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Conformality of the Apollonian Metric

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Abstract. The Apollonian metric a_D of a domain $D \subset \overline{\mathbb{R}}^n$ is rarely conformal. In fact, if it is conformal at one point then D is, up to a Möbius transformation, a complement of a convex body of constant width and if it is conformal at two points then D is a ball. We consider a quantity that measures the deviation of a_D from being conformal. This quantity is essential in comparing the Apollonian metric to hyperbolic and quasihyperbolic metrics. We show that this quantity is invariant under Möbius transformations and compute it for some standard domains. We then use it to obtain sharp estimates between any two of the Apollonian, hyperbolic and quasihyperbolic metrics on such domains.

Keywords. Apollonian metric, hyperbolic metric, quasihyperbolic metric, Möbius maps.

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1. Introduction

This paper is a continuation of our earlier work on the Apollonian metric [I1]. There we showed that for any domain $D \subset \overline{\mathbb{R}}^n$, the Apollonian metric a_D of D is conformal either at every point or at only one point or at no point of D [I1, Theorem 2]. Moreover, we established that the first two possibilities take place if and only if the domain D is, up to a Möbius transformation, the complement of a ball or a convex body of constant width, respectively [I1, Theorem 3]. The relations between the convex bodies of constant width and the Apollonian metric is studied more closely in [I2]. See [ChG] for more about convex bodies of constant width.

In this paper we consider a quantity $H_D(x) \in [1, +\infty]$ that measures the deviation of the Apollonian metric a_D from being conformal at the point x, where $H_D(x) = 1$ corresponds to a_D being conformal at x. Hence, according to [I1, Theorem 2] if D is not, up to a Möbius transformation, a complement of

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a ball or a convex body of constant width, then $H_D(x) > 1$ for all $x \in D$. Despite this fact, for many domains D one has

$$\inf_{x \in D} H_D(x) = 1. \tag{1.1}$$

Domains for which (1.1) holds are important in the light of the following observation.

Let $f\colon D \to D'$ be an Apollonian isometry, i.e., f is a homeomorphism with

$$a_{D'}(f(x), f(y)) = a_D(x, y) \qquad \text{for all } x, y \in D.$$

If a_D is conformal at $x \in D \setminus \{\infty, f^{-1}(\infty)\}$ (that is, $H_D(x) = 1$) then one can prove that f is 1-quasiconformal at x, i.e.,

$$\limsup_{r \to 0} \frac{\max\{|f(x) - f(y)| : |x - y| = r\}}{\min\{|f(x) - f(y)| : |x - y| = r\}} = 1. \tag{1.2}$$

The proof makes use of the fact that ∂D has a very simple geometry when viewed from the point x. But ∂D will still have simple geometry when viewed from a point x with $H_D(x) = 1 + \epsilon$ for small enough $\epsilon > 0$. If one shows that (1.2) holds also for such points then (1.2) holds for each point of some open subset U of D and the restriction of f to U is a Möbius transformation by Liouville's Theorem (see, for instance, [Ge] and [R]). This would be an important step in proving Beardon's Conjecture which says that all the Apollonian isometries are the restrictions of Möbius transformations. These ideas are being developed in [I2].

In Section 2 we show that the quantity $H_D(x)$ is invariant under Möbius transformations of $\overline{\mathbb{R}}^n$ and provide four types of domains for which (1.1) holds. In general, computing $H_D(x)$ is rather complicated due to the fact that its definition uses limiting process. To compensate for this difficulty we give very simple characterization of $H_D(x)$ in terms of a purely geometric quantity that uses neither limits nor the Apollonian metric a_D of D. This quantity plays an important role in studying the relations between any two of the Apollonian, hyperbolic and quasihyperbolic metrics which will be discussed in Section 3.

Finally, when this article was being prepared for publication it came to the author's attention that Peter Hästö has established results similar to those in Examples 2.8 and 2.9.

2. Möbius invariance

We denote by $\mathbb{R}^n$ the n-dimensional euclidean space and by $\{e_1, e_2, \ldots, e_n\}$ its standard basis, $n \geq 2$. The set of unit vectors in $\mathbb{R}^n$ is denoted by $\mathbb{V}^n$. The open and closed balls of radius $r > 0$ and centered at $x \in \mathbb{R}^n$ are denoted by $B^n(x, r)$ and $\overline{B}^n(x, r)$, respectively. The sphere of radius $r > 0$ and centered at $x \in \mathbb{R}^n$ is denoted by $S^{n-1}(x, r)$. The unit ball and the unit sphere in $\mathbb{R}^n$ are denoted by $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$, respectively. The upper-half space in $\mathbb{R}^n$ is denoted by $\mathbb{H}^n$. At times we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

The Möbius space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^n$. The Möbius space $\overline{\mathbb{R}}^n$, equipped with the *chordal distance*

$$\chi(x,y) := \begin{cases} \dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} & \text{if } x,y \in \mathbb{R}^n, \\ \dfrac{1}{\sqrt{1+|x|^2}} & \text{if } y=\infty, \end{cases}$$

is a metric space.

