Assume now, for the moment, that β is a positive integer, and that $f \in L_2^+$ is in the Sobolev space $\mathcal{S}^\beta$. We then obtain, using induction, that

$$C_0^+(-\mathbf{D})^\beta f(a,\mathbf{u}) = C_\beta^+ f(a,\mathbf{u}).$$

But since an arbitrary function g in L_2^+ is the limit of its Cauchy transform from above, $g = \lim_{a\downarrow 0} C^+ g(a,\cdot)$. Applying this to the previous equation, we obtain

$$(-\mathbf{D})^\beta f = \lim_{a\downarrow 0} C_\beta^+ f(a,\cdot), \qquad f \in \mathcal{S}^\beta \cap L_2^+.$$

A similar treatment for functions in $L_2^- \cap \mathcal{S}^\beta$ however shows that

$$\mathbf{D}^\beta f = \lim_{a\downarrow 0} C_\beta^- f(a,\cdot), \qquad f \in \mathcal{S}^\beta \cap L_2^-.$$

We therefore have to introduce an extra operator, the Hilbert transform $\mathcal{H}$. It is defined by

$$\mathcal{H}f = \begin{cases} +f & f \in L_2^+, \\ -f & f \in L_2^-. \end{cases}$$

It is known (see [6]) that the Hilbert transform is a singular integral transform. Obviously $\mathcal{H}^2 = Id$, and since, as follows from the reasoning above, both eigenspaces of $\mathcal{H}$, namely L_2^+ and L_2^-, are invariant under the Dirac operator, $\mathcal{H}$ commutes with $\mathbf{D}$. We can combine the results for L_2^+ and L_2^- and write

$$(-\mathcal{H}\mathbf{D})^\beta f = \lim_{a\downarrow 0}(C_\beta^- f(a,\cdot) + C_\beta^+ f(a,\cdot)), \qquad f \in \mathcal{S}^\beta.$$

But $C_\beta^- + C_\beta^+$ is, up to the factor $a^{-\beta-n/2}$, the wavelet transform T_β associated with the scalar valued, radially symmetric wavelet ψ_β, and so we can write this as

$$(-\mathcal{H}\mathbf{D})^\beta f = \lim_{a\downarrow 0} a^{-\beta-n/2} T_\beta f(a,\cdot), \qquad f \in \mathcal{S}^\beta. \tag{8}$$

Take now $\beta > 0$, not necessarily an integer (as already remarked, we can generalise the theory to complex β with $\operatorname{Re}\beta > 0$). In the proof of Theorem 4.4 the equality

$$\|C_\beta^+ f(a,\cdot)\|_{L_2}^2 = c\int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\omega})|^2 |\boldsymbol{\omega}|^{2\beta} e^{-2a|\boldsymbol{\omega}|} d\boldsymbol{\omega}$$

(where c is a constant independent of f) has bee established. Clearly this leads to the estimate

$$\|C_\beta^+ f(a,\cdot)\|_{L_2} \leq c\|f\|_{S,\beta}.$$

It follows that $C_\beta^+ f$ is a function in the Szegő space, and every function in the Szegő space has a boundary value. A similar reasoning holds for C_β^- (obviously we could have rewritten the resoning of Theorem 4.4 directly in terms of T_β). Therefore we can extend (8) to a definition of a fractional power of the Dirac operator:

$$(-\mathcal{H}\mathbf{D})^\beta f \equiv \lim_{a\downarrow 0} a^{-\beta-n/2} T_\beta f(a,\cdot), \qquad f \in \mathcal{S}^\beta.$$

Obviously, this is really useful only if it leads to a one-parameter group of operators, i.e. if $(-\mathcal{H}\mathbf{D})^\gamma \circ (-\mathcal{H}\mathbf{D})^\beta = (-\mathcal{H}\mathbf{D})^{\gamma+\beta}$. We prove that this is the case.

