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Bound-Preserving Operators and Bernstein Type Inequalities

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Abstract. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and

$$\|p\| := \max_{z\in\partial\mathbb{D}} |p(z)|,$$

where $p(z) = \sum_{k=0}^{n} a_k(p)z^k$ is a polynomial of degree at most n and $a_k(p) \in \mathbb{C}$. The following sharpening of Bernstein's inequality

$$\|p'\| + \frac{2n}{n+2}|a_0(p)| \leq n\|p\|$$

has been proved by Ruscheweyh. Our main contribution concerns the case of equality which has remained unsolved since 1982. We prove another inequality of Bernstein type that leads to an improvement of the upper bound for $\|p'\|$ under some additional condition.

Keywords. Polynomials, bound-preserving operators, Bernstein type inequalities.

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1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ the set of functions analytic on $\mathbb{D}$. Let also $\mathcal{P}_n$ be the vector space of polynomials $p(z) = \sum_{k=0}^{n} a_k(p)z^k$ of degree at most n with coefficients $a_k(p) \in \mathbb{C}$. We define for a function $f \in \mathcal{H}(\mathbb{D})$,

$$\|f\| := \sup_{z\in\mathbb{D}} |f(z)|.$$

According to the celebrated inequality of Bernstein (for references see [8] and the new book by Rahman and Schmeisser [9])

$$\|p'\| \leq n\|p\|, \qquad p \in \mathcal{P}_n,\ n \geq 1, \tag{1}$$

and equality holds in (1) if and only if $p(z)$ is a constant multiple of z^n.

In this paper, we prove the following sharpening of (1).

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Theorem 1. *Let*

$$d_n := \sup\left\{d \geq 0 : \, \|p'\| + d|a_0(p)| \leq n\|p\| \text{ for all } p \in \mathcal{P}_n\right\}.$$

Then $d_1 = 1$ and $d_n = 2n/(n+2)$ if $n \geq 2$. The equality

$$\|p'\| + d_n|a_0(p)| = n\|p\|$$

holds for all $p \in \mathcal{P}_n$ if n=1 and only for $p(z) = a_n(p)z^n$ if $n \geq 2$.

Theorem 2. *Let*

$$\delta_n := \sup\left\{\delta \geq 0 : \, \delta\|p'\| + |a_0(p)| \leq \|p\| \text{ for all } p \in \mathcal{P}_n\right\}.$$

Then $\delta_1 = 1$, $\delta_2 = \left(1+\sqrt{3}\right)^{-1}$, $\delta_3 = \left(2+\sqrt{10}\right)^{-1}$ and in general

$$\frac{2}{n(n+1)} \leq \delta_n \leq \frac{3}{n(n+2)}.$$

The equality

$$\delta_n\|p'\| + |a_0(p)| = \|p\|$$

holds for all $p \in \mathcal{P}_n$ if $n = 1$ and only for constant polynomials if $n \geq 2$.

An obvious consequence of Theorem 2 is the next corollary.

Corollary 1. *Let $p \in \mathcal{P}_n$. Then*

$$\frac{2}{n(n+1)}\|p'\| + |a_0(p)| \leq \|p\|$$

and equality holds for all $p \in \mathcal{P}_n$ if $n = 1$ and only for constant polynomials if $n \geq 2$.

The inequality

$$\|p'\| + d_n|a_0(p)| \leq n\|p\|, \qquad p \in \mathcal{P}_n \tag{2}$$

is not new and has been proved and published in 1982 by Ruscheweyh [10]. Later, several extensions of (2) were obtained by Frappier, Rahman and Ruscheweyh [4]. Our contribution concerns the case of equality which has remained unsolved since 1982 (see [10, pp. 125–126]).

The inequality

$$\delta_n\|p'\| + |a_0(p)| \leq \|p\|, \qquad p \in \mathcal{P}_n, \tag{3}$$

is a Bernstein type inequality but it also can be compared with

$$\frac{1}{2}\sec\left(\frac{\pi}{n+2}\right)|a_1(p)| + |a_0(p)| \leq \|p\|, \qquad p \in \mathcal{P}_n,$$

which has been obtained [2] recently as an extension of an inequality due to Visser [13]. Moreover, it has been proved in [2] that

$$\frac{1}{2}\sec\left(\frac{\pi}{n+2}\right) = \max\left\{\gamma \in \mathbb{R} : \, \gamma|a_1(p)| + |a_0(p)| \leq \|p\| \text{ for all } p \in \mathcal{P}_n\right\}.$$

Remark that (3) leads to an improvement of the upper bound for $\|p'\|$ obtained from (2) as soon as

$$\|p'\| \le \frac{1}{\delta_n}\left(\|p\| - |a_0(p)|\right) < n\|p\| - \frac{2n}{n+2}|a_0(p)|,$$

i.e., as soon as

$$\frac{1-n\delta_n}{1-\frac{2n}{n+2}\delta_n} < \frac{|a_0(p)|}{\|p\|}, \qquad n > 1.$$

2. Some Useful Lemmas

As in [4] (see also [10, Chapter 4]), our proofs depend on a careful study of a class of bound-preserving operators on $\mathcal{P}_n$. We recall that for

$$f(z) = \sum_{n=0}^{\infty} a_n(f) z^n \in \mathcal{H}(\mathbb{D}) \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} a_n(g) z^n \in \mathcal{H}(\mathbb{D})$$

the convolution (or Hadamard product) $f * g$, defined by

$$(f*g)(z) := \sum_{n=0}^{\infty} a_n(f) a_n(g) z^n,$$

also belongs to $\mathcal{H}(\mathbb{D})$. Let for $n \ge 1$

$$\begin{aligned} \mathcal{B}_n &:= \left\{F \in \mathcal{H}(\mathbb{D}) :\ F(0) = 1 \text{ and } \|f * p\| \le \|p\| \text{ for all } p \in \mathcal{P}_n\right\}, \\ \mathcal{B}_\infty &:= \bigcap_{n \ge 1} \mathcal{B}_n \\ &= \left\{F \in \mathcal{H}(\mathbb{D}) :\ F(0) = 1 \text{ and } \|f * f\| \le \|f\| \text{ for all } f \in \mathcal{H}(\mathbb{D})\right\}. \end{aligned}$$

We call $F \in \mathcal{B}_n$ a bound-preserving function over $\mathcal{P}_n$ because the operator

$$p \mapsto p * F, \qquad p \in \mathcal{P}_n,$$

has norm at most one. For similar reasons a function $F \in \mathcal{B}_\infty$ is called a bound-preserving function on $\mathcal{H}(\mathbb{D})$ and the corresponding operator

$$f \mapsto f * F, \qquad f \in \mathcal{H}(\mathbb{D}),$$

is called bound-preserving. The following results, due to Sheil-Small (see [11]), Carathéodory and Toeplitz (see [12, pp. 153–159] and [5, pp. 148–154]), and Szász (see [8, Chapter 16] and [9, Chapter 4]), give a complete description of the classes $\mathcal{B}_n$ and $\mathcal{B}_\infty$.