The *cross-ratio* of a quadruple a,b,c,d of points in $\overline{\mathbb{R}}^n$ with $a \neq b$ and $c \neq d$ is defined by

$$|a,b,c,d| := \frac{\chi(a,c)\chi(b,d)}{\chi(a,b)\chi(c,d)} = \frac{|a-c||b-d|}{|a-b||c-d|},$$

where we use the convention that

$$\frac{|x-\infty|}{|y-\infty|} := 1 \qquad \text{for all } x,y \in \mathbb{R}^n.$$

The group of Möbius transformations of $\overline{\mathbb{R}}^n$ is denoted by $\text{Möb}(\overline{\mathbb{R}}^n)$. A homeomorphism $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ belongs to $\text{Möb}(\overline{\mathbb{R}}^n)$ if and only if

$$|f(a),f(b),f(c),f(d)| = |a,b,c,d|$$

for all quadruples a,b,c,d in $\overline{\mathbb{R}}^n$ [B1, Theorem 3.2.7]. By a sphere or a ball in $\overline{\mathbb{R}}^n$ we mean the image of $\mathbb{S}^{n-1}$ or $\mathbb{B}^n$ under a Möbius transformation of $\overline{\mathbb{R}}^n$.

Throughout the paper we assume that $D \subset \overline{\mathbb{R}}^n$ is a domain, where ∂D contains at least three points. The Apollonian distance $a_D(x,y)$ between the points x and y in D is defined as

$$a_D(x,y) := \max_{w,z\in\partial D} \log \frac{|x-w||y-z|}{|x-z||y-w|}.$$

We say that a_D is *conformal* at $x \in D$ if the following limit

$$\lim_{y\to x} \frac{a_D(y,x)}{\chi(y,x)}$$

exists. In general, this limit does not always exist. Therefore, it is natural to consider for each $x \in D$ the quantity

$$H_D(x) := \frac{L_D(x)}{l_D(x)}$$

where

$$L_D(x) := \limsup_{y\to x} \frac{a_D(x,y)}{\chi(x,y)} \qquad \text{and} \qquad l_D(x) := \liminf_{y\to x} \frac{a_D(x,y)}{\chi(x,y)}.$$

Hence

$$1 \leq H_D(x) \leq \infty$$

for each $x \in D$ and a_D is conformal at x if and only if $H_D(x) = 1$.

First we establish the invariance of $H_D(x)$ under Möbius transformations.

Theorem 2.1. *Let $D \subset \overline{\mathbb{R}}^n$ be a domain and $f \in \text{Möb}(\overline{\mathbb{R}}^n)$. Then*

$$H_{f(D)}(f(x)) = H_D(x)$$

for all $x \in D$.

Proof. According to Proposition 7.2 in [I1], the quantity

$$\lambda := \lim_{y \to x} \frac{\chi(f(x), f(y))}{\chi(x, y)}$$

is a finite non-zero number. Let $\{y_k\}$ be a sequence in D converging to x such that

$$L_{f(D)}(f(x)) = \lim_{y_k \to x} \frac{a_{f(D)}(f(x), f(y_k))}{\chi(f(x), f(y_k))}.$$

Then

$$\lim_{y_k \to x} \frac{a_D(x, y_k)}{\chi(x, y_k)} = \lim_{y_k \to x} \frac{a_{f(D)}(f(x), f(y_k))}{\chi(f(x), f(y_k))} \cdot \frac{\chi(f(x), f(y_k))}{\chi(x, y_k)} = \lambda L_{f(D)}(f(x)).$$

Hence

$$L_D(x) \geq \lambda L_{f(D)}(f(x)).$$

In a similar fashion we obtain

$$l_D(x) \leq \lambda l_{f(D)}(f(x))$$

and consequently

$$H_D(x) \geq H_{f(D)}(f(x)).$$

By applying the above argument to f^{-1} we obtain $H_D(x) \leq H_{f(D)}(f(x))$ as required. ∎

Remark 2.2. It is evident from the proof of Theorem 2.1 that neither of the quantities $L_D(x)$ and $l_D(x)$ are invariant under Möbius transformations.

To make the computations of $H_D(x)$ easier, we next give a characterization of $H_D(x)$ in terms of a purely geometric quantity $\mu_D(x, u)$ defined as follows. Recall that if E is a compact set in $\mathbb{R}^n$, then for each $u \in \mathbb{V}^n$ there exists a unique support hyperplane of E with an outer normal vector u (see, for instance, [Sch, p. 11]). The width of E in the direction of u is defined to be

$$\text{width}(E, u) := \text{dist}(P_1, P_2),$$

where P_1 and P_2 are the support hyperplanes of E with outer normal vectors u and $-u$, respectively. Observe that

$$0 \leq \text{width}(E, u) \leq \text{diam}(E) \qquad \text{and} \qquad \text{width}(E, -u) = \text{width}(E, u).$$

Now for each $u \in \mathbb{V}^n$ we define

$$\mu_D(x,u) := \begin{cases} \operatorname{width}(\overline{\mathbb{R}}^n \setminus D, u) & \text{if } x = \infty, \\ \operatorname{width}(\overline{\mathbb{R}}^n \setminus i_x(D), u) & \text{if } x \neq \infty, \end{cases}$$

where $i_x \in \text{Möb}(\overline{\mathbb{R}}^n)$ is the inversion in $S^{n-1}(x,1)$. It immediately follows from the definition that $\mu_D(x,-u) = \mu_D(x,u)$ and that $\mu_D(x,u)$ is monotonic with respect to D, i.e., if $D \subset D'$ then

$$\mu_D(x,u) \geq \mu_{D'}(x,u)$$

for all $x \in D$ and $u \in \mathbb{V}^n$. Observe also that $B^n(x, \operatorname{dist}(x,\partial D)) \subset D$ for $x \neq \infty$ and using monotonicity we obtain

$$0 \leq \mu_D(x,u) \leq \begin{cases} \dfrac{2}{\operatorname{dist}(x,\partial D)} & \text{if } x \neq \infty \\ \operatorname{diam}(\partial D) & \text{if } x = \infty \end{cases} \tag{2.1}$$

for each $x \in D$ and for each $u \in \mathbb{V}^n$.

