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Polynomial Approximation on Compact Sets Bounded by Dini-Smooth Arcs

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Abstract. For a compact set K which is the closure of a Jordan domain, the Faber operator provides a well-known tool for deriving results on the error of uniform polynomial approximation on K. We show that the corresponding methods also work for compact sets which are not the closure of a Jordan domain. In particular, the cases of so-called touching domains and of Jordan arcs are considered.

Keywords. Faber operator, Dini-smoothness, uniform approximation, touching domains, Jackson theorem.

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1. Introduction

Let K be a compact subset of the complex plane $\mathbb{C}$ and let $A(K)$ be the Banach space of functions that are continuous on K and holomorphic in the interior K^o of K endowed with the uniform norm

$$\|F\|_K := \sup_{z\in K} |F(z)|, \qquad F \in A(K).$$

We study the rate of polynomial approximation in $A(K)$ defined by

$$E_n(F, K) := \inf_{P\in\Pi_n} \|F - P\|_K, \qquad F \in A(K),$$

where Π_n denotes the set of polynomials of degree $\leq n$.

From now on, we always suppose that K is not a single point and that the complement of K with respect to the extended plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is a simply connected region. Note that in this case, a uniquely determined best approximating polynomial $P_n^* = P_n^*(F)$ in Π_n exists for all $F \in A(K)$.

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According to the Riemann Mapping Theorem, a unique conformal mapping ψ of $\mathbb{C}_\infty \backslash \overline{\mathbb{D}}$, where $\mathbb{D} := \{z : |z| < 1\}$, onto $\mathbb{C}_\infty \backslash K$ exists so that $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. The n-th Faber polynomial $F_n := F_{n,K}$ with respect to K may be defined by

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad z \in K.$$

Then F_n is a polynomial of exact degree n. For further properties of Faber polynomials and Faber expansions, see [11], [8] or [22].

The set K is called a *Faber set* if the linear operator

$$T\colon \left(\bigcup_{n\in\mathbb{N}} \Pi_n, \|\cdot\|_{\overline{\mathbb{D}}} \right) \to \left(\bigcup_{n\in\mathbb{N}} \Pi_n, \|\cdot\|_K \right),$$

defined by

$$(Tp)(z) := \sum_{\nu=0}^{n} a_\nu F_\nu(z), \quad \text{where } p \text{ is the polynomial } p(w) = \sum_{\nu=0}^{n} a_\nu w^\nu,$$

which is always onto and one-to-one, turns out to be continuous. In this case, T extends uniquely to a continuous linear operator — also denoted by T — from $A(\overline{\mathbb{D}})$ to $A(K)$ which is also one-to-one, see e.g. [10, Lemma 1].

If K is a Faber set and if $\|T\|$ denotes the operator norm of T, then we have the following estimate which connects the uniform polynomial approximation on $\overline{\mathbb{D}}$ (or $\partial\mathbb{D}$) to the uniform polynomial approximation on K.

Suppose that $F \in A(K)$ belongs to the range $T(A(\overline{\mathbb{D}}))$ of T and $F = Tf$, say. Then, for all polynomials p,

$$\|F - Tp\|_K \leq \|T\| \cdot \|f - p\|_{\overline{\mathbb{D}}}.$$

If $p = p_n^*$ is the best approximating polynomial of degree not larger than n to f on $\overline{\mathbb{D}}$ then we get in particular

$$E_n(F, K) \leq \|T\| \cdot E_n(f, \overline{\mathbb{D}}). \tag{1}$$

Thus, according to classical results on polynomial approximation on the closed unit disk, we are able to estimate $E_n(F, K)$ if it is possible to find estimates for the modulus of continuity of f.

A large number of corresponding results already exist for the case that K is the closure of a Jordan domain, that is, for the case that $\Gamma = \partial K$ is a Jordan curve (see e.g. [2], [9], [15], or the monographs [11], [22]).

