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Growth Majorants and Quotient Representations of Meromorphic Functions

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Abstract. The classes Λ of meromorphic functions in $\mathbb{C}$ of finite λ-type were introduced and studied by L. A. Rubel and B. A. Taylor. The representation problem for meromorphic functions from Λ as quotients of entire functions from Λ without common zeroes, the so-called canonical quotient representations, is considered, in this paper. It is shown that this problem reduces to the study of growth majorant properties of entire functions with restrictions on their zero quantities.

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1. Introduction and main results

The classical representation problem for a meromorphic function in a domain consists in representing f as a quotient $f = g/h$ of holomorphic functions g and h the growth of which does not exceed the growth of f. More precisely, the growth of $\log|g(z)|$ and $\log|h(z)|$ does not exceed in some sense the growth of the Nevanlinna characteristic $T(|z|, f)$, whenever it is defined.

Some known results can be formulated as follows.

Theorem A (J. Liouville – R. Nevanlinna, [6, p. 49])**.** *A meromorphic function f in $\mathbb{C}$ can be represented as the quotient $f = g/h$ of polynomials g and h, i.e.*

$$\log(|g(z)| + |h(z)|) = O(\log|z|), \qquad |z| \to +\infty,$$

if and only if $T(r, f) = O(\log r)$, $r \to +\infty$.

Theorem B (R. Nevanlinna, [15, p. 24])**.** *A function f meromorphic in the unit disk $\mathbb{D}$ is of bounded Nevanlinna characteristic if and only if there exist functions g and h bounded holomorphic in $\mathbb{D}$ such that $f = g/h$.*

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A classical result for functions meromorphic in $\mathbb{C}$ follows from the Hadamard and Borel theorems for entire functions.

Theorem C ([15, p. 91]). *If f is a meromorphic function in $\mathbb{C}$ of order $\rho(f) = \rho < +\infty$, then there exist entire functions g and h with $\rho(g) \leq \rho$ and $\rho(h) \leq \rho$ such that $f = g/h$.*

Mention that in Theorems A, B and C, the functions g and h do not have common zeroes.

The most general result concerns the following classes of functions.

Definition 1 ([15, p. 66]). *Let λ be a positive, continuous, non-decreasing, and unbounded function on $(0, +\infty)$, called the growth function. A meromorphic function in $\mathbb{C}$ is of finite λ-type if*

$$T(r, f) \leq a\lambda(br)$$

for some constants a, b and all positive r. Let Λ denote that class of functions.

Let $\mathcal{Z} = \{a_j\}$ be a sequence of nonzero complex numbers without accumulation points in $\mathbb{C}$. The counting function $n(r, \mathcal{Z})$ denotes the quantity of a_j in the disk $\{z\colon |z| \leq r\}$, and

$$\mathcal{N}(r, \mathcal{Z}) = \int_0^r \frac{n(t, \mathcal{Z})}{t}\, dt.$$

Let λ be a growth function. By S_λ we denote the set of sequences $\mathcal{Z}$ satisfying the inequality

$$\mathcal{N}(r, \mathcal{Z}) \leq a\lambda(br)$$

for some constants a, b and all positive r.

Theorem D (L. Rubel – B. Taylor, [15, p. 75]). *Each function from Λ can be represented as a quotient of entire functions in Λ if and only if for every $\mathcal{Z}$ in S_λ there exists an entire function $f \in \Lambda$, with zero sequence $\mathcal{Z}(f)$ such that $\mathcal{Z}(f) \supset \mathcal{Z}$.*

Theorem E (J. Miles – L. Rubel – B. Taylor, [15, p. 78], [16]). *Every meromorphic function in Λ can be represented as a quotient of entire functions in Λ.*

Some extention of such representation was given by B. Khabibullin [11], [12].

In Theorem E the entire functions have, in general, common zeroes. A. Gol'dberg has considered the possibility of representation of an arbitrary meromorphic function in $\mathbb{C}$ by a quotient of entire functions without common zeroes, the restriction on growth of which is given by $T(r, f)$.