Theorem 2.3. *For each $x \in D$, we have*

$$H_D(x) = \frac{\max_{u \in \mathbb{V}^n} \mu_D(x,u)}{\min_{u \in \mathbb{V}^n} \mu_D(x,u)}.$$

Proof. We will first show that if L is a line passing through x and if $u \in \mathbb{V}^n$ is its direction vector then

$$\lim_{y \to x} \frac{a_D(y,x)}{\chi(y,x)} = \begin{cases} (1+|x|^2)\mu_D(x,u) & \text{if } x \neq \infty, \\ \mu_D(x,u) & \text{if } x = \infty, \end{cases} \tag{2.2}$$

where y varies over the line L.

Assume first that $x = \infty$. Then $\mu_D(\infty,u) = \operatorname{width}(\overline{\mathbb{R}}^n \setminus D, u)$ and

$$\begin{aligned} \lim_{y \to \infty} \frac{a_D(y,\infty)}{\chi(y,\infty)} &= \lim_{y \to \infty} \left(\sqrt{1+|y|^2} \log \frac{\max\{|y-w| : w \in \partial D\}}{\min\{|y-w| : w \in \partial D\}} \right) \\ &= \lim_{y \to \infty} \left(|y| \log \frac{\max\{|y-w| : w \in \partial D\}}{\min\{|y-w| : w \in \partial D\}} \right) \\ &= \operatorname{width}(\overline{\mathbb{R}}^n \setminus D, u) \end{aligned}$$

by [I1, Lemma 7.6].

Now assume that $x \neq \infty$. Let $f \in \text{Möb}(\overline{\mathbb{R}}^n)$ be the inversion in $S^{n-1}(x,1)$. Then

$$\lim_{y \to \infty} \frac{a_{f(D)}(y,\infty)}{\chi(y,\infty)} = \frac{1}{1+|x|^2} \lim_{y \to x} \frac{a_D(y,x)}{\chi(y,x)}$$

by Propositions 7.2 and 7.3 in [I1] and hence we have

$$\lim_{y \to x} \frac{a_D(y,x)}{\chi(y,x)} = (1+|x|^2)\mu_{f(D)}(\infty,u) = (1+|x|^2)\mu_D(x,u)$$

as required.

Now let $\{y_k\}$, $y_k \in D$, be any sequence converging to x. Then there exists a subsequence of $\{y_k\}$, also denoted by $\{y_k\}$, such that the sequence of vectors $(y_k - x)/|y_k - x| \in \mathbb{V}^n$ ($y_k/|y_k|$ if $x = \infty$) converges to some $u \in \mathbb{V}^n$. Let L be a line passing through x and parallel to u. Then using (2.2) along with the continuity argument one can show that

$$\lim_{k\to\infty} \frac{a_D(y_k, x)}{\chi(y_k, x)} = \begin{cases} (1+|x|^2)\mu_D(x,u) & \text{if } x \neq \infty, \\ \mu_D(x,u) & \text{if } x = \infty, \end{cases}$$

which completes the proof of the theorem. ∎

The following lemma is a consequence of Theorems 2.1 and 2.3.

Lemma 2.4. *Let D be an image of a convex domain in $\mathbb{R}^n$ under a Möbius transformation. Then for each $x \in D$ we have*

$$H_D(x) \leq 2.$$

This inequality is sharp.

Proof. Using Theorem 2.1 we can assume that D is a convex domain in $\mathbb{R}^n$. Let $\zeta \in \partial D$ be a point with $\operatorname{dist}(x, \partial D) = |x - \zeta|$ and let D_ζ be a half-space in $\mathbb{R}^n$ determined by the tangent hyperplane to ∂D at ζ and containing D. Then the Apollonian metric a_{D_ζ} of D_ζ coincides with its quasihyperbolic metric k_{D_ζ} [B2, Lemma 3.1] and $\operatorname{dist}(x, \partial D) = \operatorname{dist}(x, \partial D_\zeta)$. Hence by monotonicity of $\mu_D(x,u)$ along with (2.2) we obtain that

$$\mu_D(x,u) \geq \mu_{D_\zeta}(x,u) = \lim_{y\to x} \frac{a_{D_\zeta}(x,y)}{|x-y|} = \lim_{y\to x} \frac{k_{D_\zeta}(x,y)}{|x-y|} = \frac{1}{\operatorname{dist}(x,\partial D)}.$$

On the other hand, according to (2.1) we have

$$\mu_D(x,u) \leq \frac{2}{\operatorname{dist}(x,\partial D)}.$$

Using Theorem 2.3 we complete the proof. Equality holds for the domain in Example 2.7. ∎

Recall that the Apollonian distance function constitutes a metric if and only if the boundary of the underlying domain is not a proper subset of a sphere in $\overline{\mathbb{R}}^n$ [B2, Theorem 1.1]. The following lemma characterizes such domains in terms of the quantity H_D.

Lemma 2.5. *Let $D \subset \overline{\mathbb{R}}^n$ be a domain. Then $H_D(x) = \infty$ for some $x \in D$ if and only if the boundary ∂D of D is a proper subset of a sphere in $\overline{\mathbb{R}}^n$.*

Proof. Assume that $H_D(x) = \infty$ for some $x \in D$. Using Theorem 2.1 we can assume that $x = \infty$. Then Theorem 2.3 along with (2.1) implies that $\min_{u\in\mathbb{V}^n} \mu_D(\infty, u) = \min_{u\in\mathbb{V}^n} \mathrm{width}(\overline{\mathbb{R}}^n \setminus D, u) = 0$ for some $u \in \mathbb{V}^n$, i.e., the boundary ∂D of D is properly contained in a hyperplane orthogonal to u.