The boundedness of the Faber operator on more general compact sets K was studied in particular in [10]. In this paper, we use this concept to find results on polynomial approximation on compact sets bounded by a finite number of Jordan arcs or Jordan curves. In particular, we shall consider the following two cases.

a) The case of so-called touching domains as studied in [5] and [12]. Here, the compact set K consists of the closure of two Jordan domains touching at one point, for instance, the lemniscate $|z^2 - 1| \le 1$.

b) The case of a compact set K which is itself a Jordan arc, for example, two lines meeting at a point. Of course, in this case $A(K)$ equals $C(K)$, the Banach space of functions continuous on K with the uniform norm.

2. Approximation on Faber sets

In what follows we consider compact sets — with $\mathbb{C}_\infty \backslash K$ being a simply connected domain as above — having the additional property that $\Gamma := \partial K$ is a curve. This is equivalent to saying that ψ extends continuously to the unit circle $\partial\mathbb{D}$.

For $p \in \mathbb{N}$, let $\omega_p(\varphi, t)$ denote the p-th modulus of continuity of a function φ continuous on $\partial\mathbb{D}$. If φ satisfies $\int_0^1 \omega_p(\varphi, u)u^{-1}\,du < \infty$, we define

$$\omega_p^*(\varphi, t) := \omega_p(\varphi, t) + \int_0^t \frac{\omega_p(\varphi, u)}{u}\,du + t^p \int_t^1 \frac{\omega_p(\varphi, u)}{u^{p+1}}\,du.$$

Then we have the following basic result.

Proposition 1. *Let p be a positive integer. Then a constant c_p exists such that for all Faber sets K and for all $F \in A(K)$ with $\int_0^1 \omega_p(F \circ \psi, u)u^{-1}\,du < \infty$ we have*

$$E_n(F, K) \le c_p \,\|T\|\, \omega_p^*(F \circ \psi, 1/n).$$

Proof. We consider the Cauchy integral

$$f(w) := \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(F \circ \psi)(\zeta)}{\zeta - w}\,d\zeta, \qquad |w| < 1,$$

of $F \circ \psi\,(= F \circ \psi_{|\partial\mathbb{D}})$. Since $\int_0^1 \omega_p(F \circ \psi, u)u^{-1}\,du < \infty$, the function f extends continuously to $\overline{\mathbb{D}}$ and we have

$$\omega_p(f, t) \le d_p\, \omega_p^*(F \circ \psi, t)$$

for $0 < t < 1$, with some constant d_p, see e.g. [11, p. 54].

In particular, the continuity of f implies that F belongs to the range of T and that $Tf = F$, see [10, Theorem 5]. Now, according to known results of Jackson type for polynomial approximation on $\overline{\mathbb{D}}$, see e.g. [11, p. 56], there is a constant C_p (not depending on f) such that

$$E_n(f, \partial\mathbb{D}) \le C_p\, \omega_p(f, 1/n)\,.$$

A combination of these results with inequality (1) proves the proposition. ■

In order to express approximation results in terms of F instead of $F \circ \psi$ we have to impose further restrictions. As usual, $\mathrm{Lip}_M(\alpha) = \mathrm{Lip}_M(\alpha, \Gamma)$ denotes the class of functions $g\colon \Gamma \to \mathbb{C}$ satisfying a uniform Hölder condition of the form

$$|g(\zeta_1) - g(\zeta_2)| \le M|\zeta_1 - \zeta_2|^\alpha, \qquad \zeta_1, \zeta_2 \in \Gamma,$$

and

$$\mathrm{Lip}(\alpha) := \mathrm{Lip}(\alpha, \Gamma) := \bigcup_{M>0} \mathrm{Lip}_M(\alpha).$$

With these notions we can easily prove the following Jackson-type result.

Theorem 1. *Let $K \subset \mathbb{C}$ be a Faber set such that $\psi \in \mathrm{Lip}(\beta, \partial\mathbb{D})$, for some $\beta > 0$. If $\alpha \le 1$, there exists a constant $c = c(K, \alpha\beta)$ such that*

$$E_n(F, K) \le \frac{cM}{n^{\alpha\beta}},$$

for all $n \in \mathbb{N}$ and all $F \in A(K)$ which belong to $\mathrm{Lip}_M(\alpha, \Gamma)$.