Lemma 1 (Sheil-Small)**.** *We have $G \in \mathcal{B}_\infty$ if and only if there is some probability measure $d\mu$ on $\partial\mathbb{D}$ such that*

$$G(z) = \int_{\partial\mathbb{D}} \frac{1}{1-\zeta z}\, d\mu(\zeta)$$

We have $F \in \mathcal{B}_n$ *if and only if*

$$F(z) = G(z) + o(z^n) \quad \text{as } z \to 0 \text{ for some } G \in \mathcal{B}_\infty.$$

Lemma 2 (Carathéodory and Toeplitz)**.** *Let* $F(z) = 1 + \sum_{k=1}^{\infty} A_k z^k \in \mathcal{H}(\mathbb{D})$*. If*

$$\operatorname{Det}_n(A_1, \dots, A_n) := \det \begin{pmatrix} 1 & A_1 & \cdots & A_n \\ \bar{A}_1 & 1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_n & \bar{A}_{n-1} & \cdots & 1 \end{pmatrix} > 0, \qquad n = 1, 2, \dots,$$

then $F \in \mathcal{B}_\infty$*.*

Conversely, if $F \in \mathcal{B}_\infty$ *and* $\det_\nu(A_1, \dots, A_\nu) = 0$ *for some* $\nu \geq 1$*, then there exists an integer* $N \geq 1$ *such that*

$$\begin{aligned} \operatorname{Det}_n(A_1, \dots, A_n) &> 0 \qquad \text{if } n = 1, \dots, N-1, \\ \operatorname{Det}_n(A_1, \dots, A_n) &= 0 \qquad \text{if } n \geq N. \end{aligned}$$

In that case

$$F(z) = \sum_{j=1}^{N} \frac{\lambda_j}{1 - \xi_j z},$$

where the coefficients λ_j*,* $j = 1, \dots, N$*, are positive and the* N *complex numbers* $\xi_j \in \partial\mathbb{D}$*,* $j = 1, \dots, N$*, are pairwise distinct.*

Lemma 3 (Szász)**.** *Let* $F(z) = 1 + \sum_{k=1}^{\infty} A_k z^k \in \mathcal{H}(\mathbb{D})$*. Then* $F \in \mathcal{B}_n$ *if and only if the Toeplitz-Hermite form*

$$(z_0, z_1, \dots, z_n) \mapsto \sum_{\mu,\nu=0}^{n} A_{\mu-\nu} z_\mu \bar{z}_\nu, \qquad A_0 := 1,\ A_{-k} := \bar{A}_k,\ k = 1, \dots, n,$$

is non-negative definite.

The next statement shows how Lemma 2 and Lemma 3 can be applied.

Proposition 1. *A necessary and sufficient condition for a Toeplitz-Hermite form to be positive definite is that the principal minors of the corresponding hermitian matrix are all positive.*

In the discussion of equality cases of our results we use a well-known lemma whose proof can be found in [6, Theorem 1]:

Lemma 4. *Let* $f \in \mathcal{H}(\mathbb{D})$ *be non-constant and* $0 < r < 1$*. Let* $\zeta \in \mathbb{D}$ *with* $|\zeta| = r$ *be such that*

$$|f(\zeta)| = \max_{|z|=r} |f(z)|.$$

Then

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \geq \frac{|f(\zeta)| - |f(0)|}{|f(\zeta)| + |f(0)|} > 0.$$

3. Proof of Theorem 1

The result is obvious when $n = 1$, so from now on we shall assume $n \geq 2$. For the sake of completeness, we include a variant of the elegant proof first published in [10]. Let $p \in \mathcal{P}_n$ and $d \geq 0$ and $\tilde{p}(z) := z^n p(1/z)$. By using the fact that $\|p\| = \|\tilde{p}\|$ we obtain

$$\begin{aligned}
(4) \qquad & \|p'\| + d|a_0(p)| \leq n\|p\| \\
\Leftrightarrow \quad & \frac{1}{n}\|p'\| + \frac{d}{n}|a_0(p)| \leq \|p\| \\
\Leftrightarrow \quad & \left|\sum_{j=0}^{n-1} \frac{n-j}{n} a_{n-j}(p) z^j + \frac{d}{n} a_0(p) e^{i\theta} z^n\right| \leq \|p\| \quad \text{for all } z \in \mathbb{D},\ \theta \in \mathbb{R} \\
\Leftrightarrow \quad & \left|\sum_{j=0}^{n} a_{n-j}(p) z^j * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d}{n} e^{i\theta} z^n\right)\right| \leq \|p\| \quad \text{for all } z \in \mathbb{D},\ \theta \in \mathbb{R} \\
\Leftrightarrow \quad & \left|\tilde{p}(z) * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d}{n} e^{i\theta} z^n\right)\right| \leq \|\tilde{p}\| \quad \text{for all } z \in \mathbb{D},\ \theta \in \mathbb{R}.
\end{aligned}$$

The mapping $p \mapsto \tilde{p}$ is an isomorphism of $\mathcal{P}_n$ and we have

$$d_n = \sup\left\{ d \geq 0 : \sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d}{n} e^{i\theta} z^n \in \mathcal{B}_n \text{ for all } \theta \in \mathbb{R} \right\}.$$