Now assume that ∂D is a proper subset of a sphere S in $\overline{\mathbb{R}}^n$ which we can assume to be of the form $P \cup \{\infty\}$, where P is a hyperplane in $\mathbb{R}^n$. By means of a preliminary Möbius transformation, if necessary, we can assume that $\infty \in D$. Then $\mathrm{width}(\overline{\mathbb{R}}^n \setminus D, u) = 0$, where $u \in \mathbb{V}^n$ is a normal of P. Then Theorem 2.3 implies that $H_D(\infty) = \infty$. ■

We conclude this section by computing the quantities

$$\inf_{x\in D} H_D(x) \qquad \text{and} \qquad \sup_{x\in D} H_D(x)$$

for some standard domains D. Computations for these examples are straightforward, but are very long and tedious. Details are left to the reader.

Example 2.6. Let $\mathbb{B}^n$ be the unit ball. For each $x \in \mathbb{B}^n$, we have

$$\mu_{\mathbb{B}^n}(x,u) = \frac{2}{1-|x|^2} = \frac{2}{1+|x|}\cdot\frac{1}{\mathrm{dist}(x,\partial\mathbb{B}^n)}.$$

In particular, $H_{\mathbb{B}^n}(x) = 1$.

Example 2.7. Let $D_0 = \{x \in \mathbb{R}^n \colon\ 0 < x_1 < 1\}$ be a parallel strip. For each $x \in D_0$, we have

$$\frac{1}{\mathrm{dist}(x,\partial D_0)} \le \mu_{D_0}(x,u) \le \frac{2}{\mathrm{dist}(x,\partial D_0)}.$$

In particular,

$$\inf_{x\in D_0} H_{D_0}(x) = 1 \qquad \text{and} \qquad \sup_{x\in D_0} H_{D_0}(x) = 2.$$

Example 2.8. Let $r \in (0,1)$ and $D_r = \{x \in \mathbb{R}^n : r < |x| < 1\}$ be an annulus. For each $x \in D_r$, we have

$$\min\left\{\frac{1}{2}, \frac{2\sqrt{r}}{1+\sqrt{r}}\right\}\cdot\frac{1}{\mathrm{dist}(x,\partial D_r)} \le \mu(x,u) \le \frac{2}{\mathrm{dist}(x,\partial D_r)}.$$

In particular,

$$\inf_{x\in D_r} H_{D_r}(x) = 1 \qquad \text{and} \qquad \sup_{x\in D_r} H_{D_r}(x) = \frac{(1+\sqrt{r})^2}{2}\sqrt{r}.$$

Example 2.9. Let $0 < \alpha < 2\pi$ and $D_\alpha = \{z \in \mathbb{C} : 0 < \arg z < \alpha\}$ be an angular sector in $\mathbb{C}$. Let $z \in D_\alpha$ be given. If $0 < \alpha \le \pi$ then

$$\frac{1}{\mathrm{dist}(z,\partial D_\alpha)} \le \mu_{D_\alpha}(z,u) \le \frac{2\cos^2(\frac{\alpha}{4})}{\mathrm{dist}(z,\partial D_\alpha)}.$$

In particular,

$$\inf_{z\in D_\alpha} H_{D_\alpha}(z) = 1 \qquad \text{and} \qquad \sup_{z\in D_\alpha} H_{D_\alpha}(z) = 2\cos^2\Big(\frac{\alpha}{4}\Big).$$

If $\pi \le \alpha < 2\pi$ then

$$\frac{\cot(\frac{\alpha}{4})}{\mathrm{dist}(z,\partial D_\alpha)} \le \mu_{D_\alpha}(z,u) \le \frac{1}{\mathrm{dist}(z,\partial D_\alpha)}.$$

In particular,

$$\inf_{z\in D_\alpha} H_{D_\alpha}(z) = 1 \qquad \text{and} \qquad \sup_{z\in D_\alpha} H_{D_\alpha}(z) = \tan\Big(\frac{\alpha}{4}\Big).$$

3. Apollonian and quasihyperbolic metrics

In this section we study the equivalence of the Apollonian and quasihyperbolic metrics on domains in $\mathbb{R}^n$. We start with the following proposition.

Proposition 3.1. *If*

$$\mu_D(x,u) \le \frac{K}{\mathrm{dist}(x,\partial D)}$$

for all $x \in D$ then

$$a_D(x_1,x_2) \le K k_D(x_1,x_2)$$

for all $x_1, x_2 \in D$.

Proof. Using (2.2) we have

$$\lim_{r\to 0}\frac{a(x,x+ru)}{|r|} = \mu_D(x,u) \le K\frac{1}{\mathrm{dist}(x,\partial D)}.$$