Proof. If $\psi \in \mathrm{Lip}_d(\beta, \partial\mathbb{D})$ we have

$$F \circ \psi \in \mathrm{Lip}_{Md}(\alpha\beta, \partial\mathbb{D}).$$

From the definition of ω_p^* we see that there exists a constant $D = D_{\alpha\beta}$, independent of $F \circ \psi$, such that

$$\omega_1^*(F \circ \psi, t) \le DMt^{\alpha\beta}$$

in the case $\alpha\beta < 1$ and

$$\omega_2^*(F \circ \psi, t) \le DMt$$

in the case $\alpha\beta = 1$. The theorem then follows from Proposition 1. ■

To get a reasonable sufficient condition for Theorem 1 to hold, we consider the curve Γ to be of the form

$$\Gamma = \bigcup_{j=1}^{m} \gamma_j$$

where the γ_j are — possibly overlapping or even coinciding — Jordan arcs or Jordan curves so that

$$\{\psi(e^{it}) : t_{j-1} \le t \le t_j\}$$

is a Jordan parametrization of γ_j (i.e. one-to-one with the possible exception that the endpoints may coincide) for certain $t_0 < t_1 < \ldots < t_m = t_0 + 2\pi$. In addition, we suppose that the $\gamma_1, \ldots, \gamma_m$ are Dini-smooth, that is, they have a parametrization

$$\gamma_j = \{\varphi_j(t) : t \in [0, 1]\}$$

with $\varphi_j'(t) \ne 0$ being Dini-continuous, see for example, [21, Section 3.3]. Finally, we suppose that the arcs γ_j and γ_{j+1} (with $\gamma_{m+1} := \gamma_0$) join at the points $\psi(e^{it_j})$ with an exterior angle of opening $\pi\beta_j$ with $0 < \beta_j \le 2$ (exterior means that the angle between the right- and left- hand tangents at $\psi(e^{it_j})$ is taken in $\mathbb{C}_\infty \setminus K$).

We say that $\Gamma = \partial K$ is *piecewise Dini-smooth with parameter* β if Γ is as above with

$$\beta := \min(1, \min_{j=1,\ldots,m} \beta_j).$$

To ensure Hölder continuity of ψ on the boundary of $\mathbb{D}$, we consider Γ with positive parameter β. This means, that the γ_j do not join under vanishing exterior angle. A vanishing interior angle as in the case of an outward pointing cusp, however, is permitted.

Corollary 1. *Let* $K \subset \mathbb{C}$ *be a compact set having piecewise Dini-smooth boundary* Γ *as above with parameter* $\beta > 0$. *If* $\alpha \leq 1$, *a constant* $c = c(K, \alpha\beta)$ *exists such that*

$$E_n(F, K) \leq \frac{cM}{n^{\alpha\beta}}$$

for all $n \in \mathbb{N}$ *and all* $F \in A(K)$ *which belong to* $\mathrm{Lip}_M(\alpha, \Gamma)$.

Proof. It can be shown ([19], in particular Remark 4) that a sufficient condition for K to be a Faber set is that $\Gamma = \bigcup_{j=1}^{m} \gamma_j$ with Dini-smooth Jordan arcs γ_j. This is essentially a consequence of Theorems 4 and 5 of [13]. Moreover, from Theorem 3.9 in [21] it follows that ψ (extended to $\mathbb{C}_\infty \setminus \mathbb{D}$) belongs to the class $\mathrm{Lip}(\beta, \partial\mathbb{D})$. ■