Let

$$P_n(z) := \sum_{j=0}^{n-1} \frac{n-j}{n} z^j.$$

According to a result of Fejér [3]

$$\operatorname{Re} P_n(z) > \frac{1}{2}, \qquad z \in \mathbb{D},$$

and by Lemma 1, $P_n \in B_\infty$. By Lemma 2 (note that P_n is a polynomial)

$$\operatorname{Det}_k(P_n) := \operatorname{Det}_k\left(\frac{n-1}{n}, \ldots, \frac{n-k}{n}\right) > 0, \qquad k = 1, 2, \ldots, n-1,$$

and given

$$F_{d,\theta}(z) := P_n(z) + \frac{de^{i\theta}}{n} z^n, \qquad \theta \in \mathbb{R},$$

the following holds

$$F_{d,\theta} \in \mathcal{B}_n \quad \Rightarrow \quad \operatorname{Det}_n(d,\theta) := \operatorname{Det}_n\left(\frac{n-1}{n}, \ldots, \frac{1}{n}, \frac{de^{i\theta}}{n}\right) \geq 0.$$

On the other hand, by Lemma 3 and Proposition 1 we have

$$\operatorname{Det}_n(d,\theta) > 0 \quad \Rightarrow \quad F_{d,\theta} \in \mathcal{B}_n.$$

The determinant $\mathrm{Det}_n(d,\theta)$ is a quadratic form with respect to the parameter d. Let

$$\mathrm{Det}_n(d,\theta) := c_2 d^2 + c_1 d + c_0.$$

It is easily seen that

$$c_0 = \mathrm{Det}_n(0,\theta) = \frac{1}{n^{n+1}} \det \begin{pmatrix} n & n-1 & n-2 & \cdots & 1 & 0 \\ n-1 & n & n-1 & \cdots & 2 & 1 \\ n-2 & n-1 & n & \cdots & 3 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n & n-1 \\ 0 & 1 & 2 & \cdots & n-1 & n \end{pmatrix}.$$

In order to calculate $\mathrm{Det}_n(0,\theta)$ we perform a sequence of elementary operations:

i) Add -1 times row $j+1$ to row j for $j = 1, 2, \ldots, n$.
ii) Add the first column to the last column to make all entries in the last column zero, except the last one which is n.
iii) Develop the determinant obtained with respect to its last column to obtain $c_0 = 2^{n-1}/n^n$.

Analogously, we have

$$\begin{aligned} c_1 &= \left.\frac{\partial\,\mathrm{Det}_n(d,\theta)}{\partial d}\right|_{d=0} \\ &= \frac{(-1)^{n+2} 2\cos\theta}{n^{n+1}} \det \begin{pmatrix} n-1 & n-2 & \cdots & 1 & 0 \\ n & n-1 & \cdots & 2 & 1 \\ n-1 & n & \cdots & 3 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 4 & \cdots & n-1 & n-2 \\ 2 & 3 & \cdots & n & n-1 \end{pmatrix}. \end{aligned}$$

Now, add -1 times row $j+1$ to row j in the above determinant for $j = 1, 2, \ldots, n-1$ and then, add -1 times row $j+1$ to row j for $j = 1, 2, \ldots, n-2$ to obtain $c_1 = (2^{n-1}\cos\theta)/n^{n+1}$. In a similar way we calculate

$$2c_2 = \frac{\partial^2\,\mathrm{Det}_n(d,\theta)}{\partial d^2} = \frac{-2}{n^{n+1}} \det \begin{pmatrix} n & n-1 & \cdots & 2 \\ n-1 & n & \cdots & 3 \\ n-2 & n-1 & \cdots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ 3 & 4 & \cdots & n-1 \\ 2 & 3 & \cdots & n \end{pmatrix}.$$

This implies $c_2 = -2^{n-3}(n+2)/n^{n+1}$ and

$$\mathrm{Det}_n(d,\theta) = \frac{2^{n-1}}{n^n}\left(1 + \frac{\cos\theta}{n} d - \frac{n+2}{4n} d^2\right) \geq \frac{2^{n-1}}{n^n}\left(1 - \frac{d}{n} - \frac{n+2}{4n} d^2\right).$$

Therefore,

$$d_n = \frac{2n}{n+2}$$

and, in particular,

$$\mathrm{Det}_n(d_n, \theta) = 0 \quad \Leftrightarrow \quad \theta = \pi \mod 2\pi.$$

The equality case in (2)*:* Let us assume that for some polynomial $p \in \mathcal{P}_n$

$$\|p'\| + d_n|a_0(p)| = n\|p\|. \tag{5}$$

According to (4), we have

$$|(\tilde{p} * F_{d_n,\theta})(z)| \leq \|\tilde{p}\|, \qquad \theta \in \mathbb{R},$$

and there exist *an extremal direction* (a real number) θ_n and *an extremal point* $z_n \in \partial\mathbb{D}$ such that

$$|(\tilde{p} * F_{d_n,\theta_n})(z_n)| = \left|\tilde{p}(z) * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d_n}{n} e^{i\theta_n} z^n\right)\right|\Bigg|_{z=z_n} = \|\tilde{p}\|.$$

In view of Lemma 3 and Proposition 1

$$\mathrm{Det}_n(d_n, \theta_n) \geq 0.$$

Case 1. Let us assume that

$$\mathrm{Det}_n(d_n, \theta_n) > 0.$$

For θ_n fixed, $\mathrm{Det}_n(d, \theta_n)$ is a continuous function of d. Therefore, for $\eta > 0$ sufficiently small, $\mathrm{Det}_n(d_n + \eta, \theta_n) > 0$ and according to Lemma 3 and Proposition 1

$$\begin{aligned} \frac{1}{n}\|p'\| + \frac{d_n+\eta}{n}|a_0(p)| &= |(\tilde{p} * F_{d_n+\eta,\theta_n})(z_n)| \\ &= \left|\tilde{p}(z) * \left(\sum_{j=0}^{n-1} \frac{n-j}{n} z^j + \frac{d_n+\eta}{n} e^{i\theta_n} z^n\right)\right|\Bigg|_{z=z_n} \\ &\leq \|\tilde{p}\|. \end{aligned}$$

In view of (5), this is possible only if $a_0(p) = 0$. However, it is well known that the only polynomials $p \in \mathcal{P}_n$ with $a_0(p) = 0$ which are extremal for (2) have the form $p(z) = a_n(p)z^n$.

Case 2. Here we assume

$$\mathrm{Det}_n(d_n, \theta_n) = 0.$$

Then, using the explicit representation for $\mathrm{Det}_n(d_n, \theta_n)$ we have

$$\theta_n = \pi \mod 2\pi.$$

According to Lemma 1 and Lemma 2

$$F_{d_n,\theta_n}(z) = \sum_{j=1}^{n} \frac{\lambda_j}{1 - \zeta_j z} + o(z^n) \qquad \text{as } z \to 0,$$

where $\lambda_j > 0$ and the n complex numbers $\zeta_j \in \partial\mathbb{D}$ are distinct.