Theorem F (A. Gol'dberg [5]). *Every meromorphic function in $\mathbb{C}$ can be represented as the quotient $f = g/h$ of entire functions g and h without common zeroes such that*

$$\log T(r, h) + \log T(r, g) = o(T(r, f))$$

as $r \to +\infty$ outside a set of finite logarithmic measure.

In this paper we consider the representation problem for meromorphic functions from Λ as quotients of entire functions from Λ without common zeroes, the so-called *canonical quotient representations.*

Definition 2. *A class Λ admits a canonical quotient representation (CQR) if each function $f \in \Lambda$ may be represented as a quotient $f = g/h$ of entire functions $g, h \in \Lambda$ without common zeroes.*

We are going to show that the problem of CQR-admissibility is tightly connected with the properties of growth majorants for entire functions with restriction on the quantities of their zeroes.

The growth majorants for entire functions play an important role in various questions of complex analysis (see, for example, [6]–[7]). Many authors have studied the problem of choosing an entire function f with a given zero set $\mathcal{Z}$ such that there is the least in some sense majorant for $M(r, f)$,

$$M(r, f) = \max\{|f(z)| : |z| \le r\},$$

in terms of $n(r, \mathcal{Z})$ (see [1] for references).

We mention here one of the recent results. In connection with Gol'dberg's theorem F, Bergweiler has shown in [2] that for an arbitrary sequence $\mathcal{Z}$ satisfying the condition

$$\lim_{r\to+\infty} \inf \frac{\log n(r, \mathcal{Z})}{\log r} > 0$$

there exists an entire function f whose zero sequence is $\mathcal{Z}$ and such that

$$\log\log M(r, f) = o((\log n(r, \mathcal{Z}))^2)\varphi(\log n(r, \mathcal{Z}))$$

as $r \to +\infty$ outside a set of finite logarithmic measure. Here φ is a positive, non-decreasing function satisfying

$$\int_1^{+\infty} \frac{dt}{\varphi(t)t\log t} < \infty.$$

This result is best possible in some sense [2].

Further, we suppose, without loss of generality, that all growth functions λ are linear near $r = 0$, $\lambda(+0) = 0$.

Let $\mathcal{E}_\lambda$ be the set of entire function f, $f(0) = 1$, whose zero sequences $\mathcal{Z}(f)$, taking into account the multiplicities, belong to S_λ.

Definition 3. *A growth function $\widetilde{\lambda}$ is called a growth majorant for $\mathcal{E}_\lambda$ if for an arbitrary sequence $\mathcal{Z}$ from S_λ there exists an entire function f of finite $\widetilde{\lambda}$-type such that $\mathcal{Z}(f) = \mathcal{Z}$.*

The existence of a growth majorant for $\mathcal{E}_\lambda$ for arbitrary λ follows from Theorem 4 below (see also [10],[17], [21]).

Definition 4. *A growth function* $\widehat{\lambda}$ *is called the minimal growth majorant for* $\mathcal{E}_\lambda$ *if it is a growth majorant and if for each growth majorant* $\widetilde{\lambda}$ *for* $\mathcal{E}_\lambda$ *there exist constants* a, b *such that*

$$\widehat{\lambda}(r) \leq a\widetilde{\lambda}(br)$$

for all positive r.

Definition 5. *Two growth functions* λ_1 *and* λ_2 *are said to be equivalent if*

$$\lambda_1(r) \leq a\lambda_2(br) \qquad \text{and} \qquad \lambda_2(r) \leq c\lambda_1(dr)$$

for some constants a, b, c, d *and all positive* r.

Hence, the minimal growth majorant is determined up to equivalence. In this sense it is unique if it exists.

A growth function $\lambda(r)$ is said to be *convex with respect to* $\log r$ if the function $\lambda(e^x)$ is convex on $\mathbb{R}$.

The main theorem of this paper is the following.