3. Approximation on touching domains

Now we consider compact sets K which are the closure of two Jordan domains meeting at a single point (so-called touching domains, see [5], [12]). More precisely, let γ_+ be a Jordan curve with $0 \in \gamma_+$ and $\gamma_+ \setminus \{0\} \subset \{z : \mathrm{Re}, z > 0\}$ which is piecewise Dini-smooth with only one corner at 0. Moreover, let

$$\gamma_- := \{-z : z \in \gamma_+\}$$

and let G_+, G_- denote the Jordan domains bounded by γ_+, γ_-, respectively. We consider K to be the closure of $G_+ \cup G_-$ for G_+, G_- as above. A typical example is the (filled-in) lemniscate K defined as the closure of $G_+ \cup G_-$ with

$$G_+ \cup G_- = \{z \in \mathbb{C} : |z^2 - 1| < 1\}.$$

If the exterior angle $\pi\beta$ of $\Gamma = \partial K = \gamma_+ \cup \gamma_-$ at the origin is positive, then Theorem 1 implies that

$$E_n(F, K) = \mathcal{O}\left(\frac{1}{n^{\alpha\beta}}\right) \tag{2}$$

for all $F \in A(K)$ with $F \in \mathrm{Lip}(\alpha)$ on Γ. In particular, if $\beta = 1$, then K satisfies a Jackson-type property of the form

$$E_n(F, K) = \mathcal{O}\left(\frac{1}{n^{\alpha}}\right) \tag{3}$$

for all $F \in A(K)$ with $F \in \mathrm{Lip}(\alpha)$ on Γ. An example is the set

$$K := \left\{x + iy : -1 \le x \le 1,\ |y| \le x^2\sqrt{1-x^2}\right\}.$$

The special case $F = F_\alpha$ with

$$F_\alpha(z) = \begin{cases} z^\alpha, & \operatorname{Re} z > 0 \\ (-z)^\alpha, & \operatorname{Re} z < 0 \end{cases}, \qquad F_\alpha(0) := 0$$

was already studied in [5] and [12] (answering a question of Grothmann and Saff in [14]). Actually, in [5], a characterization of touching domains K with $E_n(F_1, K) = \mathcal{O}(1/n)$ is given. In [12], the approximation of F_α on touching domains is performed by approximation of $z^{\alpha/2}$ on corresponding Jordan domains. This method is, however, restricted to even functions. One motivation for our investigations was to get rid of this restriction (cf. the concluding remark in [12]).

A question that naturally arises is whether (2) and (3) turn out to be sharp. In other words, are there Bernstein-type-results for touching domains also? We apply a result of Anderson, Hinkkanen and Lesley ([1]) to prove such an inverse result, at least for even functions on K.

Theorem 2. *Let K be the closure of $G_+ \cup G_-$ with G_+, G_- as above, and with exterior angle $\pi\beta$, $\beta \in (0,1]$ at the origin. If $F \in A(K)$ is even and if*

$$E_n(F,K) = \mathcal{O}\left(\frac{1}{n^{\alpha\beta}}\right)$$

for some $\alpha \in (0,1]$, then

$$\frac{|F(z) - F(0)|}{|z|^s} \to 0, \qquad z \to 0,\ z \in K,$$

for all $s < \alpha$. That is, F satisfies a Hölder condition of order s at 0, for all $s < \alpha$.

Proof. We put $A^2 := \{z^2 : z \in A\}$ for an arbitrary set A in $\mathbb{C}$ and consider the Jordan domain G_+^2 and the compact set $K^2 = \{z^2 : z \in K_+\}$ where K_+ is the set $\gamma_+ \cup G_+$. Then G_+^2 is bounded by the Jordan curve γ_+^2 which is piecewise Dini-smooth with only one (possible) corner of exterior opening $2\pi\beta$ at the origin. If

$$\Phi\colon \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus K^2$$

is a Riemann mapping with the normalization $\Phi(\infty) = \infty$ and $\Phi(1) = 0$, say, then from Theorem 3.9 of [21] we obtain that $\Phi(w)/(w-1)^{2\beta}$ is (well-defined and) continuous in $(\mathbb{C}_\infty \setminus \mathbb{D}) \cap U$ for some neighbourhood U of 1. This implies that, for a positive constant d,

$$|\Phi^{-1}(\zeta) - 1| \sim d|\zeta|^{1/(2\beta)} \qquad \text{near } 0.$$

Let $F \in A(K)$, F even, with given $E_n(F,K) = \mathcal{O}(n^{-\alpha\beta})$. We define $H \in A(K^2)$ by