Hence for each $\epsilon > 0$ there exists $r = r(x,\epsilon) > 0$ such that

$$a(x,y) < K\frac{|x-y|}{\mathrm{dist}(x,\partial D)} + \epsilon|x-y|$$

for all $y \in D$ with $|x-y| < r$. Let Γ be the quasihyperbolic geodesic joining x_1 and x_2 in D. The existence and rectifiability of such Γ is established in [GeO]. Let $\epsilon > 0$ be fixed. Then for each $x \in \Gamma$ there exists $r(x) > 0$ such that

$$a(x,y) < K\frac{|x-y|}{\mathrm{dist}(x,\partial D)} + \epsilon|x-y|$$

for all $y \in \Gamma$ with $|x-y| < r(x)$. Due to compactness of Γ we have

$$\min\{r(x) : x \in \Gamma\} > 0.$$

Since

$$k(x_1,x_2) = \int_\Gamma \frac{|dx|}{\mathrm{dist}(x,\partial D)},$$

there exists $q > 0$ such that

$$k(x_1, x_2) > \sum_{j=0}^{n} \frac{|\xi_j - \xi_{j+1}|}{\mathrm{dist}(\xi_j, \partial D)} - \epsilon$$

for all partitions $\{x_1 = \xi_0, \xi_1, \ldots, \xi_n = x_2\}$ of Γ with

$$\max\{|\xi_j - \xi_{j+1}| : \ 0 \le j \le n\} < q.$$

Put $\delta = \min\{q, \min\{r(x) : x \in \Gamma\}\}$ and let $\{x_1 = \xi_0, \xi_1, \ldots, \xi_n = x_2\}$ be a partition of Γ with $\max\{|\xi_j - \xi_{j+1}| : \ 0 \le j \le n\} < \delta$. Then we have

$$a(x_1, x_2) \le \sum_{j=0}^{n} a(\xi_j, \xi_{j+1}) < K \sum_{j=0}^{n} \frac{|\xi_j - \xi_{j+1}|}{\mathrm{dist}(\xi_j, \partial D)} + \epsilon \cdot \mathrm{length}(\Gamma)$$

and hence

$$Kk(x_1, x_2) > K \sum_{j=0}^{n} \frac{|\xi_j - \xi_{j+1}|}{\mathrm{dist}(\xi_j, \partial D)} - K\epsilon > a(x_1, x_2) - (\mathrm{length}(\Gamma) + K)\,\epsilon,$$

which completes the proof since ϵ is arbitrary and Γ is rectifiable. ■

Remark 3.2. Using Proposition 3.1 and (2.1) we obtain that $a_D \le 2k_D$ for any domain $D \subset \mathbb{R}^n$. Another proof of this result has been given by Beardon [B2, Theorem 3.2].

Corollary 3.3. *If the complement of a domain $D \subset \mathbb{R}^n$ is convex then $a_D \le k_D$.*

Proof. Indeed, given $x \in D$ let $\zeta \in \partial D$ be such a point that $\mathrm{dist}(x, \partial D) = |x - \zeta|$. Let D_ζ be a half space determined by the hyperplane tangent to ∂D at ζ and contained in D. Then by the monotonicity property we obtain

$$\mu_D(x, u) \le \mu_{D_\zeta}(x, u) = \frac{1}{\mathrm{dist}(x, \partial D)}.$$

Then the result follows from Proposition 3.1. ■

Next we turn to estimating the Apollonian metric by the quasihyperbolic metric from below by presenting such an estimate for quasiballs (see, Theorem 3.8). For quasidisks this result was proved by Gehring and Hag [GeH2]. Both our result and our method are straightforward generalisations to higher dimensions of their result and method, but we lose the sharpness of their estimate. We begin with the following proposition whose proof is a direct extension of the proof of Lemma 2.1 in [GeH2].

Proposition 3.4. *Suppose that $K \ge 1$ and $c \ge 1$ and let $b = K^{1/(n-1)}$. If $u > 0$ and $v + 1 \le 2c^b(u+1)^b$ then*

$$v \le (2c)^{2b} \max\{u^b, u^{-b}\}.$$

Lemma 3.5. *If f is a K-quasiconformal self-mapping of $\overline{\mathbb{R}}^n$ and if D is a proper subdomain of $\overline{\mathbb{R}}^n$ then for all $x, y \in D$ we have*

$$a_{f(D)}(f(x), f(y)) \leq b \cdot a_D(x, y) + c$$

with

$$b = K^{1/(n-1)} \qquad \text{and} \qquad c = 4b \log(2\lambda_n),$$

where λ_n is a constant depending only on n.

Proof. By performing a preliminary Möbius transformation we may assume that D and $f(D)$ are in $\mathbb{R}^n$ and $f(\infty) = \infty$. Fix $x, y \in D$ and choose $p, q \in \partial D$ so that

$$a_{f(D)}(f(x), f(y)) = \log \frac{|f(x) - f(p)||f(q) - f(y)|}{|f(x) - f(q)||f(p) - f(y)|}.$$

Then by Lemma 3 [GeO] we have $v + 1 \leq 2(2{\lambda_n}^2)^b (u+1)^b$, where

$$v = \frac{|f(x) - f(p)||f(q) - f(y)|}{|f(x) - f(q)||f(p) - f(y)|} \qquad \text{and} \qquad u = \frac{|x-p||q-y|}{|x-q||p-y|}.$$

Hence $v \leq (4{\lambda_n}^2)^{2b} \max\{u^b, u^{-b}\}$ whence $\log v \leq b \cdot \log |u| + c \leq b \cdot a_D(x, y) + c$ by Proposition 3.4. ∎

Lemma 3.6. *If $D \subset \overline{\mathbb{R}}^n$ is a K-quasiball then*

$$k_D(x, y) \leq bda_D(x, y) + cd$$

for all $x, y \in D$, where $b = K^{1/(n-1)}$, $c = 4b \log(2\lambda_n)$ and $d = 4(2\lambda_n)^{2b}$.