Theorem 1. *Let a growth function* λ *be convex with respect to* $\log r$ *and let* Λ *be the class of meromorphic functions of finite* λ*-type. Then* Λ *admits CQR if and only if* λ *is the minimal growth majorant for* $\mathcal{E}_\lambda$.

We shall also prove the following propositions.

Theorem 2. *Let a growth function* λ *of finite order be convex with respect to* $\log r$. *Then the function* $\widetilde{\lambda}$ *defined by*

$$\widehat{\lambda}(r) = \begin{cases} (q-1)r^{q-1}\displaystyle\int_1^r \frac{\lambda(t)}{t^q}\,dt + r^q \int_r^{+\infty} \frac{\lambda(t)}{t^{q+1}}\,dt, & \text{if } r \geq 1, \\ C\lambda(r) & \text{if } 0 < r < 1, \end{cases} \tag{1}$$

where q *is the least integer satisfying the condition*

$$\int_1^{+\infty} \frac{\lambda(t)}{t^{q+1}}\,dt < +\infty, \tag{2}$$

and C *is the constant providing the continuity of* $\widehat{\lambda}$ *at* $r = 1$, *is the minimal growth majorant for* $\mathcal{E}_\lambda$.

Corollary 1. *Let* λ *satisfy the conditions of Theorem 2 and let* $\widehat{\lambda}$ *be given by* (1). *Then* Λ *admits CQR if and only if* $\widehat{\lambda}(r) \leq a\lambda(br)$ *for some constants* a, b *and all positive* r.

Theorem 3. *Let* λ *be a growth function and let*

i) *the function* $\log \lambda(r)$ *be convex with respect to* $\log r$,
ii) *the integral* $\int_1^{+\infty} \lambda(t)/(t\lambda(ct))\,dt$ *converges for some* $c > 1$.

Then λ *has infinite order and is the minimal growth majorant for* $\mathcal{E}_\lambda$.

Corollary 2. *Let* λ *satisfy the conditions of Theorem 3. Then* Λ *admits CQR.*

2. Proof of Theorem 1

Lemma. *If a growth function* λ *is convex with respect to* $\log r$ *and* $\widetilde{\lambda}$ *is a growth majorant for* $\mathcal{E}_\lambda$*, then*

1) $\lambda(r) \le a\widetilde{\lambda}(br)$ *for some constants* a*,* b *and all positive* r*;*
2) *each function* f *from* Λ *is representable as a quotient* $f = g/h$ *of entire functions* g *and* h *of finite* $\widetilde{\lambda}$*-type without common zeroes.*

Proof. Since λ is convex with respect to $\log r$ we may choose a sequence $\mathcal{Z}$ such that $\lambda(r) \le N(r, \mathcal{Z}) \le A\lambda(r)$, where A is a constant and $r > 0$, taking $n(r, \mathcal{Z}) = [r\lambda'(r)] + 1$ almost everywhere. So, $\mathcal{Z} \in S_\lambda$. Since $\widetilde{\lambda}$ is a growth majorant for $\mathcal{E}_\lambda$, there exists a function f from $\mathcal{E}_\lambda$ with $\mathcal{Z}(f) = \mathcal{Z}$ such that $T(r, f) \le a\widetilde{\lambda}(br)$ for some constants a, b and all positive r. Consequently,

$$\lambda(r) \le \mathcal{N}(r, \mathcal{Z}) \le T(r, f) \le a\widetilde{\lambda}(br)$$

for all positive r, and we obtain statement 1) of the lemma.

Now let $f \in \Lambda$, and let $W(f)$ be the pole sequence of f. We have

$$\mathcal{N}(r, W(f)) \le T(r, f) \le a\lambda(br)$$

for some constants a, b and all positive r, i.e. $W(f) \in S_\lambda$. Since $\widetilde{\lambda}$ is a growth majorant for $\mathcal{E}_\lambda$, there exists an entire function h of finite $\widetilde{\lambda}$-type with zero sequence $\mathcal{Z}(h) = W(f)$. The function $g = fh$ is entire and of finite $\widetilde{\lambda}$-type according to 1) and the elementary properties of Nevanlinna characteristic. This proves part 2) of the lemma. ∎

Proof of Theorem 1. Let λ be the minimal growth majorant for $\mathcal{E}_\lambda$. According to statement 2) of the lemma above, Λ admits CQR.

