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Asymptotics for Minimal Blaschke Products and Best L_1 Meromorphic Approximants of Markov Functions

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Abstract. Let μ be a positive Borel measure with support $\operatorname{supp}\mu = E \subset (-1,1)$ and let

$$\Delta_n = \inf_{B\in\mathcal{B}_n} \int_E |B(x)|^2 d\mu(x),$$

where $\mathcal{B}_n$ is the collection of all Blaschke products of degree n. Denote by $B_n \in \mathcal{B}_n$ a Blaschke product that attains the value Δ_n. We investigate the asymptotic behavior, as $n \to \infty$, of the minimal Blaschke products B_n in the case when the measure μ with support $E = [a,b]$ satisfies the Szegő condition:

$$\int_a^b \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}}\, dx > -\infty.$$

At the same time, we shall obtain results related to the convergence of best L_1 approximants on the unit circle to the Markov function

$$f(z) = \frac{1}{2\pi i}\int_E \frac{d\mu(x)}{z-x}$$

by meromorphic functions of the form P/Q, where P belongs to the Hardy space H_1 of the unit disk and Q is a polynomial of degree at most n. We also include in an appendix a detailed treatment of a factorization theorem for Hardy spaces of the slit disk, which may be of independent interest.

Keywords. Blaschke products, meromorphic approximation, Markov functions, best approximation.

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1. Introduction

Let G be the open unit disk $\{z : |z| < 1\}$ in the complex plane. We assume that the circle $\Gamma = \{z : |z| = 1\}$ is positively oriented with respect to G.

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Let $L_p(\Gamma)$, $1 \le p \le \infty$, be the Lebesgue space of functions φ measurable on Γ, with the norm

$$\|\varphi\|_p = \begin{cases} \left(\int_\Gamma |\varphi(\xi)|^p \, |d\xi|\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_\Gamma |\varphi(\xi)| & \text{if } p = \infty. \end{cases}$$

Let μ be a finite positive Borel measure. We assume that the support E of μ contains infinitely many points and that $E \subset (-1,1)$. For any nonnegative integer n, denote by $\mathcal{B}_n$ the collection of all Blaschke products of the form

$$B(z) = \prod_{i=1}^{n} \frac{z - \xi_i}{1 - \bar{\xi}_i z}, \qquad \xi_i \in G.$$

Let $L_2(\mu, E)$ be the Lebesgue space of measurable functions on E, with the norm

$$\|\varphi\|_{2,\mu} = \left(\int_E |\varphi(x)|^2 d\mu(x)\right)^{1/2} < \infty.$$

In this paper, we investigate the limiting behavior, as $n \to \infty$, of the solutions to the following extremal problem:

$$\Delta_n := \inf_{B \in \mathcal{B}_n} \int_E |B(x)|^2 \, d\mu(x), \tag{1.1}$$

and we do this in connection with best meromorphic approximation in $L_1(\Gamma)$ to the Markov function (1.2) so that the present paper may be viewed as a sequel to [3]. Beyond meromorphic approximation, such minimization problems arise, for instance, in the theory of n-widths of sets of analytic functions (see, for example, [7], [8], [16]). Moreover, they constitute a natural generalization of the Szegő theory of orthogonal polynomials to the hyperbolic setting.

Denote by $\mathbf{A}$ the restriction to E of the closed unit ball of the Hardy space $H_\infty(G)$. It is proved by Fisher and Micchelli (see [7], [8]) that

$$d_n(\mathbf{A}, L_2(\mu, E)) = d^n(\mathbf{A}, L_2(\mu, E)) = \delta_n(\mathbf{A}, L_2(\mu, E)) = \inf_{B \in \mathcal{B}_n} \|B\|_{2,\mu},$$

where d_n, d^n, and δ_n are the Kolmogorov, Gelfand, and linear n-widths of $\mathbf{A}$ in $L_2(\mu; E)$, respectively.

It is easy to show that solutions to (1.1) exist and that any solution B_n is a Blaschke product with all zeros belonging to the smallest interval $K(E)$ containing the support E of μ. It is proved in [2] that all zeros $x_{1,n}, \ldots, x_{n,n}$ of B_n are simple.

As we mentioned already, there is a close connection between the extremal constant Δ_n and the error in best meromorphic approximation to the Markov function

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z - x}, \tag{1.2}$$

in the space $L_1(\Gamma)$. This connection goes as follows.

Let $H_p(G)$, $1 \le p \le \infty$, be the Hardy space of analytic functions on G. Let $\mathcal{M}_{n,p}(G)$, $1 \le p \le \infty$, be the class of all meromorphic functions of G that can be represented in the form $h = P/Q$, where P belongs to the Hardy space $H_p(G)$ and Q is a polynomial of degree at most n, $Q \not\equiv 0$.

It is proved by Andersson [2] (see also [3]), that

$$\Delta_n = \inf_{h \in \mathcal{M}_{n,1}(G)} \|f - h\|_1$$

and there exists a best approximant h_n in $\mathcal{M}_{n,1}(G)$ to the function f in the space $L_1(\Gamma)$ such that $\Delta_n = \|f - h_n\|_1$ and $h_n = P_n/B_n$, where $P_n \in H_1(G)$ and B_n is a solution of the extremal problem (1.1). Moreover, the function h_n satisfies on Γ the following equation:

$$(1.3) \qquad ({B_n}^2(f - h_n))(\xi)\, d\xi = |(f - h_n)(\xi)|\, |d\xi|$$

(see [3]). Since f is holomorphic on Γ, it follows from (1.3) and the reflection principle that the functions $(f - h_n)$ and h_n may in fact be continued analytically across Γ (see, for example, [10]).

Let us mention that Szegő type asymptotics of minimal Blaschke products for certain extremal problems connected with n-widths of sets of analytic functions were studied by Levin and Saff [12], and Parfenov [13]–[15] (see also Fisher and Saff [9]). In another connection, the present authors investigated in [3] the convergence of the best meromorphic approximants and the limiting distribution of poles of the best approximants to Markov functions in the space $L_p(\Gamma)$, $1 < p \le \infty$ (for the case $p = 2$, see also [5]).

This paper is organized as follows. In Section 2 we present some formulas involving the extremal Blaschke products B_n as well as their orthogonality properties. Subsequently, in the case when the measure μ has support $E = [a, b]$ and satisfies the Szegő condition

$$\int_a^b \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}} dx > -\infty,$$

we state in Section 3 (Theorem 4) the asymptotic behavior of the extremal functions B_n in the doubly-connected region $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$, where E^{-1} is defined by $E^{-1} = \{x : x^{-1} \in E\}$. Section 3 also contains a description of the region and rate of convergence of the best approximants h_n to f (Theorem 5). Finally, in Sections 4 and 5, we prove Theorems 4 and 5, respectively. The methods that we use are based on an investigation of Szegő type asymptotics for orthogonal polynomials with varying weight (see Totik [24] and Stahl [21]).

In the Appendix, we derive auxiliary results on Hardy spaces of the disk slit along a real segment that are of technical use in Section 5; these results, that we were not able to locate in the literature, can be generalized to domains whose boundary consists of finitely many rectifiable Jordan arcs and curves, and may be of interest in their own right.

