

Deficient Values of Derivatives of Meromorphic Functions in the Class S

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Abstract. It is shown that if f is transcendental and meromorphic in the plane with finitely many critical and asymptotic values then $\delta(b, f') = 0$ for every $b \in \mathbb{C} \setminus \{0\}$. If, in addition, f has finite lower order then the same result holds for higher derivatives.

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1. Introduction

Let f be a function, transcendental and meromorphic in the plane. An important role in complex dynamics [1] is played by the singular values of the inverse function f^{-1} : these are the critical values of f and the asymptotic values, that is, values a such that $f(z)$ tends to a as z tends to infinity along a path γ_a . Denote by B the class of all functions f , transcendental and meromorphic in the plane for which the set of finite singular values of f^{-1} is bounded, and by S the subclass consisting of those f for which f^{-1} has finitely many singular values. Both classes have been studied extensively in iteration theory [1, 6].

The class S is also of interest from the standpoint of value distribution theory. Thus Collingwood [5] showed that if f is a transcendental meromorphic function such that all finite singular values of f^{-1} lie in a finite set $\{a_j\}$ then any finite Nevanlinna deficient value of f must be one of the a_j . This result was generalized by Teichmüller [19], who proved for such f the asymptotic equality

$$m\left(r, \frac{1}{f'}\right) = \sum m\left(r, \frac{1}{f - a_j}\right) + S(r, f)$$

in which, using standard notation from [9, 17], the term $S(r, f)$ denotes any quantity which is $\mathcal{O}(T(r, f))$ outside a set of finite measure.

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It was proved in [15] that if f is a transcendental meromorphic function in the class B and R is a rational function with a pole at infinity then

$$m\left(r, \frac{1}{f-R}\right) = S(r, f).$$

In particular $\delta(0, f(z) - z) = 0$ and f has infinitely many fixed points. In the present paper we prove some results on the deficient values of the derivatives of functions in the classes B and S . For functions of finite lower order in the class B we have the following result.

Theorem 1.1. *Let f be transcendental and meromorphic of finite lower order in the plane such that the set of finite singular values of the inverse function f^{-1} is bounded. Let n be a positive integer. Then $\delta(b, f^{(n)}) = 0$ for every $b \in \mathbb{C} \setminus \{0\}$.*

Theorem 1.1 leads to a simple proof of a special case of a conjecture of Mues [16] (see §5). It seems difficult to extend the method of Theorem 1.1 to functions of infinite lower order, but we prove the following result for the class S .

Theorem 1.2. *Let f be transcendental and meromorphic in the plane such that the inverse function f^{-1} has finitely many singular values. Then $\delta(b, f') = 0$ for every $b \in \mathbb{C} \setminus \{0\}$.*

2. Lemmas needed for Theorems 1.1 and 1.2

The following lemma, which was proved first in [6] for entire functions, has been used extensively in complex dynamics and value distribution theory (see also [1, 2, 15, 18]).

Lemma 2.1 ([6, 18]). *Let f be transcendental and meromorphic in the plane such that the set of finite singular values of the inverse function f^{-1} is bounded. Then there exist $L > 0$ and $M > 0$ such that if $|z_0| > L$ and $|f(z_0)| > M$ then*

$$(1) \quad \left| z_0 \frac{f'(z_0)}{f(z_0)} \right| \geq \frac{\log |f(z_0)/M|}{C},$$

in which C is a positive absolute constant, in particular independent of f , L and M .

The next lemma is a standard rescaled version of the distortion theorem for univalent functions in the unit disc [12, p.5, (1.3)].

Lemma 2.2 ([12]). *Let $0 < r < R < \infty$ and let h be analytic and univalent in the disc $B(a, R)$. Then*

$$\max\{|h'(z)| : |z-a| \leq r\} \leq \frac{2R^3}{(R-r)^3} |h'(a)| \leq \frac{16R^4}{(R-r)^4} \min\{|h'(z)| : |z-a| \leq r\}.$$

Proof. Assume without loss of generality that $a = 0$, and set

$$H(w) = \frac{h(Rw)}{Rh'(0)}$$

and $z = Rw$. Then $H'(0) = 1$ and [12, p. 5, (1.3)] gives, for $|z| \leq r$,

$$\frac{1 - r/R}{8} \leq |H'(w)| \leq \frac{2}{(1 - r/R)^3},$$

from which Lemma 2.2 follows at once. \blacksquare

We need Fuchs' Small Arcs Lemma [7]: the version stated here is from [11, p.721].