Conversely, suppose that Λ admits CQR. By the theorem of L. A. Rubel and B. A. Taylor ([15, p. 75], [13, p. 35]), for arbitrary sequence $\mathcal{Z}$ in S_λ there exists a function $f \in \Lambda$ the zero sequence $\mathcal{Z}(f)$ of which coincides with $\mathcal{Z}$. The CQR admissibility gives $f = g/h$, where g and h are entire functions, $\mathcal{Z} = \mathcal{Z}(f) = \mathcal{Z}(g)$ and $g \in \Lambda$. So, λ is a growth majorant for $\mathcal{E}_\lambda$. According to statement 1) of the last lemma λ is the minimal growth majorant for $\mathcal{E}_\lambda$. This completes the proof of Theorem 1. ∎

3. Minimal growth majorants of finite order

Proof of Theorem 2. Let λ satisfy the conditions of Theorem 2. Since λ has finite order of growth, there exists a natural integer q satisfying condition (2). We suppose that q is the least such integer. Let $\mathcal{Z} \in S_\lambda$. Condition (2) provides the convergence of the canonical Weierstrass product of genus $q-1$ in $\mathbb{C}$ to an entire function f. The well-known estimate for f ([6, p. 78], [8, p. 163]) is

$$T(r,f) \le C_q \left\{ (q-1)r^{q-1}\int_1^r \frac{\mathcal{N}(t,\mathcal{Z})}{t^q}\,dt + r^q \int_r^{+\infty} \frac{\mathcal{N}(t,\mathcal{Z})}{t^{q+1}}\,dt \right\}, \qquad r \ge 1,$$

where C_q is constant. Since $\mathcal{Z} \in S_\lambda$ we have $\mathcal{N}(t,\mathcal{Z}) \le a\lambda(bt)$ for some constants a, b with $b \ge 1$, and all positive $r > 0$. Therefore,

$$\begin{aligned} T(r,f) &\le aC_q \left\{ (q-1)r^{q-1}\int_1^r \frac{\lambda(bt)}{t^q}\,dt + r^q \int_r^{+\infty} \frac{\lambda(bt)}{t^{q+1}}\,dt \right\} \\ &= aC_q \left\{ (q-1)(rb)^{q-1}\int_b^{br} \frac{\lambda(t)}{t^q}\,dt + (rb)^q \int_{br}^{+\infty} \frac{\lambda(t)}{t^{q+1}}\,dt \right\} \\ &\le aC_q\widehat{\lambda}(br), \qquad r \ge 1, \end{aligned}$$

where $\widehat{\lambda}$ is given by (1).

Since $f(0) = 1$ and since $\widehat{\lambda}$ is linear near $r = 0$, we get the desired estimate for $T(r,f)$ on $(0,1)$. Therefore $\widehat{\lambda}$ is a growth majorant for $\mathcal{E}_\lambda$.

We shall show that $\widehat{\lambda}$ is minimal. For this purpose consider a sequence $\mathcal{Z}$ of positive numbers such that $\lambda(r) \le \mathcal{N}(r,\mathcal{Z}) \le A\lambda(r)$, where A is a constant and $r > 0$. Since λ is convex with respect to $\log r$, such a sequence $\mathcal{Z}$ exists (see the proof of Theorem 1). We have $\mathcal{Z} \in S_\lambda$. For an arbitrary entire function f such that $f(0) = 1$, with $\mathcal{Z}(f) = \mathcal{Z} = \{a_j\}$ the function $\log f(z)$ with $\log f(0) = 0$ is defined outside the ray $z = ta_1$, $t \ge 1$. The following relations hold [9].