2. Some formulas

2.1. An auxiliary function φ_n.

Lemma 1. *With the notation of Section 1, we have*

$$|(f-h_n)(\xi)| = \frac{1}{2\pi}\int_E \frac{1-x^2}{|\xi - x|^2} {B_n}^2(x)\,d\mu(x), \qquad \xi \in \Gamma. \tag{2.1}$$

Proof. Let u be any harmonic function in G that is continuous on the closed disk $\bar{G}$, and let v be the harmonic conjugate of u. The function $e^{\varepsilon(u+iv)}$ lies in $H_\infty(G)$ for each real ε. By (1.3), (1.2), and Cauchy's formula, we have

$$\int_E e^{\varepsilon(u+vi)(x)} {B_n}^2(x)\,d\mu(x) = \int_\Gamma e^{\varepsilon(u+iv)(\xi)} |(f-h_n)(\xi)|\,|d\xi|.$$

Differentiating this last equation with respect to ε and setting $\varepsilon = 0$, we obtain

$$\int_E u(x){B_n}^2(x)\,d\mu(x) = \int_\Gamma u(\xi)|(f-h_n)(\xi)|\,|d\xi|.$$

Since u is an arbitrary function harmonic in G and continuous on $\bar{G}$, we get

$$({B_n}^2 d\mu)^*(\xi) = |(f-h_n)(\xi)|\,|d\xi|, \qquad \xi \in \Gamma,$$

where $({B_n}^2\,d\mu)^*$ is the balayage of the measure ${B_n}^2\,d\mu$ on Γ. Using the formula for the balayage of $({B_n}^2\,d\mu)^*$ on Γ (see, for example, [20, Section II.4]), we get (2.1). ■

Lemma 2. *The Szegő function (cf.* [23, Chap. X]*)*

$$\varphi_n(z) = \exp\left(\frac{1}{4\pi}\int_\Gamma \frac{\xi+z}{\xi-z}\log\left|\frac{(f-h_n)(\xi)}{\Delta_n}\right|\,|d\xi|\right), \qquad |z|<1,$$

for $|(f-h_n)(\xi)|/\Delta_n$, $\xi \in \Gamma$, is non-vanishing and analytic in G, continuous on $\bar{G}$, and satisfies $\|\varphi_n\|_2 = 1$, $\varphi_n > 0$ on $(-1,1)$, and

$$|(f-h_n)(\xi)| = \Delta_n |\varphi_n(\xi)|^2, \qquad \xi \in \Gamma. \tag{2.2}$$

Furthermore, there exist positive constants C_1 and C_2 not depending on n such that

$$C_1 \le |\varphi_n(z)| \le C_2, \qquad z \in \bar{G}. \tag{2.3}$$

Proof. It is well-known (and easy to see) that the outer function φ_n is non-vanishing in G, and satisfies (2.2), from which it follows that $\|\varphi\|_2 = 1$. Since, by (2.1),

$$|(f-h_n)(\bar{\xi})| = |(f-h_n)(\xi)|, \qquad \xi \in \Gamma,$$

we obtain that $|\varphi_n(\bar{\xi})| = |\varphi_n(\xi)|$, $\xi \in \Gamma$, and φ_n is real on $(-1,1)$. Moreover, since $\varphi_n(0) > 0$ and $\varphi_n \neq 0$ in G, we can conclude that φ_n is positive on $(-1,1)$. Inequalities (2.3) follow easily from (2.1), (2.2) and the maximum principle applied to $\varphi_n(z)$ and $1/\varphi_n(z)$. ■

2.2. Orthogonality. The *embedding operator* $J\colon H_2(G) \to L_2(\mu, E)$ is given by restricting an element $\varphi \in H_2(G)$ to E:

$$J(\varphi) = \varphi|_E. \tag{2.4}$$

For the adjoint $J^*\colon L_2(\mu, E) \to H_2(G)$ of the embedding operator J we have the following formula (cf. [3]):

$$J^*(\psi)(z) = \frac{1}{2\pi} \int_E \frac{\psi(y)}{1 - yz}\, d\mu(y), \qquad |z| < 1,\ \psi \in L_2(\mu, E).$$

Consequently, for $\varphi \in H_2(G)$,

$$(J^*J)(\varphi)(z) = \frac{1}{2\pi} \int_E \frac{\varphi(y)}{1 - yz}\, d\mu(y), \qquad z \in G. \tag{2.5}$$

This formula will be used to establish Lemma 3 in which we show that the Szegő function φ_n can be extended analytically to $\bar{\mathbb{C}} \setminus E^{-1}$, where E^{-1} is the reflection of E in the unit circle Γ. In this lemma we also obtain an analytic continuation of the function $B_n^2(f - h_n)/\varphi_n$ and orthogonality properties of the numerator polynomial w_n^* associated with B_n.

Lemma 3. *The function $B_n^2(f - h_n)/\varphi_n$ can be extended analytically to $\bar{\mathbb{C}} \setminus E$ and*

$$\frac{{B_n}^2(\xi)(f - h_n)(\xi)}{\varphi_n(\xi)} = \frac{1}{2\pi i} \int_E \frac{{B_n}^2(x)\, d\mu(x)}{\varphi_n(x)(\xi - x)}, \qquad \xi \in \bar{\mathbb{C}} \setminus E. \tag{2.6}$$

The function φ_n can be extended analytically to $\bar{\mathbb{C}} \setminus E^{-1}$, and satisfies the equations

$$\frac{1}{2\pi} \int_E \frac{{B_n}^2(x)\, d\mu(x)}{\varphi_n(x)(1 - \xi x)} = \Delta_n \varphi_n(\xi), \qquad \xi \in \bar{\mathbb{C}} \setminus E^{-1}, \tag{2.7}$$

and

$$\frac{1}{2\pi} \int_E \frac{B_n(x) d\mu(x)}{\varphi_n(x)(1 - \xi x)} = \Delta_n (\varphi_n B_n)(\xi), \qquad \xi \in \bar{\mathbb{C}} \setminus E^{-1}. \tag{2.8}$$

Writing $B_n = w_n^*/w_n$, where $w_n^*(z) = \prod_{i=1}^n (z - x_{i,n})$ and $w_n(z) = \prod_{i=1}^n (1 - x_{i,n} z)$, the following orthogonality relations are valid:

$$\int_E x^k \frac{w_n^*}{\varphi_n(x) {w_n}^2(x)}\, d\mu(x) = 0 \qquad \textit{for } k = 0, 1, \dots, n - 1. \tag{2.9}$$

Proof. Let φ be any function in $H_2(G)$. Using (1.3) and (2.2), we can write

$$(\varphi {B_n}^2)(\xi)(f - h_n)(\xi)/\varphi_n(\xi)\, d\xi = \Delta_n \varphi(\xi) \overline{\varphi_n(\xi)}\, |d\xi|, \qquad \xi \in \Gamma. \tag{2.10}$$

Recalling that $h_n = P_n/B_n, P_n \in H_1(G)$, we get with the help of (1.2) that

$$\int_E \varphi(x) \frac{{B_n}^2(x) d\mu(x)}{\varphi_n(x)} = \Delta_n \int_\Gamma \varphi(\xi) \overline{\varphi_n(\xi)}\, |d\xi|. \tag{2.11}$$

It follows from the last formula that

$$J^*J\left(\frac{B_n^2}{\varphi_n}\right) = \Delta_n \varphi_n, \tag{2.12}$$

where $J\colon H_2(G) \to L_2(\mu, E)$ is the embedding operator. By (2.12) and (2.5), we get (2.7) for $\xi \in G$. Since the left-hand side of (2.7) is analytic in $\overline{\mathbb{C}} \setminus E^{-1}$, it provides an analytic continuation of φ_n to $\bar{\mathbb{C}} \setminus E^{-1}$.