Proof. Let $f \colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal map such that $f(\mathbb{B}^n) = D$. Then using Lemma 3.5 and Theorem 3 of [GeO] we have

$$\begin{aligned} k_D(x, y) &= k_{f(\mathbb{B}^n)}(f(f^{-1}(x)), f(f^{-1}(y))) \\ &\leq d \cdot \max\left\{k_{\mathbb{B}^n}(f^{-1}(x), f^{-1}(y)), \left[k_{\mathbb{B}^n}(f^{-1}(x), f^{-1}(y))\right]^{1/b}\right\} \\ &= d \cdot \max\left\{a_{f^{-1}(D)}(f^{-1}(x), f^{-1}(y)), \left[a_{f^{-1}(D)}(f^{-1}(x), f^{-1}(y))\right]^{1/b}\right\} \\ &\leq d \cdot \max\left\{ba_D(x, y) + c, [ba_D(x, y) + c]^{1/b}\right\} \leq bda_D(x, y) + cd. \end{aligned}$$

∎

Theorem 3.7. *If $D \subset \overline{\mathbb{R}}^n$ is a K-quasiball then*

$$k_D(x, y) \leq pa_D(x, y)$$

for all $x, y \in D$ with $a_D(x, y) \leq 2/\sqrt{p}$, where $p = 16e^{8K(K+1)\sqrt{K-1}}$.

Proof. Since D is a K-quasiball, there exists a K-quasiconformal self-mapping f of $\overline{\mathbb{R}}^n$ such that $f(\mathbb{B}^n) = D$. Choose $h \in \text{Möb}(\overline{\mathbb{R}}^n)$ with $h(x) = 0$ and $h(y) = \infty$ and let $B^n(0, r-1)$ and $\overline{\mathbb{R}}^n \setminus \overline{B}^n(0, r+1)$ be the maximal Apollonian balls about 0 and ∞ in $h(D)$, respectively. (See, [B2] or [I1] for definition of maximal Apollonian balls). Then

$$a_D(x,y) = a_{h(D)}(0,\infty) = \log\frac{r+1}{r-1} \le \frac{2}{\sqrt{p}}$$

and hence $r \ge 4e^{4K(K+1)\sqrt{K-1}}$.

Choose $T \in \text{Möb}(\overline{\mathbb{R}}^n)$ with $T(\mathbb{B}^n) = \mathbb{B}^n$ and $T(f^{-1}(y)) = 0$ [AVV, 7.34]. Then the map $g = h \circ f \circ T^{-1} \circ i$ is a K-quasiconformal self-mapping of $\overline{\mathbb{R}}^n$ and such that $g(\mathbb{B}^n) = \overline{\mathbb{R}}^n \setminus h(\overline{D})$ and $g(\infty) = \infty$ where i is the inversion in $\mathbb{S}^{n-1}$. Put $z = g(0) \in \overline{\mathbb{R}}^n \setminus h(\overline{D})$ and let $B^n(z,l)$ and $\overline{\mathbb{R}}^n \setminus \overline{B}^n(z,L)$ be the maximal Apollonian balls about z and ∞ in $\overline{\mathbb{R}}^n \setminus h(\overline{D})$, respectively. Then $l = |z-u|$ and $L = |z-v|$ for some distinct u and v in $\partial h(D)$. Then, by Theorems 14.8 and 14.18 of [AVV], we have

$$\begin{aligned}\frac{L}{l} = \frac{|z-v||u-\infty|}{|z-u||v-\infty|} &\le \eta^*_{K,n}\left(\frac{|0-g^{-1}(v)||g^{-1}(u)-\infty|}{|0-g^{-1}(u)||g^{-1}(v)-\infty|}\right)\\ &= \eta^*_{K,n}(1) \le e^{4K(K+1)\sqrt{K-1}} = \frac{\sqrt{p}}{4}.\end{aligned}$$

Since $l \le 1$, we have $L \le \sqrt{p}/4$ and $\overline{\mathbb{R}}^n \setminus h(D) \subset \overline{B}^n(z,\sqrt{p}/4)$. Now, let $B = \overline{B}^n(rz/|z|,R)$ be the smallest ball containing $\overline{\mathbb{R}}^n \setminus h(D)$. Then

$$R = L + ||z|-r| \le \frac{\sqrt{p}}{4} + 1 \le \frac{\sqrt{p}}{2}.$$

By performing a preliminary rotation, we may assume that $B = \overline{B}^n(re_1,\sqrt{p}/2)$ and $\overline{\mathbb{R}}^n \setminus h(D) \subset B$. Let

$$h_1(x) = \frac{\sqrt{p}(x-re_1)}{2|x-re_1|^2}$$

be a Möbius transformation of $\overline{\mathbb{R}}^n$. Then $h_1(h(D))$ contains the unit ball $\mathbb{B}^n$ and by Corollary 2.5 of [GeP] we have

$$\begin{aligned}k_D(x,y) &\le 2k_{h(D)}(0,\infty) \le 4k_{h_1(h(D))}\left(\frac{\sqrt{p}}{2r}e_1,0\right)\\ &\le 4k_{\mathbb{B}^n}\left(\frac{\sqrt{p}}{2r}e_1,0\right) = 4\log\frac{2r+\sqrt{p}}{2r-\sqrt{p}}.\end{aligned}$$

Hence, if we put

$$G(t) = \log\frac{2t+\sqrt{p}}{2t-\sqrt{p}} - \frac{p}{4}\log\frac{t+1}{t-1}$$

then $G'(t) > 0$ for all $t > r$ and we obtain

$$pa_D(x,y) - k_D(x,y) \geq p\log\frac{r+1}{r-1} - 4\log\frac{2r+\sqrt{p}}{2r-\sqrt{p}} \geq -4G(r)$$
$$= 4\lim_{s\to\infty}(G(s)-G(r)) = 4\int_r^\infty G'(t)\,dt > 0$$

as required. ∎

Theorem 3.8. *If $D \subset \overline{\mathbb{R}}^n$ is a K-quasiball then*

$$k_D(x,y) \leq C(K,n)a_D(x,y)$$

for each $x, y \in D$.