This leads to the following system of equations

$$\left\{\begin{array}{rcl} \displaystyle\sum_{j=1}^{n} {\zeta_j}^k \lambda_j & = & \dfrac{n-k}{n} \quad \text{if } k = 0, \dots, n-1, \\ \displaystyle\sum_{j=1}^{n} {\zeta_j}^n \lambda_j & = & -\dfrac{d_n}{n}, \end{array}\right. \tag{6}$$

together with the constraints $\lambda_j > 0$, $j = 1, \dots, n$.

Case 2.1. Suppose that the set $\{\zeta_j\}_{j=1}^n$is not closed under conjugation. The following observation is crucial: the system (6) is equivalent to the system

$$\left\{\begin{array}{rcl} \displaystyle\sum_{j=1}^{n} {\bar{\zeta}_j}^k \lambda_j & = & \dfrac{n-k}{n} \quad \text{if } k = 0, \dots, n-1, \\ \displaystyle\sum_{j=1}^{n} {\bar{\zeta}_j}^n \lambda_j & = & -\dfrac{d_n}{n}. \end{array}\right.$$

Therefore, the polynomial $\tilde{p}$ attains its maximum modulus $\|\tilde{p}\|$ on $\partial\mathbb{D}$ at each point $z_n\zeta_j$, $j = 1, \dots, n$, and also at another point $z_n\zeta_*$,

$$z_n\zeta_* \in \partial\mathbb{D}, \quad \zeta_* \in \{\bar{\zeta}_j\}_{j=1}^n, \quad \text{and} \quad \zeta_* \neq \zeta_j,\ j = 1, \dots, n.$$

This can be seen as follows: We have

$$\begin{aligned} \|\tilde{p}\| &= |(\tilde{p} * F_{d_n,\theta_n})(z_n)| = \left| \tilde{p}(z) * \sum_{j=1}^{n} \frac{\lambda_j}{1-\zeta_j z} \right| \Bigg|_{z=z_n} \\ &= \left| \sum_{j=1}^{n} \lambda_j \tilde{p}(\zeta_j z_n) \right| \le \sum_{j=1}^{n} \lambda_j |\tilde{p}(\zeta_j z_n)| \le \|\tilde{p}\|. \end{aligned}$$

Therefore, $|\tilde{p}(z_n\zeta_j)| = \|\tilde{p}\|$, $j = 1, \dots, n$. Similarly

$$\begin{aligned} \|\tilde{p}\| &= |(\tilde{p} * F_{d_n,\theta_n})(z_n)| = \left| \tilde{p}(z) * \sum_{j=1}^{n} \frac{\lambda_j}{1-\bar{\zeta}_j z} \right| \Bigg|_{z=z_n} \\ &\le \sum_{j=1}^{n} \lambda_j |\tilde{p}(\bar{\zeta}_j z_n)| \le \|\tilde{p}\|, \end{aligned}$$

and in particular $|\tilde{p}(z_n\zeta_*)| = \|\tilde{p}\|$. Hence, the trigonometric polynomial

$$t(\theta) := \|\tilde{p}\|^2 - |\tilde{p}(e^{i\theta})|^2$$

has more than $2n$ zeros counting multiplicities in $[0, 2\pi)$. It follows that the trigonometric polynomial t, being of degree at most n, must vanish identically. Therefore, $|\tilde{p}(e^{i\theta})|$ is a constant on $[0, 2\pi]$. Under this condition, it is easily seen that $\tilde{p}(z) = a_{n-k}(p)z^k$ for some $k \in \{0, 1, \dots, n\}$. Hence, if the set $\{\zeta_j\}_{j=1}^n$ is not closed under conjugation, then the only extremal polynomials for the inequality (2) must have the form $p(z) = a_k(p)z^k$ for $0 \le k \le n$. Assuming that $p(z) = a_k(p)z^k$, $a_k(p) \neq 0$ for $0 \le k \le n-1$, we are led to

$$\|p'\| + d_n|a_0(p)| = k|a_k(p)| + d_n|a_0(p)| < n\|p\|, \qquad n \ge 2$$

which shows that p is not an extremal polynomial. Hence, all extremal polynomials for (2) have the form $p(z) = a_n(p)z^n$.

Case 2.2. Suppose now that the set $\{\zeta_j\}_{j=1}^n$ *is closed under conjugation.*

Let us write the last n equations of the system (6) as

$$\mathfrak{B}\overrightarrow{\Lambda} = \overrightarrow{C},$$

where

$$\overrightarrow{\Lambda} := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}, \qquad \overrightarrow{C} := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \qquad c_k := \begin{cases} \dfrac{n-k}{n} & \text{if } k = 1, \dots, n-1, \\ -\dfrac{d_n}{n} & \text{if } k = n \end{cases}$$

and $\mathfrak{B}$ is the invertible matrix

$$\mathfrak{B} := \begin{pmatrix} \zeta_1 & \zeta_2 & \dots & \zeta_n \\ {\zeta_1}^2 & {\zeta_2}^2 & \dots & {\zeta_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\zeta_1}^n & {\zeta_2}^n & \dots & {\zeta_n}^n \end{pmatrix}.$$

By known properties of Vandermonde matrices [1, p. 64], if

$$\left(L_0^{(j)}, L_1^{(j)}, \dots, L_{n-1}^{(j)}\right)^T$$

denotes the jth column of $\left(\mathfrak{B}^T\right)^{-1} = \left(\mathfrak{B}^{-1}\right)^T$, then

$$\sum_{k=0}^{n-1} L_k^{(j)} z^k \equiv \frac{W(z)}{(z-\zeta_j)\zeta_j W'(\zeta_j)},$$

where $W(z) := \prod_{s=1}^n (z - \zeta_s)$. The above identity can be verified directly, observing that both sides are polynomials of degree at most $n-1$ which coincide for $z = \zeta_k$, $k = 1, \dots, n$. Therefore, the jth row of $\mathfrak{B}^{-1}$ equals

$$(L_0^{(j)}, L_1^{(j)}, \dots, L_{n-1}^{(j)})$$

and the unique solution of (6) is

$$\begin{aligned}\lambda_j &= \sum_{k=0}^{n-1} L_k^{(j)} c_{k+1} \\ &= \frac{W(z)}{(z-\zeta_j)\zeta_j W'(\zeta_j)} * \sum_{k=0}^{n-1} c_{k+1} z^k \bigg|_{z=1} \\ &= \frac{W(z)}{(z-\zeta_j)\zeta_j W'(\zeta_j)} * \left(\sum_{k=1}^{n-1} \frac{n-k}{n} z^{k-1} - \frac{d_n}{n} z^{n-1} \right) \bigg|_{z=1} \\ &= \frac{1}{n} \left(n \sum_{k=0}^{n-2} L_k^{(j)} - \sum_{k=0}^{n-2} (k+1) L_k^{(j)} - d_n L_{n-1}^{(j)} \right).\end{aligned}$$