$$l_0(r,f) = \mathcal{N}(r,\mathcal{Z}), \tag{3}$$

$$l_k(r,f) = \gamma_k r^k + r^k \int_0^r \frac{n(t,\mathcal{Z})}{t^{k+1}}\,dt, \qquad k \in \mathbb{N}, \tag{4}$$

where $l_k(r,f)$, $r > 0$, are the coefficients of the Fourier developement of $\log f(re^{i\theta})$. Since the integral in (2) converges and since

$$n(t,\mathcal{Z}) \le 2\mathcal{N}(2t,\mathcal{Z}) \le 2A\lambda(2t),$$

the integral

$$\int_0^{+\infty} \frac{n(t,\mathcal{Z})}{t^{q+1}}\,dt$$

also converges. Hence, $\lim_{r\to+\infty} l_q(r,f)/r^q$ exists. We denote this limit by α_q. If $\alpha_q = 0$ we have

$$\gamma_q = -\int_0^{+\infty} \frac{n(t,\mathcal{Z})}{t^{q+1}}\,dt$$

and

$$|l_q(r,f)| = r^q \int_r^{+\infty} \frac{n(t,\mathcal{Z})}{t^{q+1}}\,dt = -\mathcal{N}(r,\mathcal{Z}) + qr^q \int_r^{+\infty} \frac{\mathcal{N}(t,\mathcal{Z})}{t^{q+1}}\,dt. \tag{5}$$

If $\alpha_q \neq 0$ we have

$$|l_q(r,f)| \geq |\alpha_q| r^q - r^q \int_r^{+\infty} \frac{n(t,\mathcal{Z})}{t^{q+1}}\,dt.$$

Since the last integral tends to zero as $r \to +\infty$, and $\alpha_q \neq 0$, there exists r_0 such that

$$(6)\quad |l_q(r,f)| \geq r^q \int_r^{+\infty} \frac{n(t,\mathcal{Z})}{t^{q+1}}\,dt = qr^q \int_r^{+\infty} \frac{\mathcal{N}(t,\mathcal{Z})}{t^{q+1}}\,dt - \mathcal{N}(r,\mathcal{Z}), \qquad r > r_0.$$

For $q > 1$ we obtain, taking into account the inequality

$$\mathcal{N}(r,\mathcal{Z}) \geq \lambda(r), \qquad r \geq 1,$$

and the definition of q,

$$(7)\quad \begin{aligned} l_{q-1}(r,f) &= \gamma_{q-1} r^{q-1} + \mathcal{N}(r,\mathcal{Z}) + (q-1) r^{q-1} \int_0^r \frac{\mathcal{N}(t,\mathcal{Z})}{t^q}\,dt \\ &= \mathcal{N}(r,\mathcal{Z}) + (q-1)(1+o(1)) r^{q-1} \int_0^r \frac{\mathcal{N}(t,\mathcal{Z})}{t^q}\,dt, \end{aligned}$$

as $r \to +\infty$.

Using (5)–(7) we may choose r_0 such that

$$(8)\qquad |l_{q-1}(r,f)| + |l_q(r,f)| \geq \frac{\widehat{\lambda}(r)}{2}, \qquad r > r_0.$$

By the aid of (3) and (5) we obtain (8) for $q = 1$.

Now if $\widetilde{\lambda}$ is a growth majorant, then there exists an entire function f with $\mathcal{Z}(f) = \mathcal{Z}$ such that $T(r,f) \leq A\widetilde{\lambda}(Br)$ for some constants A, B and all positive r. For the Fourier coefficients of its logarithm a similar inequality holds [4]:

$$(9)\qquad |l_{q-1}(r,f)| + |l_q(r,f)| \leq a\widetilde{\lambda}(br), \qquad r > 0,$$

where a, b are constants.

On the other hand, for these coefficients inequality (8) holds. Hence,

$$\widehat{\lambda}(r) \leq 2a\widetilde{\lambda}(br), \qquad r > r_0.$$

Since $\widehat{\lambda}$ and $\widetilde{\lambda}$ are linear near $r = 0$ and continuous on $(0, r_0]$ we obtain a similar inequality on $(0, r_0]$. Therefore, $\widehat{\lambda}$ is the minimal growth majorant for $\mathcal{E}_\lambda$.