Proof. Let $x, y \in D$. If $a_D(x,y) \leq 2/\sqrt{p}$ then $k_D(x,y) \leq pa_D(x,y)$ by Theorem 3.7, and if $a_D(x,y) > 2/\sqrt{p}$ then

$$k_D(x,y) \leq \left[dK^{1/(n-1)} + 2dK^{1/(n-1)}\sqrt{p}\log(2\lambda_n)\right] a_D(x,y)$$

by Lemma 3.6. Here $p = 16e^{8K(K+1)\sqrt{K-1}}$ and $d = 4(2\lambda_n)^{2K^{1/(n-1)}}$. Hence the theorem holds with

$$C(K,n) = \max\left\{p,\ dK^{1/(n-1)} + 2d\cdot K^{1/(n-1)}\sqrt{p}\log(2\lambda_n)\right\}.$$

∎

We conclude this section by presenting some sharp estimates between any two of the Apollonian, hyperbolic and quasihyperbolic metrics on the plane angular domain D_α, $0 \leq \alpha \leq \pi$. For simplicity these metrics are denoted by a, h, and k, respectively. We begin with the following lemma.

Lemma 3.9. *Let $0 < \alpha < \pi$ and $0 < \phi < \alpha$. Then*

$$\frac{\alpha}{\pi} < \sin\left(\frac{\alpha}{2}\right) < \frac{\alpha}{2} \tag{3.1}$$

and

$$\frac{1}{\sin(\frac{\alpha}{2})} < \frac{\sin(\frac{\pi}{\alpha}\phi)}{\sin\beta} < \frac{\pi}{\alpha} \qquad \text{where } \beta := \min\{\phi, \alpha-\phi\}. \tag{3.2}$$

The proof of this lemma is left to the reader.

Theorem 3.10. *For all $z_1, z_2 \in D_\alpha$, we have*

$$k(z_1,z_2) \leq h(z_1,z_2) \leq \frac{\pi}{\alpha}\sin\left(\frac{\alpha}{2}\right) k(z_1,z_2).$$

These inequalities are sharp.

Proof. Let h_{D_α} and k_{D_α} be the density functions for the hyperbolic and quasihyperbolic metrics on D_α, respectively. Then using the same argument as in the proof of Proposition 3.1 it is enough to show that

$$k_{D_\alpha}(z) \le h_{D_\alpha}(z) \le \frac{\pi}{\alpha}\sin\Big(\frac{\alpha}{2}\Big)\, k_{D_\alpha}(z) \tag{3.3}$$

for all $0 \le \alpha \le \pi$.

Case1: $\alpha = 0$. Then the density of the quasihyperbolic metric is given by $k_{D_0}(z) = 1/\operatorname{dist}(z, \partial D_0)$. To compute the density of the hyperbolic metric we use the conformal map $\omega_1 \colon D_0 \to \mathbb{B}^{n2}$ given by

$$\omega_1(z) = \frac{e^{\pi z - i\pi/2} - 1}{e^{\pi z - i\pi/2} + 1}.$$

Then

$$h_{D_0}(z) = h_{\mathbb{B}^2}(\omega_1(z))|\omega_1{}'(z)| = \frac{\pi}{\sin(\pi \operatorname{Im} z)} = \frac{\pi}{\sin \pi(\operatorname{dist}(z, \partial D_0))}$$

and (3.3) follows from (3.1).

Case2: $\alpha > 0$. Then the density of the quasihyperbolic metric is given by

$$k_{D_\alpha}(z) = \frac{1}{\operatorname{dist}(z, \partial D_\alpha)} = \begin{cases} \dfrac{1}{|z|\sin(\arg(z))} & \text{if } \arg(z) \le \dfrac{\alpha}{2}, \\[2ex] \dfrac{1}{|z|\sin(\alpha - \arg(z))} & \text{if } \arg(z) \ge \dfrac{\alpha}{2}. \end{cases}$$

and we use the conformal map $\omega_2 \colon D_\alpha \to D_\pi$, $\omega_2(z) = z^{\pi/\alpha}$, to compute the density of the hyperbolic metric in D_α. We have

$$h_{D_\alpha}(z) = h_{D_\pi}(\omega_2(z))|\omega_2{}'(z)| = \frac{\pi}{\alpha}|z|^{\pi/\alpha - 1}\frac{1}{\operatorname{Im} z^{\pi/\alpha}} = \frac{\pi}{\alpha|z|\sin(\frac{\pi}{\alpha}\arg(z))}.$$

Then using (3.2) we obtain

$$\sin\Big(\frac{\pi}{\alpha}\arg(z)\Big) \ge \begin{cases} \dfrac{\sin(\arg(z))}{\sin(\frac{\alpha}{2})} & \text{if } 0 < \arg(z) \le \dfrac{\alpha}{2}, \\[2ex] \dfrac{\sin(\alpha - \arg(z))}{\sin(\frac{\alpha}{2})} & \text{if } \dfrac{\alpha}{2} \le \arg(z) < \alpha \end{cases}$$

and

$$\sin\Big(\frac{\pi}{\alpha}\arg(z)\Big) \le \begin{cases} \dfrac{\pi}{\alpha}\sin(\arg(z)) & \text{if } 0 < \arg(z) \le \dfrac{\alpha}{2}, \\[2ex] \dfrac{\pi}{\alpha}\sin(\alpha - \arg(z)) & \text{if } \dfrac{\alpha}{2} \le \arg(z) < \alpha \end{cases}$$

which implies (3.3). The proof is complete. ∎

As an immediate consequence of Theorem 3.10, Proposition 3.1 and Example 2.9 we obtain the following corollary.