A short computations yields

$$\begin{aligned}\lambda_j = {} & \frac{W(1)}{n} \frac{1}{\zeta_j W'(\zeta_j)} \frac{\zeta_j}{(1-\zeta_j)^2} \times \\ & \left[\frac{2d_n}{W(1)} - (n-1) + \frac{W'(1)}{W(1)} + \zeta_j \left(-\frac{d_n}{W(1)} \right) \right. \\ & \left. + \bar{\zeta}_j \left(n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)} \right) \right]\end{aligned} \tag{7}$$

under the supposition that $W(1) \neq 0$.

Remark. A proper modification of formula (7) should be used in the case $W(1) = 0$. Assuming that $\zeta_1 = 1$, then (7) has the form

$$\lambda_j = \begin{cases} \dfrac{1}{n} \left\{ n - 1 - \dfrac{W''(1)}{2W'(1)} - \dfrac{d_n}{W'(1)} \right\} & \text{if } j = 1, \\ \dfrac{1}{n} \dfrac{1}{\zeta_j W'(\zeta_j)} \dfrac{\zeta_j}{(1-\zeta_j)^2} \times & \\ \quad \left\{ 2d_n + W'(1) - d_n \zeta_j - (W'(1) + d_n) \bar{\zeta}_j \right\} & \text{if } 2 \leq j \leq n, \end{cases}$$

and the proof goes along the same lines as in the case $W(1) \neq 0$.

We have

$$\|\tilde{p}\| = |(\tilde{p} * F_{d_n,\theta_n})(z_n)| = \left| \sum_{j=1}^{n} \lambda_j \tilde{p}(\zeta_j z_n) \right| = \sum_{j=1}^{n} \lambda_j |\tilde{p}(\zeta_j z_n)| = \|\tilde{p}\|.$$

This means that at each point $\zeta_j z_n$, $j = 1, \ldots, n$, the value $\tilde{p}(\zeta_j z_n)$ has constant argument and $|\tilde{p}(\zeta_j z_n)| = \|\tilde{p}\|$. Without any loss of generality, we may suppose that $z_n = 1$ and $\tilde{p}(\zeta_j) = \|\tilde{p}\|$, $j = 1, \dots, n$, i.e., we have the representation

$$\tilde{p}(z) = cW(z) + \|\tilde{p}\|, \qquad c \in \mathbb{C}.$$

Assuming $c \neq 0$ ($\tilde{p}$ is non-constant) we use Lemma 4 to obtain

$$0 < \frac{\zeta_j \tilde{p}'(\zeta_j)}{\tilde{p}(\zeta_j)} = \frac{c\zeta_j W'(\zeta_j)}{\|\tilde{p}\|}. \tag{8}$$

A combination of (7) and (8), together with the facts that $\lambda_j > 0$, $W(1) > 0$ (this is true because we assume that $W(1) \neq 0$ and the set $\{\zeta_j\}_{j=1}^n$ of zeros of W is closed under conjugation), and $(1-\zeta_j)^2/\zeta_j = 2\operatorname{Re}(\zeta_j) - 2 < 0$, yield that

$$\arg \beta(\zeta_j) \equiv \arg(-\bar{c}), \qquad j = 1, 2, \dots, n,$$

where

$$\beta(u) := \frac{2d_n}{W(1)} - (n-1) + \frac{W'(1)}{W(1)} + u\left(-\frac{d_n}{W(1)}\right) + \bar{u}\left(n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)}\right).$$

In other words, $\arg \beta(\zeta_j)$ is constant for $j = 1, 2, \dots, n$.

Case 2.2.1. We shall first assume that

$$\left|\frac{d_n}{W(1)}\right| \neq \left|n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)}\right|.$$

Then the image of the unit circle $\partial\mathbb{D}$ by the function $\beta(u)$ is a once-covered non-degenerate ellipse and such a curve meets any line at most twice. Hence, $\arg \beta(\zeta_j)$ cannot be constant for $n > 2$.

Case 2.2.2. We now discuss the case

$$\left|\frac{d_n}{W(1)}\right| = \left|n - \frac{W'(1)}{W(1)} - \frac{d_n}{W(1)}\right|.$$

By assumption, the point set $\{\zeta_j\}_{j=1}^n$ is closed under conjugation. This implies $W(1) > 0$ and $W'(1) > 0$. Then either $nW(1) = W'(1)$ or $nW(1) = W'(1) + 2d_n$. It follows that either

$$n = \frac{W'(1)}{W(1)} = \sum_{j=1}^{n} \operatorname{Re} \frac{1}{1-\zeta_j} = \frac{n}{2}$$

which is impossible or

$$\beta(\zeta_j) = 1 - \zeta_j \frac{d_n}{W(1)} + \frac{1}{\zeta_j}\frac{d_n}{W(1)} = 1 - 2i\operatorname{Im}(\zeta_j)\frac{d_n}{W(1)}.$$

This implies that $\operatorname{Im}\zeta_j$ is constant. Clearly this can occur only if $n = 2$.

We complete the proof of Theorem 1 by considering the case $n = 2, p \in \mathcal{P}_2$. From the system (6), taking into account that $\bar{\zeta}_2 = \zeta_1$, we obtain $\zeta_1 = 1/2 + i\sqrt{3}/2$ and $\beta(\zeta_1) = 3/2 - i\sqrt{3}/2 = \bar{\beta}(\zeta_2)$. Hence,

$$\arg \beta(\zeta_1) \neq \arg \beta(\zeta_2).$$

Hence, $\tilde{p}$ is a constant, i.e., the only extremal polynomials $p \in \mathcal{P}_2$ for (2) are $p(z) = a_2(p)z^2$.