Let $\varphi \in H_2(G)$. Multiplying equation (2.10) by $\overline{B_n(\xi)}$ yields

$$(\varphi B_n)(\xi)(f - h_n)(\xi)/\varphi_n(\xi)d\xi = \Delta_n \varphi(\xi)\overline{\varphi_n(\xi)B_n(\xi)}\,|d\xi|, \qquad \xi \in \Gamma.$$

Using now the last formula, we get

$$\begin{aligned} \int_E \varphi(x)\frac{B_n(x)d\mu(x)}{\varphi_n(x)} &= \Delta_n \int_\Gamma \varphi(\xi)\overline{\varphi_n(\xi)B_n(\xi)}\,|d\xi|, \\ J^*J\left(\frac{B_n}{\varphi_n}\right) &= \Delta_n \varphi_n B_n, \end{aligned}$$

from which (2.8) follows.

Taking $\xi = x_{i,n}$ in (2.8), we obtain that

$$\int_E \frac{B_n(x)d\mu(x)}{\varphi_n(x)(1 - x_{i,n}x)} = 0, \qquad i = 1, \dots, n,$$

which implies (2.9).

It follows from (1.3) and (2.2) that

$$\left(\frac{{B_n}^2(f - h_n)}{\varphi_n}\right)(\xi) = \Delta_n \overline{\varphi_n(\xi)}\frac{1}{i\xi} = \Delta_n \varphi_n\left(\frac{1}{\xi}\right)\frac{1}{i\xi}, \qquad \xi \in \Gamma, \tag{2.13}$$

where we used the fact that φ_n is real on $(-1, 1)$. Using (2.13), we can conclude that ${B_n}^2(f - h_n)/\varphi_n$ can be extended analytically to $\bar{\mathbb{C}} \setminus E$. Hence, by (2.13) and (2.7), we get (2.6). ∎

3. Asymptotic results

We assume hereafter that the support of μ is a closed interval $E = [a, b]$ and the measure μ satisfies the Szegő condition

$$\int_E \frac{\log(d\mu/dx)}{\sqrt{(x - a)(b - x)}}\,dx > -\infty. \tag{3.1}$$

It will be assumed without loss of generality that $E \subset (0, 1)$ so that $E^{-1} = [1/b, 1/a]$. Denote by Φ the conformal mapping of the region $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$ onto $\{w : r < |w| < 1/r\}$ with $\Phi(1) = 1$. Then

$$r = \exp\left(-\frac{\pi K}{2K'}\right),$$

where

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\tau^2 x^2)}}, \qquad \tau^2 = \frac{(1-a^2)(1-b^2)}{(1-ab)^2},$$

and K' is the corresponding elliptic integral for $\tau' = \sqrt{1-\tau^2}$ (see, e.g., [20]).

Let $g_E(z,\xi) = \log\left|(1-\bar{\xi}z)/(z-\xi)\right|$ be the Green's function of G with singularity at the point $\xi \in G$. Denote by ω the unit equilibrium measure for E corresponding to the Green's potential

$$V_{g_E}^{\omega}(z) = \int_E g_E(z,\xi)\, d\omega(\xi).$$

As is well-known the Green's potential $V_{g_E}^{\omega}$ has the following equilibrium property:

$$V_{g_E}^{\omega} = \log(1/r) \qquad \text{on } E,$$

and the measure $d\omega$ can be represented in the form

$$d\omega(x) = \frac{k}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}}\, dx \qquad \text{for } x \in E, \tag{3.2}$$

where $k = (1-ab)/2K'$ (cf. [20, Section II.5]).

Let $d\mu = \psi d\omega + d\mu_s$ be the Riesz decomposition of μ, where $\psi = d\mu/d\omega$ denotes the Radon-Nikodym derivative of μ with respect to the equilibrium measure ω, and μ_s is a singular measure. Denote by $\mathcal{G}_\omega(\psi)$ the geometric mean of the function ψ with respect to ω:

$$\mathcal{G}_\omega(\psi) = \exp\left(\int_E \log\psi\, d\omega\right).$$

It follows from the Szegő condition that $\mathcal{G}_\omega(\psi) > 0$.

Denote by $\mathcal{D}_\psi(z)$ the *Szegő function* of ψ for the doubly-connected domain $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$ (cf. [3]). We have

$$\begin{aligned}\mathcal{D}_\psi(z) \;\; = \;\; & \sqrt{\mathcal{G}_\omega(\psi)}\exp\Bigg(\sqrt{(z-a)(z-b)(1-az)(1-bz)} \\ & \times \frac{1}{2\pi}\int_a^b \frac{\log(\psi(x)/\mathcal{G}_\omega(\psi))}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}}\,\frac{1-2xz+x^2}{(z-x)(1-xz)}\,dx\Bigg).\end{aligned}$$

Here and in what follows we take that branch of the root that is positive on the positive part of the real line. $\mathcal{D}_\psi(z)$ has the following properties:

1. $\mathcal{D}_\psi(z)$ is analytic and non-vanishing in $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$; $\mathcal{D}_\psi(z)$ is an outer function in the Hardy space $H_2(\bar{\mathbb{C}} \setminus (E \cup E^{-1}))$ (cf. the appendix);
2. $|\mathcal{D}_\psi(x)|^2 = \psi(x)$ a.e. (almost everywhere) on E;
3. $|\mathcal{D}_\psi(z)|^2 = \mathcal{G}_\omega(\psi)$ on Γ.

We remark that in property 2), $|\mathcal{D}_\psi(x)|^2, x \in E$, is defined as the nontangential limit of $|\mathcal{D}_\psi(z)|^2$ as $z \to x$, $z \in \bar{\mathbb{C}} \setminus (E \cup E^{-1})$.

The rate of convergence to zero, as $n \to \infty$, of Δ_n was investigated by the authors in [3] (see also [4]). As consequence of results related to the convergence of the best meromorphic approximants to Markov functions in the space $L_p(\Gamma)$, $1 < p \leq \infty$, we proved that

$$\frac{\Delta_n}{2r^{2n}\mathcal{G}_\omega(\psi)} \to 1 \qquad \text{as } n \to \infty. \tag{3.3}$$

We now describe the limiting behavior of the functions B_n and φ_n.

Theorem 4. *The sequence of measures* $({B_n}^2/\Delta_n)\, d\mu$ *converges weak** *to* $d\omega$*:*

$$\frac{{B_n}^2}{\Delta_n}\, d\mu \xrightarrow{*} d\omega \qquad \text{as } n \to \infty. \tag{3.4}$$

Furthermore, for $z \in \bar{\mathbb{C}} \setminus (E \cup E^{-1})$

$$\frac{(B_n\mathcal{D}_\psi)(z)}{\Phi^n(z)\sqrt{\mathcal{G}_\omega(\psi)}} \to 1 \tag{3.5}$$

and for $z \in \bar{\mathbb{C}} \setminus E^{-1}$

$$\varphi_n(z) \to \hat{\varphi}(z) = \frac{\sqrt{k/2}}{\sqrt{(1-az)(1-bz)}}, \tag{3.6}$$

where the limits as $n \to \infty$ *are locally uniform.*

The next assertion describes the convergence of h_n to f.