$$H(\zeta) := H(z^2) := F(z), \qquad z \in K_+.$$

Note that $z \mapsto z^2$ is one-to-one on K_+. Let (P_n^*) be the sequence of best approximating polynomials of degree not greater than n to F on K. Since f is even and K equals $-K$, the polynomials P_n^* are even. Writing $Q_n(z^2) = P_n^*(z)$ where Q_n is a polynomial of degree $n/2$, we conclude that

$$\begin{aligned}\sup_{\zeta\in K^2} |H(\zeta) - Q_n(\zeta)| &= \sup_{z\in K^+} |F(z) - P_n^*(z)| \\ &= \sup_{z\in K} |F(z) - P_n^*(z)| = E_n(F,K)\end{aligned}$$

and thus, by our assumption,

$$E_n(H,K^2) = \mathcal{O}\left(\frac{1}{n^{\alpha\beta}}\right).$$

According to Theorem 3 of [1], applied with 0 instead of 1, the function H satisfies

$$\frac{|H(\zeta) - H(0)|}{|\zeta|^{s/2}} \to 0, \qquad \zeta \to 0,\ \zeta \in K^2,$$

for all $s < \alpha$, which implies

$$\frac{|F(z) - F(0)|}{|z|^{s}} \to 0, \qquad z \to 0,\ z \in K,$$

for all $s < \alpha$. ■

Remark 1. Theorem 2 implies, in particular, that, for the functions F_α considered above, the rate

$$E_n(F_\alpha,K) = \mathcal{O}\left(\frac{1}{n^{\alpha\beta}}\right)$$

cannot be improved, for K with exterior angle β at 0 and $\alpha \le 1$, in the sense that

$$\sup_{n\in\mathbb{N}} n^\delta E_n(F_\alpha,K) = \infty$$

for all $\delta > \alpha\beta$. Using Corollary 2 from [7] instead of Theorem 3 from [1], it may be derived in a similar way as in the proof of Theorem 2 that in the case $\alpha < 1$ and $\beta < 1$ we actually have

$$E_n(F_\alpha,K) \ge \frac{c}{n^{\alpha\beta}}$$

for some positive constant c.

In [12], the case $\alpha > 1$ was also considered for the functions F_α. We shall prove a similar result for arbitrary functions satisfying a certain smoothness condition on the boundary of K and for more general sets K. We note that a path is understood to be a piecewise continuously differentiable curve.

Theorem 3. *Let K be the closure of $G_+ \cup G_-$ with G_+, G_- as above and exterior angle $\pi\beta$, $\beta \in (0,1]$ at the origin. If $p \in \mathbb{N}$ and if $\alpha < 1$, then there exists a constant $c_p = c_p(K, \alpha\beta)$ such that*

$$E_n(F,K) \le \frac{c_p M}{n^{\beta(p+\alpha)}}$$

for all $n > p$ and all $F \in A(K)$ with $F^{(p)}$ extending continuously to K and $F^{(p)} \in \mathrm{Lip}(\alpha)$ on $\Gamma = \partial K$.

Proof. Let c be as in Theorem 1. We show that there exists a constant d such that

$$E_{n+1}(F,K) \le cd\frac{E_n(F',K)}{n^\beta}. \tag{4}$$

Then the assertion follows by induction on p and application of Theorem 1, since if $F^{(p)}$ extends continuously to K then the same is true for $F^{(0)}, \dots, F^{(p-1)}$.

Let P_n^* be the best approximating polynomial of degree not greater than n to F' on K. We define

$$F_1(z) = F_{1,n}(z) := \int_0^z (F'(\zeta) - P_n^*(\zeta))\, d\zeta, \qquad z \in K,$$

where the integration is performed along an arbitrary path from 0 to z in K. Then $F_1 \in A(K)$ and

$$F_1(z) = F(z) - Q_{n+1}(z), \qquad z \in G_+ \cup G_-,$$

with a polynomial $Q_{n+1} \in \Pi_{n+1}$. Moreover, $F_1' = F' - P_n^*$ in $G_+ \cup G_-$ and, by continuous extension, also on K.