Summing up, if a polynomial $p \in \mathcal{P}_n$, $n \geq 2$ is extremal for (2), i.e., p satisfies (5), then $\tilde{p}$ must be a constant, i.e., *the only extremal polynomials* $p \in \mathcal{P}_n$, $n \geq 2$, *for* (2) *are of the form* $p(z) = a_n(p)z^n$.

4. Proof of Theorem 2

We consider the case $n \geq 2$. Clearly,

$$\begin{aligned} & \delta\|p'\| + |a_0(p)| \leq \|p\| \\ \Leftrightarrow \quad & \left| a_0(p) + \sum_{k=1}^{n} \delta e^{i\theta} k a_k(p) z^k \right| \leq \|p\| \quad \text{for all } z \in \mathbb{D},\ \theta \in \mathbb{R} \\ \Leftrightarrow \quad & \left| p(z) * \left(1 + \delta e^{i\theta} \sum_{k=1}^{n} k z^k\right) \right| \leq \|p\| \quad \text{for all } z \in \mathbb{D},\ \theta \in \mathbb{R} \end{aligned}$$

and by the definition of $\mathcal{B}_n$

$$\delta_n = \sup\left\{ \delta \geq 0 :\ 1 + \delta e^{i\theta} \sum_{k=1}^{n} k z^k \in \mathcal{B}_n \text{ for all } \theta \in \mathbb{R} \right\}.$$

According to Lemma 2, Lemma 3, and Proposition 1 we need to study the determinants

$$\operatorname{Det}_m(\delta e^{i\theta}) := \det \begin{pmatrix} 1 & \delta e^{i\theta} & \cdots & m\delta e^{i\theta} \\ \delta e^{-i\theta} & 1 & \cdots & (m-1)\delta e^{i\theta} \\ \vdots & \vdots & \ddots & \vdots \\ m\delta e^{-i\theta} & (m-1)\delta e^{-i\theta} & \cdots & 1 \end{pmatrix}$$

for $m = 1, 2, \ldots, n$.

We shall prove that there exists a positive number $\tilde{\delta}_n$ such that

a) $\operatorname{Det}_m(\delta e^{i\theta}) > 0$ for $m = 1, 2, \ldots, n-1$, $\delta \in [0, \tilde{\delta}_n]$, and $\theta \in \mathbb{R}$;
b) $\operatorname{Det}_n(\delta e^{i\theta}) > 0$ for $\delta \in [0, \tilde{\delta}_n)$ and $\theta \in \mathbb{R}$;
c) $\operatorname{Det}_n(-\tilde{\delta}_n) = 0$ and $\operatorname{Det}_n(\tilde{\delta}_n e^{i\theta}) > 0$ for $\theta \neq \pi \mod 2\pi$;
d) $\operatorname{Det}_n(-\tilde{\delta}_n - \varepsilon) < 0$ for $\varepsilon \in (0, \varepsilon_o)$, $\varepsilon_o > 0$, and ε_o sufficiently small.

Then by Lemma 2, Lemma 3, and Proposition 1 we conclude that

$$\delta_n = \tilde{\delta}_n.$$

For each $m \geq 1$ and $\theta \in \mathbb{R}$ we define a symmetric $m \times m$ matrix $\mathcal{M}_m := \mathcal{M}_m(\delta\cos\theta)$ by

$$(\mathcal{M}_m)_{sj} := \begin{cases} -(2s+2)j\delta\cos\theta + (s+1)(j+1) & \text{if } 1 \leq j < s \leq m \\ -(2s+2)j\delta\cos\theta + (s+1)(j+1) + 1 & \text{if } 1 \leq j = s \leq m. \end{cases}$$

The following recursion formula will be useful.

$$\operatorname{Det}_m(\delta e^{i\theta}) = \operatorname{Det}_{m-1}(\delta e^{i\theta}) - \delta^2 \operatorname{Det}\left(\mathcal{M}_{m-1}(\delta\cos\theta)\right). \tag{9}$$

The above identity can be obtained by performing a sequence of elementary operations on the Toeplitz matrix

$$\mathfrak{Mat}_m(\delta e^{i\theta}) := \begin{pmatrix} 1 & \delta e^{i\theta} & \cdots & m\delta e^{i\theta} \\ \delta e^{-i\theta} & 1 & \cdots & (m-1)\delta e^{i\theta} \\ \vdots & \vdots & \ddots & \vdots \\ m\delta e^{-i\theta} & (m-1)\delta e^{-i\theta} & \cdots & 1 \end{pmatrix}.$$

(i) Add proper multiples of the first row of $\mathfrak{Mat}_m$ to obtain a $(m+1)\times(m+1)$ matrix $\mathfrak{Mat}'_m$ with $(\mathfrak{Mat}'_m)_{s1} = 0$ for $1 < s \leq m+1$. Delete the first row and the first column of $\mathfrak{Mat}'_m$ to obtain an $m\times m$ matrix $\mathfrak{Mat}''_m$ with $\operatorname{Det}_m(\delta e^{i\theta}) = \det\left(\mathfrak{Mat}''_m\right)$.

(ii) For each $2 \leq j \leq m$, add $-j$ times the first column of $\mathfrak{Mat}''_m$ to the jth column of $\mathfrak{Mat}''_m$. Call the matrix obtained in this way $\mathfrak{Mat}'''_m$.

(iii) We have

$$\operatorname{Det}_m(\delta e^{i\theta}) = \det(\mathfrak{Mat}'''_m) = \operatorname{Det}_{m-1}(\delta e^{i\theta}) + \det\left(\mathcal{M}'_m\right)$$

where $\mathcal{M}'_m$ is an $m\times m$ matrix. Use now the first row of $\mathcal{M}'_m$ to cancel all entries in the first column of $\mathcal{M}'_m$ except entry $(\mathcal{M}'_m)_{11}$ which is equal to $-\delta^2$. Finally, we obtain (9).