Theorem 5. *We have*

$$\frac{(f-h_n)(z)}{\frac{\mathcal{D}_\psi^2(z)r^{2n}}{\Phi^{2n}(z)}} \to \frac{k}{i\sqrt{(1-az)(1-bz)(z-a)(z-b)}}, \tag{3.7}$$

uniformly on compact subsets of $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$ *as* $n \to \infty$.

Concerning the asymptotic distribution of the zeros $x_{i,n}$ of the Blaschke products B_n, we have the following easy consequence of (3.5).

Theorem 6. *The sequence of discrete probability measures*

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_{i,n}},$$

where δ_x *denotes the unit point mass at* x*, converges weak** *to* $d\omega$ *as* $n \to \infty$.

We remark Theorem 6 is valid under much weaker assumptions on the measure μ. For example, it suffices that $d\mu/dx > 0$ a.e. on $E = [a, b]$.

4. Proof of Theorem 4

4.1. Auxiliary results. Denote by Ψ the conformal mapping of the region $\bar{\mathbb{C}} \setminus E$ onto $\{w : |w| > 1\}$ such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. Let $C(E)$ be the logarithmic capacity of the interval $E = [a, b]$. It is well-known that $C(E) = (b-a)/4$ (see e.g. [20]). Let ω_1 be the unit equilibrium measure for E:

$$d\omega_1(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}\,dx \qquad \text{for } x \in E, \tag{4.1}$$

with corresponding logarithmic potential

$$V^{\omega_1}(z) = \int_E \log\frac{1}{|z-x|}\,d\omega_1(x).$$

We have for $x \in E$

$$d\omega(x) = \frac{\pi k}{\sqrt{(1-ax)(1-bx)}}\,d\omega_1(x) = \pi\sqrt{2k}\,\hat{\varphi}(x)\,d\omega_1(x), \tag{4.2}$$

where

$$\hat{\varphi}(z) = \frac{\sqrt{k/2}}{\sqrt{(1-az)(1-bz)}}.$$

Let $d\mu = \psi_1 d\omega_1 + d\mu_{s,1}$ be the Riesz decomposition of μ, where $\psi_1 = d\mu/d\omega_1$ is the Radon-Nikodym derivative of μ with respect to the equilibrium measure ω_1, and $\mu_{s,1}$ is a singular measure.

Let $\mathcal{G}_{\omega_1}(\psi_1)$ be the geometric mean of the function ψ_1 with respect to ω_1:

$$\mathcal{G}_{\omega_1}(\psi_1) = \exp\left(\int_E \log\psi_1\,d\omega_1\right).$$

Since the measure μ satisfies the Szegő condition (3.1), $\mathcal{G}_{\omega_1}(\psi_1) > 0$.

Denote by $D_{\psi_1}(z)$ the *classical Szegő function* of ψ_1 for the simply connected domain $\bar{\mathbb{C}} \setminus E$. $D_{\psi_1}(z)$ has the following properties:

1. $D_{\psi_1}(z)$ is analytic and non-vanishing in $\bar{\mathbb{C}} \setminus E$; $D_{\psi_1}(z)$ is an outer function in the Hardy space $H_2(\bar{\mathbb{C}} \setminus E)$ (cf. the appendix);
2. $|D_{\psi_1}(z)|^2 = \psi_1(z)$ a.e. on E and ${D_{\psi_1}}^2(\infty) = \mathcal{G}_{\omega_1}(\psi_1)$.

We have (see e.g. [24])

$$D_{\psi_1}(z) = \exp\left(\sqrt{(z-a)(z-b)}\,\frac{1}{2\pi}\int_a^b \frac{\log(\psi_1(x))}{\sqrt{(x-a)(b-x)}}\,\frac{1}{z-x}\,dx\right). \tag{4.3}$$

4.2. The convergence of φ_n. Let

$$\delta_n = \int_E \frac{w_n^{*2}(x)d\mu(x)}{w_n{}^2(x)\varphi_n(x)} = \int_E \frac{B_n{}^2(x)d\mu(x)}{\varphi_n(x)}.$$

Since, from (2.9), the polynomial $w_n^*(z) = \prod_{i=1}^n (z - x_{i,n})$ is the n-th orthogonal polynomial with respect to the measure $d\mu/\varphi_n w_n{}^2$, and φ_n is bounded from below and above on E by constants not depending on n, by virtue of the results of Totik [24, Sections 14 and 16] and Stahl [21],

$$\frac{\delta_n}{2C(E)^{2n}\mathcal{G}_{\omega_1}(\psi_1/w_n{}^2\varphi_n)} \to 1 \qquad \text{as } n \to \infty \tag{4.4}$$

and

$$\frac{w_n^{*2} D_{\psi_1/w_n{}^2\varphi_n}{}^2(z)}{C(E)^{2n}\Psi^{2n}(z)\mathcal{G}_{\omega_1}(\psi_1/w_n{}^2\varphi_n)} \to 1 \tag{4.5}$$

uniformly on compact subsets of $\bar{\mathbb{C}} \setminus E$ as $n \to \infty$. Moreover, the sequence of measures $(B_n{}^2/\varphi_n\delta_n)\, d\mu$ weak* converges to $d\omega_1$:

$$\frac{B_n{}^2}{\varphi_n\delta_n}\, d\mu \xrightarrow{*} d\omega_1 \qquad \text{as } n \to \infty \tag{4.6}$$

(see [3, page 407] for more details concerning the application of results from [24] and [21]).

We now prove that, as $n \to \infty$,

$$\varphi_n(z) \to \hat{\varphi}(z)$$

uniformly on compact subsets of $\bar{\mathbb{C}} \setminus E^{-1}$, where $\hat{\varphi}(z)$ is defined in (3.6). By (2.7) and (4.6), for $z \in \bar{\mathbb{C}} \setminus E^{-1}$, we have

$$\begin{aligned} \frac{\Delta_n}{\delta_n}\varphi_n(z) &= \frac{1}{2\pi}\int_E \frac{B_n{}^2(x)d\mu(x)}{\varphi_n(x)\delta_n(1-zx)} \\ &\to \frac{1}{2\pi}\int_E \frac{d\omega_1(x)}{1-zx} = \frac{1}{2\pi\sqrt{(1-az)(1-bz)}} \qquad \text{as } n \to \infty. \end{aligned}$$

Since $\|\varphi_n\|_2 = 1$ and $\|1/\sqrt{(1-az)(1-bz)}\|_2 = \sqrt{2/k}$,

$$\frac{\Delta_n}{\delta_n} \to \frac{1}{\pi\sqrt{2k}} \qquad \text{as } n \to \infty, \tag{4.7}$$

and for $z \in \bar{\mathbb{C}} \setminus E^{-1}$

$$\varphi_n(z) \to \hat{\varphi}(z) = \frac{\sqrt{k/2}}{\sqrt{(1-az)(1-bz)}}, \tag{4.8}$$

where the limit as $n \to \infty$ is locally uniform. We remark that it follows from (3.3) and (4.7) that

$$\frac{\delta_n}{\pi\sqrt{2k}\, 2r^{2n}\mathcal{G}_\omega(\psi)} \to 1 \quad \text{as} \quad n \to \infty. \tag{4.9}$$

4.3. The limiting distribution of zeros of B_n. In this subsection we obtain Szegő type asymptotics for the Blaschke products B_n. We have

$$\frac{{B_n}^2 d\mu}{\Delta_n} = \frac{\delta_n}{\Delta_n}\, \varphi_n \frac{{B_n}^2}{\delta_n \varphi_n}\, d\mu.$$

From this, on account of (4.7), (4.8), (4.6), and (4.2), we get (3.4).