This completes the proof of Theorem 2. ∎

Proof of Corollary 1. This corollary follows immediately from Theorem 1 and Theorem 2. ∎

Corollary 3. *If λ is a growth function of slow growth, i.e. there exists a constant M such that $\lambda(2r) \leq M\lambda(r)$ for all positive r, then the minimal growth majorant $\widehat{\lambda}$ for $\mathcal{E}_\lambda$ is also of slow growth.*

Proof. It is easy to see that the slow growth implies order finitness. Hence, according to Theorem 2, $\widehat{\lambda}$ is given by (1). Consider $\widehat{\lambda}(2r)$. Changing the variables in the integrals of (1) we obtain $\widehat{\lambda}(2r) \le M_q\widehat{\lambda}(r)$ for $r \ge 1$, where M_q is a constant. By the aid of (1) such an inequality is also valid for $r > 0$. ■

4. Minimal growth majorants of infinite order

Proof of Theorem 3. We may assume that $\lambda(t)$ is smooth. Denote $\tau(t) = \log\lambda(t)$. Then $t\tau'(t)$ is non-decreasing, since τ is convex with respect to $\log t$. According to condition ii)

$$\lim_{t\to+\infty} t\tau'(t) \to +\infty.$$

Indeed, if $t\tau'(t)$ were bounded by a constant C we would obtain

$$\log\frac{\lambda(ct)}{\lambda(t)} = (c-1)t_c\tau'(t_c) \le C(c-1), \qquad t \le t_c \le ct.$$

But this contradicts ii). Consequently, the function λ must have infinite order. We may assume without loss of generality $t\tau'(t)$ increasing for large t.

Since we assume that λ is linear near $r = 0$ and well defined up to equivalence, we may also assume

$$ct\tau'(ct) = \frac{1}{2}, \qquad 0 < t \le 1.$$

Since $t\tau'(t)$ increases to $+\infty$ as $t \to +\infty$, for every $k \in \mathbb{N}$ there exists an unique point $r_k > 1$ such that

$$cr_k\tau'(cr_k) = k.$$

The point r_k is the unique point where the function $\tau(ct) - k\log t$ attains its minimum on $(0, +\infty)$.

Let $\{a_j\} = \mathcal{Z} \in S_\lambda$. By Definition 5 and the inequality $n(r, \mathcal{Z}) \le 2\mathcal{N}(2r, \mathcal{Z})$ we may assume $|a_1| \ge 1$ and $n(r, \mathcal{Z}) \le \lambda(r)$ for $r > 0$. In order to construct an entire function f of finite λ-type with $\mathcal{Z}(f) = \mathcal{Z}$ we put

$$\gamma_k = -\frac{1}{k}\sum_{|a_j|\le r_k}\left(\frac{1}{a_j}\right)^k, \qquad k \in \mathbb{N},$$

and

$$g(z) = \sum_{k\in\mathbb{N}} \gamma_k z^k. \tag{10}$$

This series converges in the unit disc. Indeed, since the function $\tau(ct) - k\log t$ decreases on $(1, r_k)$ we obtain

$$\begin{aligned}
|\gamma_k| &\leq \frac{1}{k}\sum_{|a_j|\leq r_k}\frac{1}{|a_j|^k} = \frac{1}{k}\int_1^{r_k}\frac{dn(t,\mathcal{Z})}{t^k} = \frac{n(r_k,\mathcal{Z})}{k{r_k}^k} + \int_1^{r_k}\frac{n(t,\mathcal{Z})}{t^{k+1}}\,dt \\
&\leq \frac{\lambda(r_k)}{k{r_k}^k} + \int_1^{r_k} e^{\tau(c\tau)-k\log t}\frac{\lambda(t)}{t\lambda(ct)}\,dt \leq \frac{\lambda(r_k)}{k{r_k}^k} + \lambda(c)\int_1^{+\infty}\frac{\lambda(t)}{t\lambda(ct)}\,dt \\
&\leq \lambda(1) + \lambda(c)\int_1^{+\infty}\frac{\lambda(t)}{t\lambda(ct)}\,dt.
\end{aligned}$$

Therefore, the sequence $\{\gamma_k\}$ is bounded, and the series (10) converges in the unit disc.