Clearly, $\det\left(\mathcal{M}_m(\delta\cos\theta)\right)$ is a polynomial of degree m in the variable $\delta\cos\theta$. For example

$$\begin{aligned} \det(\mathcal{M}_1) &= -4\delta\cos\theta + 5, \\ \det(\mathcal{M}_2) &= 12\delta^2\cos^2\theta - 28\delta\cos\theta + 14 \end{aligned}$$

and the following recursion formula holds for $m \geq 3$

$$\begin{aligned} \det(\mathcal{M}_m) &= \left[-2\delta\cos\theta\left(1+\frac{1}{m}\right) + 1 + \left(1+\frac{1}{m}\right)^2\right]\det(\mathcal{M}_{m-1}) \\ &\quad - \left(1+\frac{1}{m}\right)^2 \det\left(\mathcal{M}_{m-2}\right). \end{aligned}$$

Let us define

$$\det(\mathcal{M}_m(\delta\cos\theta)) =: \sum_{k=0}^{m} a_{k,m}\delta^k\cos^k\theta.$$

We can easily compute $a_{m,m} = (-1)^m 2^m (m+1)$. We shall prove that

$$\operatorname{sign}\left(a_{k,m}\right) = (-1)^k, \qquad 0 \leq k \leq m. \tag{10}$$

We have

$$a_{0,m} = a_{0,m-1} + \left(1 + \frac{1}{m}\right)^2 (a_{0,m-1} - a_{0,m-2}), \qquad m \geq 2.$$

Using the definition $a_{0,j} := 0$ if $j \leq 0$ it follows by induction on m that

$$a_{0,m} > 0 \quad \text{and} \quad a_{0,m} > a_{0,m-1}, \qquad m \geq 0.$$

We also have

$$a_{1,m} = -2\left(1 + \frac{1}{m}\right) a_{0,m-1} + a_{1,m-1} + \left(1 + \frac{1}{m}\right)^2 (a_{1,m-1} - a_{1,m-2}),$$

if $m \geq 1$ and $a_{1,j} := 0$ if $j < 1$. It follows by induction on m that

$$a_{1,m} < 0 \quad \text{and} \quad a_{1,m} < a_{1,m-1}, \qquad m \geq 1.$$

As a last example

$$a_{2,m} = -2\left(1 + \frac{1}{m}\right) a_{1,m-1} + a_{2,m-1} + \left(1 + \frac{1}{m}\right)^2 (a_{2,m-1} - a_{2,m-2}), \quad m \geq 2.$$

A double induction on m and k yields (10).

By making use of (10) we obatin

$$\det(\mathcal{M}_m(\delta \cos\theta)) \leq \det(\mathcal{M}_m(-\delta)) = \sum_{k=0}^{m} |a_{k,m}|\delta^k, \qquad \delta > 0, \tag{11}$$

with equality if and only if $\theta = \pi \mod 2\pi$. This and (9) imply

$$\mathrm{Det}_m(\delta e^{i\theta}) \geq \mathrm{Det}_m(-\delta) = 1 - \delta^2 - \delta^2 \sum_{j=1}^{m-1} \det(\mathcal{M}_j(-\delta)), \quad \delta > 0,\, \theta \in \mathbb{R}, \tag{12}$$

with equality if and only if $\theta = \pi \mod 2\pi$.

We conclude that $\tilde{\delta}_n = \delta_n$ is the only positive root of the equation $\mathrm{Det}_n(-\delta) = 0$ and that the sequence $\{\delta_m\}_{m\geq 1}$ is strictly decreasing. Furthermore,

$$\mathrm{Det}_n\left(\delta_n e^{i\theta}\right) = 0 \qquad \Leftrightarrow \qquad \theta = \pi \mod 2\pi,$$

and $\mathrm{Det}_n\left(\delta_n e^{i\theta}\right) > 0$ for $\theta \neq \pi \mod 2\pi$.

Clearly, $\mathrm{Det}_1(-\delta) = 1 - \delta^2$. Some straightforward computation shows that $\mathrm{Det}_2(-\delta) = 1 - 6\delta^2 - 4\delta^3$ and $\mathrm{Det}_3(-\delta) = 1 - 20\delta^2 - 32\delta^3 - 12\delta^4$. This proves

$$\delta_1 = 1, \qquad \delta_2 = \frac{1}{1+\sqrt{3}}, \qquad \delta_3 = \frac{1}{2+\sqrt{10}}.$$

However, it does not seem easy to obtain an explicit formula for δ_n expressed as a function of n. Therefore, we only shall give some growth estimates.

It is seen from (11) and (12) that $\mathrm{Det}_n(-\delta)$ is a polynomial in δ whose coefficients, except for the constant one, are negative. One easily computes

$$\begin{aligned}\mathrm{Det}_n(-\delta) &= 1-\left(\frac{1}{6}\sum_{k=1}^{n} k(k+1)(2k+1)\right)\delta^2+o\big(\delta^2\big)\\ &= 1-\frac{n(n+1)^2(n+2)}{12}\delta^2+o(\delta^2), \qquad \text{as } \delta\to 0\end{aligned}$$

which yields

$$1-\frac{n(n+1)^2(n+2)}{12}{\delta_n}^2 > \mathrm{Det}_n(-\delta_n)=0,$$

i.e.,

$$\delta_n \le \frac{2\sqrt{3}}{(n+1)\sqrt{n(n+2)}}.$$

For $\delta \neq 0$ we have,

$$\mathrm{Det}_n(\delta)=\delta^{n+1}\det\begin{pmatrix} 1-(1-1/\delta) & 1 & \cdots & n\\ 1 & 1-(1-1/\delta) & \cdots & n-1\\ 2 & 1 & \cdots & n-2\\ \vdots & \vdots & \ddots & \vdots\\ n & n-1 & \cdots & 1-(1-1/\delta)\end{pmatrix}.$$

Hence, for any root δ of $\mathrm{Det}_n(\delta)=0$, the number $1-1/\delta$ is an eigenvalue of the symmetric $(n+1)\times(n+1)$ matrix

$$\mathfrak{A} := \begin{pmatrix} 1 & 1 & 2 & \cdots & n\\ 1 & 1 & 1 & \cdots & n-1\\ 2 & 1 & 1 & \cdots & n-2\\ \vdots & \vdots & \ddots & \vdots & \\ n & n-1 & n-2 & \cdots & 1\end{pmatrix}. \tag{13}$$

In particular, all the roots of $\mathrm{Det}_n(-\delta)=0$, except $\delta_n>0$, are real and negative. Moreover, suppose that $\mathrm{Det}_n(\delta^*)=0$ and $\delta^*\neq -\delta_n$. Then, (11) and (12) imply $\delta^*>0$, $\delta^*>\delta_n$, and

$$1+\frac{1}{\delta_n} > \left|1-\frac{1}{\delta^*}\right|.$$

Hence, $1+1/\delta_n$ *is the spectral radius of the symmetric matrix* (13).