Theorem 5.1. *Take $f \in \mathcal{S}^{\gamma+\beta}$. Then*

$$(-\mathcal{H}\mathbf{D})^\gamma \circ (-\mathcal{H}\mathbf{D})^\beta f = (-\mathcal{H}\mathbf{D})^{\gamma+\beta} f.$$

Proof. We actually shall prove that, for $f \in \mathcal{S}^\beta$,

$$T_\gamma \circ (T_0)^{-1} \circ T_\beta f = a^{-\beta} T_{\gamma+\beta} f.$$

If $f \in \mathcal{S}^{\beta+\gamma}$ then the boundary values of both sides exist, and the theorem simply states their equality.

We have already seen that if $f \in \mathcal{S}^\beta$, $a^{-\beta-n/2}T_\beta f$ has a boundary value then this boundary value equals $(T_0)^{-1} \circ T_\beta f$. Now, fix a_1 for a moment, and let F be the function F_{a_1} defined as $F_{a_1}(\mathbf{x}) = a_1^{-\beta-n/2} T_\beta f(a_1, \mathbf{x})$. We can then take the wavelet transform of F_{a_1}; in the calculations we need the fact that ψ_β is a radially symmetric function, from which the equality

$$W(a_1, \mathbf{x})\psi_\beta(\mathbf{u}_1) = W(a_1, \mathbf{u}_1)\psi_\beta(\mathbf{x})$$

follows. So, with expression (3) for Q_γ^β

$$\begin{aligned}
&T_\gamma F_{a_1}(a_2,\mathbf{u}_2)\\
&= \int_{\mathbb{R}^n} d\mathbf{x}\, W(a_2,\mathbf{u}_2)\psi_\gamma(\mathbf{x})F_{a_1}(\mathbf{x})\\
&= a_1^{-\beta-n/2}\int_{\mathbb{R}^n} d\mathbf{x}\, W(a_2,\mathbf{u}_2)\psi_\gamma(\mathbf{x})\int_{\mathbb{R}^n} d\mathbf{u}_1\, W(a_1,\mathbf{x})\psi_\beta(\mathbf{u}_1)f(\mathbf{u}_1)\\
&= a_1^{-\beta-n/2}\int_{\mathbb{R}^n} d\mathbf{x}\, W(a_2,\mathbf{u}_2)\psi_\gamma(\mathbf{x})\int_{\mathbb{R}^n} d\mathbf{u}_1\, W(a_1,\mathbf{u}_1)\psi_\beta(\mathbf{x})f(\mathbf{u}_1)\\
&= \int_{\mathbb{R}^n} d\mathbf{u}_1 \left(a_1^{-\beta-n/2}\int_{\mathbb{R}^n} d\mathbf{x}\, W(a_2,\mathbf{u}_2)\psi_\gamma(\mathbf{x})W(a_1,\mathbf{u}_1)\psi_\beta(\mathbf{x})\right) f(\mathbf{u}_1)\\
&= \int_{\mathbb{R}^n} d\mathbf{u}_1 a_1^{-\beta-n/2}Q_\gamma^\beta(a_1,\mathbf{u}_1;a_2,\mathbf{u}_2)f(\mathbf{u}_1)\\
&= \frac{a_2^{\gamma+n/2}}{(a_1+a_2)^{\beta+\gamma+n/2}}\langle W(a_1+a_2,\mathbf{u}_2)\psi_{\beta+\gamma},f\rangle_{L_2}\\
&= \frac{a_2^{\gamma+n/2}}{(a_1+a_2)^{\beta+\gamma+n/2}}T_{\beta+\gamma}f(a_1+a_2,\mathbf{u}_2).
\end{aligned}$$

The change of integration order is permitted because $f \in L_2$. Since $f \in \mathcal{S}^\beta$, T_β is in the analogue of the Szegő space, and the limit $\lim_{a_1\downarrow 0} F_{a_1}$, taken in the L_2 sense, exists, and it equals $(-\mathcal{H}\mathbf{D})^\beta f$. Moreover the wavelet transform taken in a fixed point $(a_2,\mathbf{u}_2)$ is continuous, and therefore we can take the limit of the equality above,

$$T_\gamma(-\mathcal{H}\mathbf{D})^\beta f(a_2,\mathbf{u}_2) = a_2^{-\beta}T_{\gamma+\beta}f(a_2,\mathbf{u}_2).$$

This proves the theorem. ■

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