We now prove (3.5). Let

$$u_n = \frac{{B_n}^2 {\mathcal{D}_\psi}^2}{\Phi^{2n}\mathcal{G}_\omega(\psi)}.$$

We remark that u_n is analytic in $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$. By properties of the Szegő function $\mathcal{D}_\psi$, we have

$$|u_n| = 1 \qquad \text{on } \Gamma$$

and

$$|u_n| = \frac{{B_n}^2 \psi}{r^{2n}\mathcal{G}_\omega(\psi)} \tag{4.10}$$

a.e. on E. We claim that $u_n \to 1$ uniformly on compact subsets of $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$ as $n \to \infty$. Since $|u_n| = 1$ on Γ, it is sufficient to show that, for $z \in G \setminus E$, $u_n \to 1$ where the limit as $n \to \infty$ locally uniform.

Setting

$$m_n(z) = \frac{2{B_n}^2(z){w_n}^2(z)(D_{\psi_1/{w_n}^2\varphi_n})^2(z)}{\delta_n \Psi^{2n}(z)}, \tag{4.11}$$

it follows from (4.4) and (4.5) that

$$m_n(z) \to 1 \tag{4.12}$$

uniformly on compact subsets of $\bar{\mathbb{C}} \setminus E$ as $n \to \infty$.

Let

$$v_n(z) = \frac{{B_n}^2(z){w_n}^2(z){D_\psi}^2(z)}{\Psi^{2n}(z)(D_{{w_n}^2})^2(z) r^{2n}\mathcal{G}_\omega(\psi)}.$$

It is not hard to see that the function $v_n(z)$ is analytic in $\bar{\mathbb{C}} \setminus E$ and

$$|v_n| = {B_n}^2\psi/(r^{2n}\mathcal{G}_\omega(\psi)) \tag{4.13}$$

a.e. on E. With the help of (4.2), we can represent $v_n(z)$ in the form

$$v_n(z) = m_n(z) \cdot \frac{\delta_n}{\pi\sqrt{2k}\, 2r^{2n}\mathcal{G}_\omega(\psi)} \cdot \frac{{D_{\varphi_n}}^2(z)}{{D_{\hat{\varphi}}}^2(z)}.$$

By (4.8), (4.9), and (4.12), we get

$$v_n(z) \to 1 \tag{4.14}$$

uniformly on compact subsets of $\bar{\mathbb{C}} \setminus E$ as $n \to \infty$.

Let $g_n = u_n/v_n$. Then g_n is analytic and nonzero in $\bar{\mathbb{C}} \setminus (E \cup E^{-1})$. By (4.10) and (4.13), we get $|g_n| = 1$ a.e. on E. From this, since $\mathcal{D}_\psi \in H_2(\bar{\mathbb{C}} \setminus (E \cup E^{-1}))$ and $D_\psi \in H_2(\bar{\mathbb{C}} \setminus E)$ are outer functions in the corresponding Hardy spaces, we obtain by Remark 10 and Theorem 11 of the Appendix that $g_n \in H_\infty(G \setminus E)$ and $1/g_n \in H_\infty(G \setminus E)$. Using now a normal families argument, and the facts that $|g_n| \to 1$ uniformly on Γ as $n \to \infty$, and $(u_n/v_n)(x) > 0$ for $x \in (b, 1)$, we get $g_n \to 1$ uniformly on compact subsets of $G \setminus E$ as $n \to \infty$. Therefore, by (4.14), $u_n(z) \to 1$ uniformly on compact subsets of $G \setminus E$ as $n \to \infty$. Hence, (3.5) is proved.

5. Proof of Theorem 5

Using (2.6), we can write

$$(f - h_n)(z) = \frac{\varphi_n(z)}{{B_n}^2(z)} \cdot \frac{1}{2\pi i} \int_E \frac{{B_n}^2(x) d\mu(x)}{\varphi_n(x)(z - x)}, \qquad z \in \bar{\mathbb{C}} \setminus E.$$

By (4.6),

$$\int_E \frac{{B_n}^2(x) d\mu(x)/\delta_n}{\varphi_n(x)(z - x)} \to \int_E \frac{d\omega_1(x)}{z - x} = \frac{1}{\sqrt{(z-a)(z-b)}},$$

uniformly on compact subsets of $\bar{\mathbb{C}} \setminus E$ as $n \to \infty$. Hence, by (3.5), (3.6) and (4.9), we get (3.7).

Appendix A. Hardy spaces of a slit disk

A.1. Definition and boundary values. For $1 \le p < \infty$, the Hardy space $H_p(D)$ of a domain $D \subset \bar{\mathbb{C}}$ is the space of those analytic functions g in D such that $|g|^p$, which is subharmonic in D, has a harmonic majorant there; the Hardy space $H_\infty(D)$ is simply the space of bounded analytic functions in D. Clearly, this definition is conformally invariant.

Hardy spaces of the unit disk are well-known and we shall refer freely to their classical properties (see, for instance, [6, 10, 11]). From them, one can proceed to Hardy spaces of simply connected domains whose boundary is a rectifiable Jordan curve via conformal mapping, and basic properties remain valid in this context with minor modifications, especially when the boundary is analytic (see for instance [6, Chapter 10, Section 10.2] for an introduction). For multiply connected domains things get more difficult, but the case of boundaries consisting of finitely many analytic Jordan curves has been considerably worked out, both in its functional-analytic aspects and from the geometric point of view on Riemann surfaces (see, for instance, [25, 26] and also [8, Chapters 3 and 4] for a self-contained introduction and further references). However, even in this case, it seems that factorization theory has not come to the degree of completeness it has reached on the circle. For instance, when needed on a slit disk for the proof of Theorem 4, the authors could not locate a published analog on such a domain to the classical fact that the ratio of a $H_p(D)$-function by an outer factor lies in $H_\infty(D)$ if its boundary values lie in $L_\infty(\partial D)$. The result is carried out indeed on the annulus in [19, Theorem 6, Section 6], with a different (but equivalent on the annulus) definition of Hardy spaces, but we could not even find in the literature a treatment of (generally twofold) boundary values when some components of the boundary of D are Jordan *arcs* rather than curves. Moreover, in the latter case, Hardy spaces defined through integral means differ from those defined through harmonic majorants, essentially because summability with respect to harmonic measure and with respect to arclength become distinct notions. This is the raison d'être for the present appendix. Although the results below generalize to finitely connected domains whose boundary consists of rectifiable Jordan arcs and curves, we make no attempt at being general and we shall proceed as quickly as possible with the result we need, namely Theorem 11 below.