For $\mathbf{x}^T := (x_0,\dots,x_n)\in\mathbb{R}^{n+1}$ and $\mathbf{y}^T := (y_0,\dots,y_n)\in\mathbb{R}^{n+1}$ let

$$\langle \mathbf{x},\mathbf{y}\rangle := \mathbf{x}^T\mathbf{y}=\sum_{k=0}^{n} x_k y_k$$

denote the inner product of the vectors $\mathbf{x}$ and $\mathbf{y}$. Rayleigh's Quotient Theorem (see [7, p. 156]) applied to the vector $\mathbf{x}_o^T := (1, \ldots, 1) \in \mathbb{R}^{n+1}$ implies

$$1 + \frac{1}{\delta_n} \geq \frac{\langle \mathbf{x}_o^T \mathfrak{A}, \mathbf{x}_o \rangle}{\langle \mathbf{x}_o, \mathbf{x}_o \rangle} = \left(n + 1 + 2 \sum_{k=1}^{n} k(n-k+1) \right) \frac{1}{n+1} = 1 + \frac{n(n+2)}{3}.$$

Hence,

$$\delta_n \leq \frac{3}{n(n+2)}.$$

Applying Gerschgorin's estimates (see [7, p. 227]) to the first row of (13) and taking into account that the first Gerschgorin disk contains all the others as subsets, we obtain

$$\frac{1}{\delta_n} = \left| 1 - \left(1 + \frac{1}{\delta_n} \right) \right| \leq \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

and conclude that

$$\delta_n \geq \frac{2}{n(n+1)}.$$

The equality case in (3)*:* Now we determine all polynomials $p \in \mathcal{P}_n$, $n > 1$, such that

$$\delta_n \|p'\| + |a_0(p)| = \|p\|. \tag{14}$$

We define

$$F_{\delta,\theta} := 1 + \delta e^{i\theta} \sum_{k=1}^{n} k z^k.$$

If (14) holds for $p \in \mathcal{P}_n$, $n \geq 1$, then

$$\begin{aligned} |p * F_{\delta_n,\theta_n}(z_n)| &= |\delta_n e^{i\theta_n} z_n p'(z_n) + a_0(p)| \\ &= \delta_n |p'(z_n)| + |a_0(p)| \\ &= \delta_n \|p'\| + |a_0(p)| = \|p\| \end{aligned} \tag{15}$$

for some real θ_n and $z_n \in \partial\mathbb{D}$.

The sequence $\{\delta_n\}_{n\geq 1}$ is strictly decreasing, δ_n is the only positive root of the equation $\operatorname{Det}_n(-\delta) = 0$, δ_n is a simple zero of $\operatorname{Det}_n(-\delta)$ and δ_n is the unique solution of the minimum problem

$$\min \{ |z| : \operatorname{Det}_n(z) = 0 \text{ for all } z \in \mathbb{C} \}.$$

Similarly as in the proof of Theorem 1 we conclude that the direction θ_n is such that $\operatorname{Det}_n(\delta_n e^{i\theta_n}) = 0$, i.e., $\theta_n = \pi \mod 2\pi$. Then, by Lemma 1 and Lemma 2 we have

$$F_{\delta_n,\theta_n}(z) = 1 - \delta_n \sum_{k=1}^{n} k z^k = \sum_{j=1}^{n} \frac{\lambda_j}{1 - z\zeta_j} + o(z^n) \qquad \text{as } z \to 0$$

with $\lambda_j > 0$, $j = 1, \ldots, n$, and pairwise distinct $\zeta_j \in \partial\mathbb{D}$, $j = 1, \ldots, n$. From (15) we obtain

$$|p(z_n\zeta_j)| = \|p\|, \qquad j = 1, 2, \ldots, n.$$

Without loss of generality we may assume $z_n = 1$. Let

$$p(z) := cW(z) + \|p\|, \qquad c \in \mathbb{C},\ c \neq 0,$$

where $W(z) := \prod_{j=1}^{n}(z - \zeta_j)$.

We now consider the system of equations

$$\sum_{j=1}^{n} \lambda_j \zeta_j{}^k = -\delta_n k, \qquad k = 1, 2, \ldots, n, \tag{16}$$

where $\lambda_j > 0$, $j = 1, \ldots, n$, $\lambda_1 + \cdots + \lambda_n = 1$ and the n complex numbers $\xi_j \in \partial\mathbb{D}$, $j = 1, \ldots, n$, are pairwise distinct. As in the proof of Theorem 1 we have

$$\begin{aligned} 0 < \lambda_j &= -\delta_n \sum_{k=0}^{n-1} L_k^{(j)}(k+1) \\ &= \frac{-\delta_n}{\zeta_j W'(\zeta_j)}\left[\frac{-\zeta_j}{(1-\zeta_j)^2}W(1) + \frac{1}{1-\zeta_j}W'(1)\right] \\ &= \frac{\delta_n}{\zeta_j W'(\zeta_j)}\frac{\zeta_j}{(1-\zeta_j)^2}\left[W(1) + (\bar{\zeta}_j - 1)W'(1)\right]. \end{aligned} \tag{17}$$

By Lemma 4, $\arg(\zeta_j W'(\zeta_j))$ does not depend of j and we deduce from (17) that

$$\arg\left(W(1) + (\bar{\zeta}_j - 1)W'(1)\right)$$

is a constant for $j = 1, \ldots, n$. This can occur only if $n = 2$.

Now consider a polynomial $p \in \mathcal{P}_2$ such that

$$\delta_2\|p'\| + |a_0(p)| = \frac{1}{1+\sqrt{3}}\|p'\| + |a_0(p)| = \|p\|.$$

Without loss of generality we may assume that

$$p(z) = c(z - e^{i\theta})(z - e^{-i\theta}) + \|p\|.$$

We use (16) for $n = 2$ to obtain $\cos\theta = -\delta_2$. Hence,

$$p(z) = cz^2 + 2\delta_2 cz + c + \|p\|, \qquad \|p'\| = 2|c| + 2\delta_2|c|$$

and

$$\delta_2\|p'\| + |a_0(p)| = \|p\| \qquad \Leftrightarrow \qquad |c + \|p\|| = \|p\| - |c|.$$

This means that

$$|a_2(p)| + |a_0(p)| = |c + \|p\|| + |c| = \|p\|, \qquad p \in \mathcal{P}_2,$$

and we have equality in Visser's inequality (see [2] and [13]) for $n = 2$. Therefore, $c = 0$ and p is constant.

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