Lemma 2.3 ([11]). *Let g be meromorphic in $|z| \leq R$, with $g(0) = 1$. Let η_1, η_2 be positive with $\eta_1 + \eta_2 < 1$. Then there exists a subset E_R of $[0, R(1 - \eta_1)]$, having measure greater than $R(1 - \eta_1 - \eta_2)$, with the following property: if $r \in E_R$ and F_r is a subinterval of $[0, 2\pi]$ of length m then*

$$(2) \quad \int_{F_r} \left| \frac{rg'(re^{i\theta})}{g(re^{i\theta})} \right| d\theta \leq \frac{400}{\eta_1^2 \eta_2} T(R, g) m \log \frac{2\pi e}{m}.$$

We require the following lemma, which for functions of finite lower order is completely standard.

Lemma 2.4. *Let h be transcendental and meromorphic in the plane, such that $\delta(\infty, h) > 2\delta > 0$. Then there exist a sequence $r_k \rightarrow +\infty$ and for each k an arc Ω_k of the circle $S(0, r_k)$ with center 0 and radius r_k , such that*

$$(3) \quad \log |h(z)| > \delta T(r_k, h), \quad z \in \Omega_k,$$

and such that the angular measure m_k of Ω_k satisfies

$$(4) \quad m_k (\log T(r_k, h))^5 \rightarrow \infty.$$

If, in addition, h has finite lower order μ then $m_k \geq c$, in which c is a positive constant depending only on δ and μ .

Proof. The arc Ω_k is chosen so as to have $\log |h(z)| \geq 2\delta T(r_k, h)$ at one of its endpoints, and the angular measure m_k of Ω_k is determined by Lemma 2.3 as follows. Suppose first that h has finite lower order μ , and denote by c_j positive constants depending only on μ and δ . By [10] there exists a sequence $s_k \rightarrow +\infty$ with $T(2s_k, h) \leq c_1 T(s_k, h)$. Set $R = 2s_k$ and $\eta_1 = \eta_2 = 1/8$, and apply Lemma 2.3 to $h_1(z) = a_1 z^{n_1} h(z)$, with a_1, n_1 chosen so that $h_1(0) = 1$. Since

$$\left| \frac{h'(z)}{h(z)} \right| \leq \left| \frac{h_1'(z)}{h_1(z)} \right| + \frac{n_1}{|z|}, \quad T(2s_k, h_1) \leq T(2s_k, h) + \mathcal{O}(\log s_k),$$

this gives $r_k \in E_R \cap [s_k, 2s_k]$ such that (2) becomes, for the interval F_{r_k} corresponding to the arc Ω_k ,

$$\begin{aligned} \int_{F_{r_k}} \left| \frac{r_k h'(r_k e^{i\theta})}{h(r_k e^{i\theta})} \right| d\theta &\leq c_2(T(2s_k, h) + \mathcal{O}(\log s_k))m_k \log \frac{2\pi e}{m_k} \\ &\leq c_3 T(r_k, h)m_k \log \frac{2\pi e}{m_k} < \delta T(r_k, h) \end{aligned}$$

provided $m = m_k$ is chosen small enough and r_k is sufficiently large. But we have $\log |h(z)| \geq 2\delta T(r_k, h)$ at one endpoint of Ω_k , and (3) now follows.

The case of infinite lower order requires a result of Bergweiler and Bock [3, Lemma 1]. Let $\Phi(x) = \log T(e^x, h)$. Then by [3, Lemma 1] there exist sequences $x_k \rightarrow +\infty$, $M_k \rightarrow +\infty$, $\varepsilon_k \rightarrow 0+$ such that

$$\Phi(x_k + u) \leq \Phi(x_k) + \Phi'(x_k)u + \varepsilon_k, \quad |u| \leq \frac{M_k}{\Phi'(x_k)}.$$