We assume hereafter that $[a, b] \subset G$ is a real segment, and we set for simplicity $\Omega = G \setminus [a, b]$ so that $\partial\Omega = \Gamma \cup [a, b]$. Occasionally in the paper, we also refer to some of the definitions and results below for the domain $\bar{\mathbb{C}} \setminus ([a, b] \cup [1/a, 1/b])$, and we leave it to the reader to check that everything carries over to the latter with obvious modifications, using *e.g.* reflection across the circle.

Lemma 7. *For $1 \leq p \leq \infty$, any member of $H_p(\Omega)$ has nontangential limit almost everywhere on Γ and nontangential limits from above and below almost everywhere on $[a, b]$ with respect to Lebesgue measure. The boundary functions thus defined are measurable with respect to Lebesgue measure.*

Proof. From [6, Theorem 10.2], it follows that each function in $H_p(\Omega)$ is the sum of a function in $H_p(G)$ and a function in $H_p(\bar{\mathbb{C}} \setminus [a, b])$; actually, the theorem that we refer to is stated for domains whose boundary consists of a finite union of Jordan *curves*, but examination of the proof shows that is still valid for increasing unions (keeping connectivity fixed) of such domains and therefore it is valid in our case. Now, the existence of measurable nontangential limits on Γ for $H_p(G)$-functions is well-known, and from it the existence of measurable nontangential limits from above and below on $[a, b]$ for $H_p(\bar{\mathbb{C}} \setminus [a, b])$-functions follows by composition with the conformal mapping

$$\alpha(z) = \frac{(b-a)}{4}\left(z + \frac{1}{z}\right) + \frac{a+b}{2} \tag{A.1}$$

from G onto $\bar{\mathbb{C}} \setminus [a, b]$, granted the conformal invariance of Hardy spaces. ■

It is well-known that Hardy functions on the disk can be recovered from their boundary values through a Cauchy integral and through a Poisson integral as well. On Ω, the analog of the Poisson integral will be more useful for our purposes, and this is why we need now introduce harmonic measure. On a domain D, the harmonic measure from a point $z \in D$, denoted by $\omega_{z,D}$, is the unique Borel probability measure on ∂D such that every harmonic function h on D that is continuous on $\bar{D}$ satisfies

$$h(z) = \int_{\partial D} h(\xi)\, d\omega_{z,D}(\xi).$$

Such a measure exists for every $z \in D$ whenever ∂D has positive logarithmic capacity, see e.g. [18, Chapter 4, Section 3] or [20, Appendix 3], and this is the case in particular when $D = \Omega$. Moreover, although $\omega_{z,D}$ depends on z, there is an inequality of the form $\omega_{z_1,D} \le C\omega_{z_2,D}$, for some constant $C = C(z_1, z_2)$, whenever $z_1, z_2 \in D$ (see [18, Corollary 4.3.5] or [20, Appendix 3, Section V]), so we simply speak of subsets of harmonic measure zero on ∂D and likewise of measurable and summable functions with respect to harmonic measure.

Lemma 8.

(i) *The harmonic measure of Ω is absolutely continuous with respect to Lebesgue measure on $\partial\Omega = \Gamma \cup [a, b]$.*

(ii) *If $\lambda \in L_1(\omega_{z,\bar{\mathbb{C}}\setminus[a,b]}, [a, b])$, then $\lambda \in L_1(\omega_{z,\Omega}, [a, b])$, and if*

$$h(z) = \int_{[a,b]} \lambda(t)\, d\omega_{z,\bar{\mathbb{C}}\setminus[a,b]}(t), \qquad z \in \bar{\mathbb{C}} \setminus [a, b],$$

designates the corresponding solution to the generalized Dirichlet problem, then

$$h(z) = \int_{\Gamma} h(\xi)\, d\omega_{z,\Omega}(\xi) + \int_{[a,b]} \lambda(t)\, d\omega_{z,\Omega}(t), \qquad z \in \Omega. \tag{A.2}$$

(iii) *If $\eta \in L_1(\omega_{z,G}, \Gamma)$, then $\eta \in L_1(\omega_{z,\Omega}, \Gamma)$, and if*

$$u(z) = \int_{\Gamma} \eta(\xi)\, d\omega_{z,G}(\xi), \qquad z \in G,$$

designates the corresponding solution to the generalized Dirichlet problem, then

$$u(z) = \int_{\Gamma} \eta(\xi)\, d\omega_{z,\Omega}(\xi) + \int_{[a,b]} u(t)\, d\omega_{z,\Omega}(t), \qquad z \in \Omega. \tag{A.3}$$

Proof. Let Z be a Borel subset of Γ or $[a, b]$. If we discard one of the connected components of $\partial\Omega$, we get a simply connected domain Ω_1 such that $\omega_{z,\Omega_1}(Z) \ge \omega_{z,\Omega}(Z)$ for each $z \in \Omega$ [18, Corollary 4.3.9]. This shows that $L_1(\omega_{z,\bar{\mathbb{C}}\setminus[a,b]}, [a, b]) \subset L_1(\omega_{z,\Omega}, [a, b])$ and also that $L_1(\omega_{z,G}, \Gamma) \subset L_1(\omega_{z,\Omega}, \Gamma)$ for $z \in \Omega$, which accounts for the summability assertion in (ii), (iii), and reduces the proof of (i) to showing that $\omega_{z,G}$ and $\omega_{z,\bar{\mathbb{C}}\setminus[a,b]}$ are absolutely continuous with respect to Lebesgue measure on Γ and $[a,b]$ respectively. The disk case is trivial since $d\omega_{z,G}(\theta) = P_z(\theta)d\theta$, where P_z is the familiar Poisson kernel, and the case of $\bar{\mathbb{C}} \setminus [a,b]$ follows for instance from the fact that $\omega_{\infty,\bar{\mathbb{C}}\setminus[a,b]}$ is the equilibrium measure of the segment $[a,b]$ for the logarithmic potential [18, Theorem 4.3.14] which is given by (4.1). This proves (i).

As to (A.2), we observe by Brelot's theorem [20, Appendix 3] that the right-hand side of (A.2) is the generalized solution to the Dirichlet problem on Ω with boundary values h on Γ and λ on $[a,b]$, and therefore it is the lower (resp. upper) envelope of those superharmonic (resp. subharmonic) functions S in Ω that are bounded below (resp. above) and satisfy

$$\liminf_{z\to t\in[a,b]} S(z) \geq \lambda(t) \qquad \text{and} \qquad \liminf_{z\to\xi\in\Gamma} S(z) \geq h(\xi)$$

(resp.

$$\limsup_{z\to t\in[a,b]} S(z) \leq \lambda(t) \qquad \text{and} \qquad \limsup_{z\to\xi\in\Gamma} S(z) \leq h(\xi)).$$

Since h is itself the generalized solution to the Dirichlet problem on $\bar{\mathbb{C}} \setminus [a,b]$ with boundary values λ on $[a,b]$, it can be characterized, using Brelot's theorem again, as a lower (resp. upper) envelope of a family of superharmonic (resp. subharmonic) functions which is easily checked to be included in the previous one. Therefore h is at the same time not smaller and not bigger than the right-hand side of (A.2). The proof of (A.3) is similar. ∎

We are now in position to prove the following theorem.