It can be shown that a constant $d > 0$ exists such that, for all $z, z' \in K$, there exists a path γ in K connecting z and z' with the property that the length of γ is bounded by $d|z - z'|$. Note that vanishing exterior angles do not occur. Since

$$\|F_1'\|_K \le E_n(F', K),$$

this implies that

$$|F_1(z) - F_1(z')| \le dE_n(F',K)|z - z'|$$

for all $z, z' \in K$, and in particular $F_1 \in \mathrm{Lip}_{M_1}(1)$ with $M_1 := dE_n(F',K)$. Since F_1 and F differ only by a polynomial of degree $\le n+1$, the Corollary 1 implies (4). ∎

4. Approximation on Jordan arcs

In this section, we consider another — for the Faber operator non-classical — situation: we suppose that $K = \partial K = \Gamma$ is a Jordan arc so that

$$\Gamma = \bigcup_{j=1}^{2\ell} \gamma_j$$

where the γ_j are Jordan arcs as in Section 2 with $\gamma_{j+\ell} = \gamma_j$ for $j = 1, \dots, \ell$ (each γ_j is traversed twice by $\psi(e^{it})$ for $t \in [t_0, t_0 + 2\pi]$). If the γ_j are Dini-smooth, K is now itself a piecewise Dini-smooth arc and we again suppose that it has parameter $\beta > 0$ as in Section 2. We note that at the endpoints $\psi(e^{it_\ell})$ and $\psi(e^{it_{2\ell}})$ of Γ the exterior angles are 2π, that is, $\beta_\ell = \beta_{2\ell} = 2$, so that these cusps behave nicely and do not contribute to the value of β.

Again, from the Corollary 1 we obtain the existence of a constant $c = c(K, \alpha\beta)$ with

$$E_n(F, \Gamma) = \frac{c}{n^{\alpha\beta}} \tag{5}$$

for all $F \in \mathrm{Lip}_1(\alpha, \Gamma)$. Note that $A(K) = C(\Gamma)$ now.

Example 1. Let Γ be the union of two line segments joining at the origin under an angle $\pi\beta$ of the form

$$\Gamma_\beta = [0,1]e^{\pi i\beta/2} \cup [0,1]e^{-\pi i\beta/2}$$

for some $\beta \in (0,1)$. Then Γ is piecewise Dini-smooth with parameter β, so that (5) holds.

If the Jordan arc Γ is Dini-smooth (without corners), then we have $\beta = 1$ and (5) implies the existence of a constant $c = c(\Gamma, \alpha)$ with

$$E_n(F, \Gamma) = \frac{c}{n^{\alpha}} \tag{6}$$

for all $F \in \mathrm{Lip}_1(\alpha, \Gamma)$.

Provided that (6) holds — in the case of an arbitrary Jordan arc — only for $\alpha = 1$, it is known, see for example [20] or [6], that a constant $d = d(\Gamma)$ exists with

$$E_n(F, \Gamma) \le d \cdot \omega\left(F, \frac{1}{n}\right) \tag{7}$$

for all $F \in C(\Gamma)$, where $\omega(F, \cdot)$ denotes the modulus of continuity of F on Γ. An important feature here is that the constant c in (6) is independent of F.

A Jordan arc Γ is said to have the *Jackson-property* (J-property for brevity) if (7) holds for all $F \in C(\Gamma)$, in other words, if the analogue of Jackson's First Theorem holds for Γ. Several papers have investigated necessary or sufficient conditions for a Jordan arc Γ to have the J-property. Newman [20] has shown that $\Gamma \in C^{1+\delta}$ for an arbitrary positive δ is sufficient for the J-property and he posed the problem of characterizing the arcs Γ having the J-property. Andersson [3] proved in particular that arcs as in Example 1 do not have the J-property. Moreover, it is known that smoothness is not sufficient, see [16]. From (6) we see that a little more, namely Dini-smoothness, turns out to be sufficient.

Maimeskul [17] has proved that $\psi \in \mathrm{Lip}(1, \partial\mathbb{D})$ is necessary for Γ to have the J-property, and now it is conjectured that this is also sufficient (cf. [18], [1]). As a consequence of Theorem 1 and the above remarks we have the following result.