Moreover, the sequence (x_k) may be chosen so that $\Phi'(x_k)$ tends to infinity but satisfies $\Phi'(x_k) = \mathcal{O}(\Phi(x_k))^{3/2}$, see [3, pp. 328–329]. With $\rho_k = e^{x_k}$ this gives

$$\begin{aligned} (5) \quad T(r, h) &\leq 2T(\rho_k, h) \left(\frac{r}{\rho_k} \right)^{\lambda_k}, \\ \left| \log \frac{r}{\rho_k} \right| &\leq \frac{4}{\lambda_k}, \\ \lambda_k = \Phi'(x_k) &= \mathcal{O}(\log T(\rho_k, h))^{3/2}. \end{aligned}$$

Let

$$\begin{aligned} (6) \quad R &= R_k = \rho_k e^{4/\lambda_k}, \\ \eta_1 &= 1 - e^{-2/\lambda_k} \sim \frac{2}{\lambda_k} = \mathcal{O}(1), \\ \eta_2 &= e^{-2/\lambda_k} - e^{-4/\lambda_k} \sim \frac{2}{\lambda_k}. \end{aligned}$$

Then $R(1 - \eta_1 - \eta_2) = \rho_k$, and Lemma 2.3, (5) and (6) then give r_k in $E_R \cap [\rho_k, R_k]$ such that (2) becomes

$$\begin{aligned} \int_{F_{r_k}} \left| \frac{r_k h'(r_k e^{i\theta})}{h(r_k e^{i\theta})} \right| d\theta &\leq 400\lambda_k^3(T(R_k, h) + \mathcal{O}(\log R_k))m \log \frac{2\pi e}{m} \\ &= \mathcal{O}\left(T(r_k, h)(\log T(r_k, h))^{9/2}m \log \frac{2\pi e}{m}\right), \end{aligned}$$

from which it is clear that $m = m_k$ may be chosen so as to satisfy (4). ■

3. Proof of Theorem 1.1

Suppose that f is as in the hypotheses but that $\delta(b, f^{(n)}) > 0$ for some positive integer n and some $b \in \mathbb{C} \setminus \{0\}$. Assume without loss of generality that $b = n!$.

By Lemma 2.4 there exist positive constants c_1, c_2 and a sequence $r_k \rightarrow +\infty$ such that

$$(7) \quad |f^{(n)}(z) - b| < \exp(-c_1 T(r_k, f^{(n)})), \quad z \in \Omega_k,$$

in which Ω_k is an arc of $S(0, r_k)$ of angular measure $m_k \geq c_2$. Integration of (7) gives a monic polynomial

$$(8) \quad P(z) = P_k(z) = z^n + \cdots = \prod_{j=1}^n (z - d_j),$$

possibly depending on r_k , such that

$$(9) \quad |f^{(q)}(z) - P^{(q)}(z)| < \exp(-c_3 T(r_k, f^{(n)})), \quad z \in \Omega_k, \quad q = 0, 1, \dots, n,$$

denoting by c_3, c_4, \dots positive constants which do not depend on r_k .

With c_4 small, let r_k be large and choose

$$z_0 \in \Omega_k \setminus \bigcup_{|d_j| \leq 2r_k} B(d_j, c_4 r_k).$$

Then (9) gives

$$(10) \quad \log |f(z_0)| \geq n \log r_k - c_5.$$

Hence, using (8), (9) and (10),

$$(11) \quad \begin{aligned} \left| \frac{z_0 P'(z_0)}{P(z_0)} \right| &= \left| \sum_{j=1}^n \frac{z_0}{z_0 - d_j} \right| = \mathcal{O}(1), \\ \left| \frac{z_0 f'(z_0)}{f(z_0)} \right| &= \left| \frac{z_0 P'(z_0) + \mathcal{O}(1)}{P(z_0) + \mathcal{O}(1)} \right| = \mathcal{O}(1). \end{aligned}$$

But (10) and (11) together contradict (1), and Theorem 1.1 is proved. \blacksquare

The example e^{e^z} shows that for infinite lower order the angular measure m_k cannot in general be expected to exceed $c_6(\log T(r_k))^{-1}$, leading to at best $|z_0 f'(z_0)/f(z_0)| \leq c_7 \log T(r_k)$. Comparing this with (10) then suggests that the method of Theorem 1.1 cannot be extended to the case of infinite lower order.