Theorem 9. *Let $1 \leq p \leq \infty$ and $g \in H_p(\Omega)$. If g^* denotes the nontangential boundary function of g on Γ and $g^{\pm}$ its nontangential boundary function on $[a,b]$ from above and below, then $|g^*|^p$ concatenated with $|g^{\pm}|^p$, as well as $\log|g^*|$ concatenated with $\log|g^{\pm}|$, are summable with respect to the harmonic measure $\omega_{z,\Omega}$. Moreover, it holds that*

$$\text{(A.4)} \qquad \begin{aligned} g(z) + g(\bar{z}) \;=\; & \int_\Gamma (g^*(\xi) + g^*(\bar{\xi}))\, d\omega_{z,\Omega}(\xi) \\ & + \int_{[a,b]} (g^+(t) + g^-(t))\, d\omega_{z,\Omega}(t), \qquad z \in \Omega. \end{aligned}$$

Proof. Note that g^* concatenated with $g^{\pm}$ is indeed measurable with respect to (the completion of) harmonic measure by Lemmas 7 and 8. If $g \in H_\infty(\Omega)$, then g^* and $g^{\pm}$ are obviously bounded, and the summability of their log-modulus, as well as (A.4), will follow from the case $p < \infty$ since $H_\infty(\Omega) \subset H_p(\Omega)$. Thus we assume that $p < \infty$ and, upon decomposing g as the sum of a function in $H_p(G)$ and of a function in $H_p(\bar{\mathbb{C}} \setminus [a,b])$ like we did in the proof of Lemma 7, it is enough to prove the theorem when g belongs to one of these spaces.

The case where $g \in H_p(G)$ is clear from Lemma 8(iii), the fact that functions in $H_p(G)$ are Poisson integrals of their boundary values, and the subsequent fact that $g(\bar{z})$ is the solution to the generalized (complex) Dirichlet problem in G with boundary values $h(e^{i\theta}) = g^*(e^{-i\theta})$.

To prove the result when $g \in H_p(\bar{\mathbb{C}} \setminus [a,b])$, we consider again the conformal map $\alpha\colon G \to \bar{\mathbb{C}} \setminus [a,b]$ defined by (A.1), and we observe that it extends to a continuous map $\bar{G} \to \bar{\mathbb{C}}$ that covers $[a,b]$ twice, each subset of Γ having the same image as its conjugate, with the restrictions $\alpha\colon \Gamma \cap \{\pm \mathrm{Im} z \geq 0\} \to [a,b]$ being homeomorphisms. Therefore, if h is a harmonic function on $\bar{\mathbb{C}} \setminus [a,b]$ that is continuous on $\bar{\mathbb{C}}$, then $H = h \circ \alpha$ is a continuous function on $\bar{G}$ which is harmonic in G and satisfies $H(e^{i\theta}) = H(e^{-i\theta})$; conversely, any H with these properties is of the form $h \circ \alpha$ for some harmonic function h on $\bar{\mathbb{C}} \setminus [a,b]$ that is continuous on $\bar{\mathbb{C}}$. Hence we get by definition of harmonic measure that for $z \in G$

$$\text{(A.5)} \qquad \int_\Gamma H(\xi)\, d\omega_{z,G}(\xi) = H(z) = h(\alpha(z)) = \int_{[a,b]} h(t)\, d\omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}(t).$$

Moreover, since it has connected boundary, $\bar{\mathbb{C}} \setminus [a,b]$ is a regular domain for the Dirichlet problem [18, Theorem 4.2.4], which means that any continuous function on $[a,b]$ extends continuously to a harmonic function on $\bar{\mathbb{C}} \setminus [a,b]$. Consequently (A.5) implies that

$$\int_\Gamma h \circ \alpha(\xi)\, d\omega_{z,G} = \int_{[a,b]} h(t)\, d\omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}(t)$$

holds for every continuous function h on $[a,b]$, implying that the inverse image of $\omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}$ under α is the Poisson integral at z on those Borel subsets of Γ that are invariant under conjugation. In particular, it holds that

$$\text{(A.6)} \qquad \omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}(\alpha(B)) = \int_{B\cup\bar{B}} P_z(\theta)\, d\theta, \qquad B \subset \Gamma \cap \{\pm\, \mathrm{Im}\, z \geq 0\}.$$

From this, we deduce that if $g \in H_p(\bar{\mathbb{C}} \setminus [a,b])$ and $g^{\pm}$ is the boundary function from above or below on $[a,b]$, then

$$\begin{aligned}
\int_{[a,b]} |g^{\pm}(t)|^p\, d\omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}(t) &= \int_0^{\pi} |g^{\pm}(\alpha(e^{i\theta}))|^p\, P_z(\theta)\, d\theta \\
&\quad + \int_{\pi}^{2\pi} |g^{\pm}(\alpha(e^{-i\theta}))|^p\, P_z(\theta)\, d\theta, \\
\int_{[a,b]} \log|g^{\pm}(t)|\, d\omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}(t) &= \int_0^{\pi} \log|g^{\pm}(\alpha(e^{i\theta}))|\, P_z(\theta)\, d\theta \\
&\quad + \int_{\pi}^{2\pi} \log|g^{\pm}(\alpha(e^{-i\theta}))|\, P_z(\theta)\, d\theta.
\end{aligned}$$

Because $f \mapsto f \circ \alpha$ establishes a one-to-one correspondence between $H_p(\bar{\mathbb{C}} \setminus [a,b])$ and $H_p(G)$, and since α preserves non-tangential convergence to the boundary at every point of $\Gamma \setminus \{-1,1\}$, the announced summability now follows from the well-kown properties of $H_p(G)$. To establish (A.4), we use (A.6) to compute

$$\begin{aligned}\int_{[a,b]} \big(g^+(t)+g^-(t)\big)\, d\omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}(t) &= \int_0^{\pi} \big(g^+(\alpha(e^{i\theta}))+g^-(\alpha(e^{i\theta}))\big)\, P_z(\theta)d\theta \\ &+ \int_{\pi}^{2\pi} \big(g^+(\alpha(e^{-i\theta}))+g^-(\alpha(e^{-i\theta}))\big)\, P_z(\theta)d\theta.\end{aligned}$$

Taking into account that $\alpha(e^{-i\theta}) = \alpha(e^{i\theta})$, we reorder terms in the right-hand side to obtain:

$$\begin{aligned}\int_{[a,b]} \big(g^+(t)+g^-(t)\big)\, d\omega_{\alpha(z),\bar{\mathbb{C}}\setminus[a,b]}(t) &= \left\{\int_0^{\pi} g^+(\alpha(e^{i\theta}))\, P_z(\theta)\, d\theta + \int_{\pi}^{2\pi} +g^-(\alpha(e^{i\theta}))\, P_z(\theta)\, d\theta\right\} \\ &+ \left\{\int_0^{\pi} g^-(\alpha(e^{i\theta}))\, P_z(\theta)\, d\theta + \int_{\pi}^{2\pi} g^+(\alpha(e^{i\theta})))\, P_z(\theta)\, d\theta\right\}.\end{aligned}$$