Corollary 2. *If the Jordan arc* Γ *is a Faber set with* $\psi \in \mathrm{Lip}(1, \partial\mathbb{D})$ *then* Γ *has the* J*-property.*

We briefly turn to the question of sharpness of (5) or, in other words, to Bernstein-type results. We apply the following theorem from [3].

Theorem A. *If* K *is an arbitrary compact plane set (but not a single point) such that* $\mathbb{C}_\infty \setminus K$ *is a simply connected region, and if, for some* $F \in A(K)$ *and some* $s \in (0,1]$,

$$E_n(F,K) = \mathcal{O}\left(\frac{1}{n^s}\right)$$

then the Cauchy integral f *of* $F \circ \psi_{|\partial\mathbb{D}}$ *extends continuously to* $\overline{\mathbb{D}}$*. Furthermore,* $f \in \mathrm{Lip}(s-\varepsilon, \partial\mathbb{D})$*, for all* $\varepsilon > 0$.

If $K = \Gamma$ is a Jordan arc which is symmetric with respect to the real axis and if we choose in particular $F(z) = \overline{z}^p$, where $z \in \Gamma$ and p is a positive integer, then

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\overline{\psi}^p(\zeta)}{\zeta - w}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\psi^p(\frac{1}{\zeta})}{\zeta - w}\, d\zeta = \psi^p\left(\frac{1}{w}\right) - R\left(\frac{1}{w}\right) \end{aligned}$$

for $0 < |w| \le 1$, where R is a polynomial of degree $\le p$. If now Γ is piecewise Dini-smooth with parameter $\beta \in (0,1)$, then $\psi \in \mathrm{Lip}(\beta, \partial\mathbb{D})$ but $\psi \notin \mathrm{Lip}(\tilde{\beta}, \partial\mathbb{D})$ for all $\tilde{\beta} > \beta$ (this follows e.g. from Theorem 3.9 in [21]). Thus, if $p\beta < 1$ then $\psi^p \notin \mathrm{Lip}(s, \partial\mathbb{D})$ for all $s > p\beta$.

If, in particular, Γ_β is as in Example 1 with $\beta \in (0,1)$, and if $G_\alpha \in \mathrm{Lip}(\alpha, \Gamma_\beta)$, with $G_\alpha(z) = \overline{z}^\alpha$ on Γ_β, where $\alpha = p/q$ is a rational number so that $\alpha\beta < 1$, then

$$E_n(G_\alpha, \Gamma_\beta) = \mathcal{O}\left(\frac{1}{n^{\alpha\beta}}\right)$$

cannot be improved in the sense that

$$\sup_{n\in\mathbb{N}} n^\delta E_n(G_\alpha, \Gamma_\beta) = \infty$$

for all $\delta > \alpha\beta$. This follows from Theorem A and the fact that

$$\sup_{z\in\Gamma_\beta} |\overline{z}^\alpha - P(z)| = \sup_{\zeta\in\Gamma_{\beta/q}} |\overline{\zeta}^p - P(\zeta^q)|$$

for all polynomials P. Note that $\Gamma_{\beta/q}$ has parameter β/q.

Finally, we remark that a Bernstein-type result similar to Theorem 3 from [1] or to Theorem 2 cannot be expected here. To see this, let us consider, for $\alpha < 1$, the function z^α, coinciding with $F_\alpha(z)$ in the right half plane, instead of G_α. We put

$$K := \left\{ z = re^{i\theta} : 0 \le r \le 1, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \right\}.$$

By taking into account the local behaviour of the Riemann mapping

$$\psi\colon \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus K,$$

we may obtain from Proposition 1 that

$$E_n(F_\alpha, \Gamma_{1/2}) \leq E_n(F_\alpha, K) = \mathcal{O}\left(\frac{1}{n^{3\alpha/2}}\right),$$

although F_α satisfies the same Hölder condition at the origin as G_α. In order to obtain inverse results it is necessary to impose additional conditions, for example on the growth of derivatives of the approximating polynomials, see e.g. [4].

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