4. Proof of Theorem 1.2

Let f be transcendental and meromorphic of infinite lower order in the plane, such that the inverse function f^{-1} has finitely many distinct finite singular values a_j . Let

$$(12) \quad \eta = \min\{|a_j - a_{j'}| : j \neq j'\},$$

and let the positive constants L, M be as in Lemma 2.1. Assume without loss of generality that M is large.

Assume further that $\delta(1, f') > 0$, and apply Lemma 2.4. Then for large k there is an arc Ω_k of the circle $S(0, r_k)$, joining α_k to β_k , say, and of arc length

$$(13) \quad 32\varepsilon_k = \frac{32}{(\log T(r_k, f'))^5},$$

on which

$$(14) \quad |f'(z) - 1| < \exp(-c_1 T(r_k, f')).$$

Here and henceforth c_j, d_j, C_j denote positive constants not depending on r_k . From (13) and (14) it follows that

$$(15) \quad 64\varepsilon_k \geq |f(\alpha_k) - f(\beta_k)| \geq 16\varepsilon_k$$

and since ε_k tends to 0 as $k \rightarrow \infty$ there is no loss of generality in assuming that

$$(16) \quad |f(\alpha_k) - a_j| \geq 8\varepsilon_k \quad \text{for all } j.$$

Lemma 4.1. *Let ϕ be that branch of the inverse function f^{-1} mapping $f(\alpha_k)$ to α_k . Then ϕ extends to a univalent analytic function on $B(f(\alpha_k), 2\varepsilon_k)$ and satisfies there*

$$(17) \quad |\phi'(w) - 1| < \exp(-c_2 T(r_k, f')).$$

Proof. By (16) the function ϕ may be extended analytically and univalently to $B(f(\alpha_k), 8\varepsilon_k)$, and (14) gives

$$(18) \quad |\phi'(f(\alpha_k))| = \frac{1}{|f'(\alpha_k)|} \leq 2.$$

It follows from Lemma 2.2 that

$$(19) \quad |\phi'(w)| \leq c_3, \quad w \in B(f(\alpha_k), 4\varepsilon_k).$$

Furthermore, (15) shows that there exists a simple subarc L_k of $f(\Omega_k)$ joining $f(\alpha_k)$ to $S(f(\alpha_k), 4\varepsilon_k)$, and (14) gives

$$(20) \quad |\phi'(w) - 1| < 2 \exp(-c_1 T(r_k, f'))$$

on L_k . Using (19), (20) and the standard harmonic measure estimate

$$\omega(w, L_k, B(f(\alpha_k), 4\varepsilon_k) \setminus L_k) \geq c_4, \quad w \in B(f(\alpha_k), 2\varepsilon_k) \setminus L_k,$$

the estimate (17) follows by applying the Two Constants Theorem [17, p.42] to the subharmonic function $\log |\phi'(w) - 1|$. This proves Lemma 4.1. ■

Lemma 4.2. *Let σ be positive but small compared to $\min\{1, \eta\}$. Then provided r_k is large enough there exists ζ_k with*

$$(21) \quad |\zeta_k - f(\alpha_k)| = \sigma$$

such that ϕ can be continued analytically and univalently to $B(\zeta_k, \sigma)$, and satisfies there

$$(22) \quad |\phi'(w) - 1| < \exp\left(\frac{-d_1 T(r_k, f')}{(\log T(r_k, f'))^{10}}\right)$$

and

$$(23) \quad |\phi(w) - \alpha_k| < 1.$$

Proof. Let a_ν be the nearest a_j to $f(\alpha_k)$. Choose ζ_k satisfying (21) so that $a_\nu, f(\alpha_k), \zeta_k$ are collinear, with $f(\alpha_k)$ separating a_ν from ζ_k .

We assert that no a_j lies in $B(\zeta_k, \sigma + 2\varepsilon_k)$. Assuming the existence of such an a_μ gives $|f(\alpha_k) - a_\mu| < 3\sigma$ and so $a_\mu = a_\nu$ by (12), since σ is small compared to η . This is a contradiction since $|a_\nu - \zeta_k| \geq \sigma + 8\varepsilon_k$ by (16) and the choice of ζ_k .