Let

$$g_R(z) = \sum_{k\in\mathbb{N}} c_k(R)z^k,$$

where

$$c_k(R) = \gamma_k + \frac{1}{k}\sum_{|a_j|<R}\left(\frac{1}{a_j}\right)^k, \qquad R \geq 1. \tag{11}$$

We shall show that $g_R(z)$ is holomorphic in the disk $\mathbb{D}_R = \{z : |z| < R\}$.

The representation

$$c_k(R) = \begin{cases} -\dfrac{1}{k}\displaystyle\sum_{R\leq|a_j|\leq r_k}\left(\frac{1}{a_j}\right)^k & \text{for } R \leq r_k, \\ \dfrac{1}{k}\displaystyle\sum_{r_k<|a_j|<R}\left(\frac{1}{a_j}\right)^k, & \text{for } R > r_k, \end{cases}$$

implies, having in mind $\int_b^a = -\int_a^b$,

$$\begin{aligned}
|c_k(R)| &\leq \frac{1}{k}\left|\int_R^{r_k}\frac{dn(t,\mathcal{Z})}{t^k}\right| \leq \frac{n(r_k,\mathcal{Z})}{k{r_k}^k} + \frac{n(R,\mathcal{Z})}{kR^k} + \left|\int_R^{r_k}\frac{n(t,\mathcal{Z})}{t^{k+1}}\,dt\right| \\
&\leq \frac{2\lambda(cR)}{R^k} + \left|\int_R^{r_k} e^{\tau(ct)-k\log t}\frac{\lambda(t)}{t\lambda(ct)}\,dt\right| \\
&\leq \frac{\lambda(cR)}{R^k}\left(2 + \int_1^{+\infty}\frac{\lambda(t)}{t\lambda(ct)}\,dt\right)
\end{aligned}$$

for $k \in \mathbb{N}$ and $R \geq 1$.

Hence, the function g_R is holomorphic in $\mathbb{D}_R$ and g_1 coincides with g in $\mathbb{D}_1$ according to (10) and (11). Define

$$f_R(z) = \exp(g_R(z)) \cdot \prod_{|a_j|<R}\left(1 - \frac{z}{a_j}\right).$$

In $\mathbb{D}_1$ the function f_R coincides with the function $f = \exp(g)$. In fact, the functions $\log f$ with $\log f(0) = 0$, and $\log f_R$ with $\log f_R(0) = 0$ have the same Taylor expansions in $\mathbb{D}_1$, since

$$\gamma_k = \gamma_k + \frac{1}{k} \sum_{|a_j|<R} \left(\frac{1}{a_j}\right)^k - \frac{1}{k} \sum_{|a_j|<R} \left(\frac{1}{a_j}\right)^k.$$

Hence, f admits an analytic continuation in $\mathbb{C}$ and this continuation in $\mathbb{D}_R$ is given by $f_R(z)$.

We shall show that $f \in \Lambda$. Indeed, for $R = 2r$ and $|z| = r$ we have

$$\log|f(z)| = \operatorname{Re} g_{2r}(z) + \sum_{|a_j|<2r} \log\left|1 - \frac{z}{a_j}\right|.$$

Let $a^+ = \max(a, 0)$ and $|z| = r$. A routine computation gives

$$\begin{aligned}
\log^+|f(z)| &\le |g_{2r}(z)| + \sum_{|a_j|<2r} \log^+\left|1 - \frac{z}{a_j}\right| \\
&\le A_1\lambda(2cr) \sum_{k\in\mathbb{N}} \frac{1}{2^k} + \sum_{|a_j|<2r} \left(\log^+ \frac{r}{|a_j|} + \log 2\right) \\
&= A_1\lambda(2cr) + \mathcal{N}(2r, \mathcal{Z}) + (\log 2) n(2r, \mathcal{Z}) \\
&\le A_2\lambda(2cr),
\end{aligned}$$

where A_1 and A_2 are constants. From this, it follows that $T(r, f) \le A_2\lambda(2cr)$. Therefore, λ is a growth majorant for $\mathcal{E}_\lambda$.