Corollary 3.11. *For all $z_1, z_2 \in D_\alpha$, we have*

$$a(z_1, z_2) \le 2\cos^2\left(\frac{\alpha}{4}\right) h(z_1, z_2).$$

This inequality is sharp.

Given any simply connected plane domain D, by [B2, Theorem 1.2] one has $a_D \le 2h_D$, where 2 is best possible. But obtaining sharp inequalities in the other direction prove to be more difficult. Below we present an estimate in this direction which holds locally.

Theorem 3.12. *For each $z_0 \in D_\alpha$, there exists a neighborhood U_{z_0} of z_0 in D_α such that*

$$\frac{\alpha}{\pi} h(z_1, z_2) \le a(z_1, z_2)$$

for all $z_1, z_2 \in U_{z_0}$. This inequality is sharp.

Proof. Using Möbius transformations of $\overline{\mathbb{C}}$, if necessary, we can assume that $|z_0| = 1$ and $0 < \arg(z_0) \le \alpha/2$. Put

$$U_{z_0} = B^2\big(z_0, |e^{i\alpha/2} - e^{i\alpha\pi/(\alpha+\pi)}|\big) = B^2\left(z_0, 2\sin\left(\frac{\alpha(\pi-\alpha)}{4(\pi+\alpha)}\right)\right).$$

Then we have

$$\arg(z_1) < \frac{\pi\alpha}{\pi+\alpha} \quad \text{and} \quad \arg(z_2) < \frac{\pi\alpha}{\pi+\alpha} \qquad \text{for all } z_1, z_2 \in U_{z_0}.$$

Observe also that $\sin\phi \le \sin(\pi\phi/\alpha)$ for all $\phi \in \big(0, \pi\alpha/(\pi+\alpha)\big)$.

Now we consider the sector D_α as a subset of the upper-half plane $D_\pi = \mathbb{H}^2$. Since the set U_{z_0} is hyperbolically convex [GeH1, Lemma 3.4], for any two points $z_1, z_2 \in U_{z_0} \subset D_\pi$ the hyperbolic geodesic $\Gamma_{z_1z_2}$ joining z_1 and z_2 in D_π is contained in U_{z_0}. Then using monotonicity of the Apollonian metric [B2, p.93] we have

$$\begin{aligned}
\frac{\alpha}{\pi} h_{D_\alpha}(z_1, z_2) &= \frac{\alpha}{\pi} h_{D_\pi}(\omega(z_1), \omega(z_2)) \le \frac{\alpha}{\pi}\int_{\omega(\Gamma_{z_1z_2})} \frac{|d\omega|}{\operatorname{Im}\omega} \\
&= \frac{\alpha}{\pi}\int_{\Gamma_{z_1z_2}} \frac{\frac{\pi}{\alpha}|z|^{\pi/\alpha-1}|dz|}{|z|^{\pi/\alpha}\sin(\frac{\pi}{\alpha}\arg(z))} = \int_{\Gamma_{z_1z_2}} \frac{|dz|}{|z|\sin(\frac{\pi}{\alpha}\arg(z))} \\
&\le \int_{\Gamma_{z_1z_2}} \frac{|dz|}{\operatorname{Im} z} = h_{D_\pi}(z_1, z_2) = a_{D_\pi}(z_1, z_2) \le a_{D_\alpha}(z_1, z_2)
\end{aligned}$$

where $\omega(z) = z^{\pi/\alpha}$. The proof is complete. ■

References

AVV. G. Anderson, M. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, A Wiley-Interscience Publ., 1997.

B1. A. Beardon, *The Geometry of Discrete Groups*, Springer, New York, 1995.

B2. ———, The Apollonian metric of a domain in $\overline{\mathbb{R}}^n$, in: *Quasiconformal Mappings and Analysis* (Ann Arbor, MI, 1995), Springer, New York, 1998, 91–108.

ChG. G. D. Chakerian and H. Groemer, Convex bodies of constant width, in: *Convexity and its Applications*, Birkhäuser, Basel, 1983, 49–96.

Ge. F. W. Gehring, Rings and quasiconformal mappings in space, *Trans. Amer. Math. Soc.* **103** (1962), 353–393.

GeH1. F. W. Gehring and K. Hag, Hyperbolic geometry and disks, *J. Comput. Appl. Math.* **105** (1999), 275–284.

GeH2. ———, The Apollonian metric and quasiconformal mappings, *Contemp. Math.* **256** (2000), 143–163.

GeP. F. W. Gehring and B. B. Palka, Quasiconformally homogeneous domains, *J. d'Analyse Math.* **30** (1976), 172–199.

GeO. F. W. Gehring and B. G. Osgood, Uniform domains and the quasihyperbolic metric, *J. d'Analyse Math.* **36** (1979), 50–74.

I1. Z. Ibragimov. On the apollonian metric of domains in $\overline{\mathbb{R}}^n$, *Complex Variables*, **48** No.10 (2003), 837–855.

I2. ———, Convex bodies of constant width and the apollonian metric, to appear.

R. Yu. G. Reshetnyak. Liouville's theorem on conformal mappings under minimal regularity assumptions, *Sib. Mat. Zh.* **8** (1967), 835–840.

Sch. R. Schneider. *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.

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