Hence ϕ can be continued analytically and univalently into $B(\zeta_k, \sigma + 2\varepsilon_k)$, starting at $f(\alpha_k)$. By (18), (21) and Lemma 2.2, ϕ satisfies

$$(24) \quad |\phi'(w)| \leq \frac{d_2}{\varepsilon_k^4}, \quad w \in B(\zeta_k, \sigma + \varepsilon_k).$$

Moreover the disc $B(f(\alpha_k), 2\varepsilon_k)$ intersects the circle $S(\zeta_k, \sigma + \varepsilon_k)$ in an open arc Σ_k of angular measure at least $d_3\varepsilon_k$, and (17) gives

$$(25) \quad |\phi'(w) - 1| < \exp(-c_2 T(r_k, f')), \quad w \in \Sigma_k.$$

Poisson's formula yields

$$\omega(w, \Sigma_k, B(\zeta_k, \sigma + \varepsilon_k)) \geq d_4 \varepsilon_k^2$$

for $w \in B(\zeta_k, \sigma)$, and so (22) follows from (13), (24), (25), on applying the Two Constants Theorem again to $\log |\phi'(w) - 1|$. Since $\phi(f(\alpha_k)) = \alpha_k$, integration of (22) starting from $f(\alpha_k) \in \partial B(\zeta_k, \sigma)$ gives (23). This proves Lemma 4.2. ■

Lemma 4.3. *Let τ be positive, but small compared to σ/q , where q is the number of finite singular values a_j . Choose*

$$(26) \quad y_k \in \left[\operatorname{Im}(\zeta_k) - \frac{\sigma}{4}, \operatorname{Im}(\zeta_k) + \frac{\sigma}{4} \right]$$

such that the strip $\{w \in \mathbb{C} : |\operatorname{Im}(w) - y_k| < 4\tau\}$ contains none of the a_j . Then provided r_k is large enough ϕ may be continued analytically and univalently to the domain

$$(27) \quad D_k = \{w \in \mathbb{C} : |\operatorname{Re}(w) - \operatorname{Re}(\zeta_k)| < 8M, |\operatorname{Im}(w) - y_k| < \tau\},$$

starting from the point

$$(28) \quad W_k = \operatorname{Re}(\zeta_k) + iy_k \in B(\zeta_k, \sigma),$$

and satisfies, for $w \in D_k$,

$$(29) \quad |\phi'(w) - 1| < \exp\left(\frac{-C_1 T(r_k, f')}{(\log T(r_k, f'))^{10}}\right)$$

and

$$(30) \quad |\phi(w) - \alpha_k| < 1 + 16M.$$

Proof. By the choice of y_k and τ the function ϕ may in fact be continued analytically and univalently to

$$D_k'' = \{w \in \mathbb{C} : |\operatorname{Re}(w) - \operatorname{Re}(\zeta_k)| < 32M, |\operatorname{Im}(w) - y_k| < 4\tau\},$$

which has center W_k . Since (22) gives $|\phi'(W_k)| \leq 2$, repeated application of Lemma 2.2 on discs of radius C_2 shows that

$$|\phi'(w)| \leq C_3, \quad w \in D_k' = \{w \in \mathbb{C} : |\operatorname{Re}(w) - \operatorname{Re}(\zeta_k)| < 16M, |\operatorname{Im}(w) - y_k| < 2\tau\}.$$

Since (22) holds on the vertical arc $w = \operatorname{Re}(\zeta_k) + iy, |y - y_k| \leq 2\tau$, the estimate (29) follows from the Two Constants Theorem, and again (30) is obtained by integration, starting from W_k and using (23). This proves Lemma 4.3. \blacksquare

Let

$$A_k = W_k - 4M - \frac{r_k}{8}, \quad U_k = B\left(A_k, \frac{r_k}{8}\right), \\ B_k = W_k + 4M + \frac{r_k}{8}, \quad V_k = B\left(B_k, \frac{r_k}{8}\right).$$

Then since $\operatorname{dist}(U_k, V_k) = 8M$ at least one of the discs U_k and V_k lies in the region $\{w \in \mathbb{C} : |w| > 4M\}$. Assume that this is true of U_k , the subsequent argument in the contrary case being identical.