Let now $\widetilde{\lambda}$ be a growth majorant. In order to obtain the inequality $\lambda(r) \le a\widetilde{\lambda}(br)$ with some constants a, b and all positive r, note first that $\lambda(r)$ is convex with respect to $\log r$. In fact, $r\lambda'(r) = \lambda(r) r\tau'(r)$, and the function at the right side of this equality increases. Using part 1) of the lemma, we obtain the required inequality. Hence, λ is the minimal growth majorant for $\mathcal{E}_\lambda$. This completes the proof. ■

Proof of Corollary 2. Since $\lambda(r)$ is convex with respect to $\log r$ under the conditions of Theorem 3, we obtain Corollary 2 from Theorem 1 and 3. ■

Theorem 4. *Let λ be a growth function such that S_λ is not empty. Then the function λ_0 defined by*

$$\lambda_0(r) = \begin{cases} \exp\left(\int_1^r \frac{\log^+ \lambda(t)}{t}\, dt\right) & \textit{if } r \ge 1, \\ r & \textit{if } 0 < r < 1, \end{cases}$$

has infinite order and is a growth majorant for $\mathcal{E}_\lambda$.

If, moreover,

$$\int_0^r \frac{\log^+ \lambda(t)}{t}\,dt \le \log^+ \lambda(br), \qquad r > 0, \tag{12}$$

with some constant b, then λ has infinite order and is the minimal growth majorant for $\mathcal{E}_\lambda$.

Proof. The function $\log \lambda_0(r)$ is, evidently, convex with respect to $\log r$. It satisfies condition *ii*) of Theorem 3. In fact,

$$\log \frac{\lambda_0(3r)}{\lambda_0(r)} \ge \int_r^{3r} \frac{\log^+ \lambda(t)}{t}\,dt \ge (\log 3) \log^+ \lambda(r), \qquad r > 1. \tag{13}$$

Since S_λ is not empty, we have $\lambda(r) \ge \alpha \log r$ for $r > r_0 > 0$, where $\alpha > 0$ is a constant. Hence, according to (13)

$$\frac{\lambda_0(r)}{\lambda_0(3r)} \le (\lambda(r))^{-\log 3} \le (\alpha \log r)^{-\log 3}, \qquad r > r_0.$$

This implies condition ii) of Theorem 3. Hence λ_0 has infinite order and is the minimal growth majorant for $\mathcal{E}_{\lambda_0}$. Furthermore, (13) implies

$$\log \frac{\lambda_0(2r)}{\lambda_0(r)} \ge (\log 2) \log^+ \lambda(r), \qquad r > 1.$$

Hence, $\lambda(r) \le \lambda_0(2r)$ and $\mathcal{E}_\lambda \subset \mathcal{E}_{\lambda_0}$. Therefore, λ_0 is a growth majorant for $\mathcal{E}_\lambda$.

If, in addition, λ satisfies condition (12), then $\lambda_0(r) \le \lambda(br)$ for $r > 0$. In this case, λ_0 and λ are equivalent and $\mathcal{E}_\lambda = \mathcal{E}_{\lambda_0}$. Since λ_0 is the minimal growth majorant for $\mathcal{E}_{\lambda_0}$, λ is the minimal growth majorant for $\mathcal{E}_\lambda$. ■

The next proposition follows immediately from Theorems 1 and 4.

Corollary 4. *If a growth function λ satisfies the conditions of Theorem 4, including* (12)*, then Λ admits CQR.*

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