Since $g \circ \alpha \in H_p(G)$ has boundary values $g^+(\alpha(e^{i\theta}))$ on the upper half of Γ and $g^-(\alpha(e^{i\theta}))$ on the lower half, and since moreover $\overline{\alpha(z)} = \alpha(\bar{z})$, the above quantity is just $g(\alpha(z))+g\big(\overline{\alpha(z)}\big)$. By Lemma 8 (ii) equation (A.2), we now conclude upon renaming $\alpha(z)$ as z that (A.4) indeed holds in this case too. ∎

We can now define an *outer function* in $H_p(\Omega)$ to be some $g \in H_p(\Omega)$ with the following three properties:

- g has no zeros in Ω,
- $|g|$ has a well-defined boundary function on $\partial\Omega$, that is to say the moduli of the upper and lower nontangential limits of $|g|$ must agree a.e. on $[a,b]$,
- $\log|g|$ solves the generalized Dirichlet problem with respect to its nontangential boundary values:

(A.7) $$\log|g|(z) = \int_{\partial\Omega} \log|g|(\xi)\, d\omega_{z,\Omega}(\xi), \qquad z \in \Omega,$$

  where we have kept in this equation the notation $|g|$ to denote the nontangential boundary function since this is non-ambiguous here.

Remark 10. Note that (A.7) is well-defined in view of Theorem 9. Note also, by Lemma 8(ii), that an outer function in $H_p(\bar{\mathbb{C}} \setminus [a,b])$ (similar definition) restricts to a outer function in $H_p(\Omega)$. The same applies via the same proof to an outer function of $H_p(\bar{\mathbb{C}} \setminus \{[a,b] \cup [1/a,1/b]\})$.

Our goal is to establish the following result:

Theorem 11. *Let $1 \leq p \leq \infty$ and $g_1, g_2 \in H_p(\Omega)$ with g_2 outer. Then, $g_1/g_2 \in H_\infty(\Omega)$ if, and only if, its nontangential boundary values on Γ and on $[a,b]$ from above and below lie in $L_\infty(\partial\Omega)$.*

To proceed with the proof, we need a few facts from uniformization theory that we outline briefly. As Ω is a planar domain with more than two boundary points, recall from the uniformization theorem (see for instance [1, Theorem 10.4] or [8, Theorem 2.3]) that there exists an analytic covering $\Upsilon\colon G \to \Omega$, and a group Σ of Möbius transformations $G \to G$ acting transitively on the fibers called the *automorphic group* of G, with the property that $\Upsilon \circ \sigma = \Upsilon$ for each $\sigma \in \Sigma$. The automorphic group is isomorphic to the fundamental group of Ω, namely to the group of integers $\mathbf{Z}$. The covering Υ can be normalized by picking $z_0 \in \Omega$ and requiring that $\Upsilon(0) = z_0$, $\Upsilon'(0) > 0$, and it is then unique. Now, using elementary properties of harmonic functions, it is not difficult to check from the definitions (see for instance [6, Theorem 10.11]) that $g \mapsto g \circ \Upsilon$ establishes a one-to-one correspondence between $H_p(\Omega)$ and those functions in $H_p(G)$ that are invariant under right composition by members of Σ.

Note, since Ω is bounded, that Υ belongs to $H_\infty(G)$ and thus it has a non-tangential boundary function Υ^* on Γ, which is clearly invariant under right composition by members of Σ.

We *claim* that Υ^* maps Γ into $\partial\Omega$.

Indeed, if we had $\Upsilon^*(e^{i\theta_0}) = z_0 \in \Omega$, then by the covering property there would be a connected neighborhood V of z_0 such that $\Upsilon^{-1}(z_0)$ consists of a disjoint union of connected open sets, each of which is homeomorphic to V under Υ. By connectedness, one of them, say W, would contain a ray $\{re^{i\theta_0};\ r_0 \leq r < 1\}$ on which we can pick a sequence z_n converging to $e^{i\theta_0}$ in $\mathbb{C}$; but z_n does not converge in W whereas $\Upsilon(z_n)$ converges to z_0 in V, contradicting homeomorphy. This proves the claim.

Now, since $\Omega \subset G$, the function Υ can be viewed as a map $G \to G$, so the Julia-Wolff lemma [17, Proposition 4.13] implies that the angular derivative $\Upsilon'(\xi)$ exists at every $\xi \in \Gamma$ where $\Upsilon^*(\xi) \in \Gamma$. In the same vein, composing Υ with

$$z \mapsto \frac{2z-(a+b)}{b-a} - \sqrt{\left(\frac{2z-(a+b)}{b-a}\right)^2 - 1},$$

which conformally maps $\bar{\mathbb{C}} \setminus [a,b] \to G$ and sends the upper (resp. lower) half of the cut onto the lower (resp. upper) half of Γ, and which is clearly conformal at every point of (a,b) from above and below, we conclude that the angular derivative $\Upsilon'(\xi)$ exists also at every $\xi \in \Gamma$ where $\Upsilon^*(\xi) \in (a,b)$. Altogether, Υ is conformal a.e. on Γ.

If now h is a harmonic function on Ω that is continuous on $\bar{\Omega}$, then $H = h \circ \Upsilon$ is a bounded harmonic function on G and as such the Poisson integral of its nontangential boundary values H^*. As $H^* = h \circ \Upsilon^*$ by the continuity of h, we get by definition of harmonic measure that

$$\text{(A.8)} \qquad \frac{1}{2\pi}\int_\Gamma h(\Upsilon^*(e^{i\theta}))\, P_z(\theta)\, d\theta = H(z) = h(\Upsilon(z)) = \int_{\partial\Omega} h(t)\, d\omega_{\Upsilon(z),\Omega}(t),$$

so the image of Υ^* has full measure on $\partial\Omega$, and since Ω is regular for the Dirichlet problem we conclude that the inverse image of $\omega_{\Upsilon(z),\Omega}$ under Υ^* is $\omega_{z,G}$ on those Borel subsets of Γ that are invariant under Σ.

The key to the proof of Theorem 11 is the following lemma.

Lemma 12. *Let $g \in H_p(\Omega)$ and $f \in H_p(G)$ be such that $f = g \circ \Upsilon$. If g is outer in $H_p(\Omega)$, so is f in $H_p(G)$.*

Proof. Clearly f has no zeros in G, and it remains to show, granted (A.7), that

$$\text{(A.9)} \qquad \log|g \circ \Upsilon|(z) = \int_\Gamma \log|(g \circ \Upsilon)^*|(\xi)\, d\omega_{z,G}(\xi), \qquad z \in G.$$

As we just saw that the inverse image of $\omega_{\Upsilon(z),\Omega}$ under Υ^* is $\omega_{z,G}$, (A.9) will follow from (A.7) if we can show that $(g \circ \Upsilon)^* = g^* \circ \Upsilon^*$. This in turn follows from the previously observed fact that Υ is conformal a.e. on Γ. ∎

Proof of Theorem 11. If we introduce the functions h_1 and h_2 in $H_p(G)$ such that $h_1 = g_1 \circ \varphi$ and $h_2 = g_2 \circ \varphi$, and if we observe that h_2 is outer by Lemma 12, we are left to establish the corresponding property on the disk where it is is well-known (cf. for instance [11, Chapter IV, Section E]). ∎

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