Lemma 4.4. *The function ϕ may be continued analytically and univalently to U_k , with*

$$(31) \quad \phi'(w) = 1 + o(1), \quad |\phi(w) - \alpha_k| \leq 2 + 16M + \frac{r_k}{4}, \quad w \in U_k.$$

Proof. Since U_k lies in $\{w \in \mathbb{C} : |w| > 4M\}$ the function ϕ extends analytically and univalently to $B(A_k, r_k/8 + 2M)$, starting at $W_k - 4M \in D_k$. Then we have $|\phi'(W_k - 4M)| \leq 2$ by (29), and Lemma 2.2 gives

$$(32) \quad |\phi'(w)| \leq C_4 r_k^4, \quad w \in B\left(A_k, \frac{r_k}{8} + M\right).$$

Also D_k intersects $S(A_k, r_k/8 + M)$ in an arc Γ_k of angular measure at least $C_5 r_k^{-1}$, so that

$$(33) \quad \omega\left(w, \Gamma_k, B\left(A_k, \frac{r_k}{8} + M\right)\right) \geq \frac{C_6}{r_k^2}, \quad w \in U_k.$$

But (29) holds on Γ_k , and so the Two Constants Theorem, (32) and (33) give

$$(34) \quad |\phi'(w) - 1| \leq \exp\left(\frac{-C_7 T(r_k, f')}{r_k^2 (\log T(r_k, f'))^{10}}\right) = \frac{o(1)}{r_k}, \quad w \in U_k,$$

since f has infinite lower order and consequently so has f' . Integrating (34) starting from the point $W_k - 4M \in D_k \cap \partial U_k$, together with (30), leads to the second estimate of (31). This proves Lemma 4.4. ■

To complete the proof of Theorem 1.2 choose $w_0 \in U_k$ with $|w_0| \geq r_k/16$. Since $|\alpha_k| = r_k$, the estimate (31) gives, provided r_k is large enough,

$$L < \frac{r_k}{2} \leq |z_0| \leq \frac{3r_k}{2}, \quad z_0 = \phi(w_0).$$

But now (1) and (31) give

$$\frac{1}{48} = \frac{(r_k/16)(1/2)}{3r_k/2} \leq \left| w_0 \frac{\phi'(w_0)}{\phi(w_0)} \right| = \left| \frac{f(z_0)}{z_0 f'(z_0)} \right| \leq \frac{C}{\log |w_0/M|} \leq \frac{C}{\log(r_k/16M)},$$

which is obviously impossible if r_k is large enough. ■

We conclude this section by noting that the method of Theorem 1.2 appears to break down for higher derivatives. If $\delta(2, f'') > 0$ then $\phi = f^{-1}$ can in principle be continued to some region U , with $f(z)$ close to a monic quadratic on $\phi(U)$. However, it seems difficult to exclude the possibility that ϕ' is small on U , so that $\phi(U)$ has small diameter, which may in turn preclude obtaining a good upper bound for $|zf'(z)/f(z)|$.

5. A special case of the Mues conjecture

Let f be transcendental and meromorphic in the plane and let n be a positive integer. It was observed by Hayman [8, 9] that since all poles of $f^{(n)}$ have multiplicity at least $n + 1$ we have

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq \frac{n+2}{n+1}.$$

This was improved to

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq \frac{n^2 + 5n + 4}{n^2 + 4n + 2}$$

by Mues [16]. If, in addition, all poles of f are simple then Mues [16] showed that

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq 1,$$

and conjectured that this remains true in the general case. The strongest general result appears to be [14, 23] (see also [20, 21, 22])

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq \frac{2n+2}{2n+1}.$$

We prove here that if f has finite lower order and $f^{(n)}$ has a finite Picard value a , that is, a finite value taken finitely often in the plane, then $f^{(n)}$ has no other finite deficient value.

Theorem 5.1. *Let f be transcendental and meromorphic of finite lower order in the plane, let n be a positive integer, and assume that $f^{(n)}$ has a finite Picard value a . Then $\delta(b, f^{(n)}) = 0$ for all $b \in \mathbb{C} \setminus \{a\}$.*

Proof. Set $g(z) = f^{(n-1)}(z) - az$. Then g is transcendental and meromorphic of finite lower order, and $g' = f^{(n)} - a$ has finitely many zeros. By Hinchliffe's extension [13] to functions of finite lower order of a result of Bergweiler and Eremenko [4], the function g has finitely many asymptotic values and so is in the class S . The result then follows from Theorem 1.1. ■

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