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Extremal Point Methods for Robin Capacity

Marcus Stiemer

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Abstract. The Robin capacity $\delta(A)$ of a compact, non-empty set $A \subset \partial\Omega$ with respect to a domain $\Omega \subset \widehat{\mathbb{C}}$ containing ∞ is defined by

$$\delta(A) = \delta(A, \Omega) = \exp\Big(\lim_{z\to\infty} -R(z) + \log|z|\Big),$$

where $R(z) = R(z, \infty)$ is the fundamental solution of a mixed boundary value problem with pole at ∞, where Dirichlet conditions are imposed on A and Neumann conditions on $\partial\Omega \setminus A$. P. Duren and M. Schiffer have discovered that it coincides with the minimal logarithmic capacity of $f(A)$ over all conformal mappings f of Ω with $f(z) = z + \mathcal{O}(1)$, $z \to \infty$.

In this article, effective methods for the numerical determination of $\delta(A)$ are developed. For this purpose the conformal invariant $\delta(A)/\operatorname{cap}(\partial\Omega)$ is related to other moduli of the given configuration like harmonic measure or conformal modulus. Then an effective extremal point discretization for these moduli based on Menke points is derived. If Ω is analytically bounded, the discretizations presented provide geometrically fast converging approximations to the considered moduli and thus to Robin capacity.

Keywords. Robin capacity, conformal invariants, extremal points.

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1. Introduction

Let Ω be a domain on the Riemann sphere $\widehat{\mathbb{C}}$ containing ∞. To begin with, assume that the boundary of Ω consists of K Dini smooth Jordan curves with $1 \le K < \infty$. Let $A \ne \emptyset$ be a closed subset of $\partial\Omega$, each component of which is a curve or an arc. The complement of A with respect to $\partial\Omega$ is denoted as B. A function R that is continuous on $\overline{\Omega}$ is called the Robin function in Ω with Dirichlet boundary A and pole at ∞, if it possesses the following four properties:

1. R is harmonic in $\Omega \setminus \{\infty\}$;
2. $\lim_{z\to\infty} R(z) - \log|z|$ exists;
3. $R(\zeta) = 0$ for $\zeta \in A$;

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4. $\partial R/\partial n(\zeta) = 0$ for $\zeta \in B$.

Here $\partial R/\partial n$ denotes the derivative in the direction of the outer normal in ζ. In particular, we claim that the gradient of R has a continuous extension to the Neumann boundary B. The definition of the Robin function can be extended to non-smooth boundaries by conformal invariance (see [11]): in such a situation, we define R to be $\tilde{R} \circ \Psi$, where Ψ is a conformal map from Ω on a domain with smooth boundary fixing ∞ and $\tilde{R}$ is the Robin function in the image domain with corresponding Dirichlet boundary. The Robin capacity $\delta(A)$ of the set A with respect to Ω is defined by

$$\delta(A) = \delta(A, \Omega) = \exp\Big(\lim_{z\to\infty} -R(z) + \log|z|\Big)\,.$$

Thus, it is defined similar to logarithmic capacity. The role of the Green function in the definition of the logarithmic capacity $\operatorname{cap}(A)$ has been taken by the Robin function.

Duren and Schiffer revealed that $\delta(A)$ coincides with the minimal logarithmic capacity of $f(A)$, where the minimum is taken over all conformal mappings f of Ω with $f(z) = z + \mathcal{O}(1)$, $z \to \infty$ (see [11, 12, 28, 31]). Here $f(A)$ has to be understood in the sense of boundary correspondence. The connection between the Robin function and Robin capacity and other quantities from geometric function theory and potential theory has been thoroughly investigated by P. Duren, J. Pfaltzgraff, M. Schiffer, R. Thurman and others (see e.g. [4, 8, 9, 10, 11, 12, 32, 33]).

The Robin capacity $\delta(A)$ with respect to Ω and the logarithmic capacity $\operatorname{cap}(\partial\Omega)$ of $\partial\Omega$ show the same scaling behavior under a conformal map f of Ω with $f(z) = az + \mathcal{O}(1)$, $z \to \infty$:

$$\delta(f(A)) = |a| \cdot \delta(A) \qquad \text{and} \qquad \operatorname{cap}(\partial f(\Omega)) = |a| \cdot \operatorname{cap}(\partial\Omega).$$

Consequently, $\delta(A)/\operatorname{cap}(\partial\Omega)$ is invariant under all conformal mappings of Ω that fix ∞.

In [3, p. 70], L. V. Ahlfors introduces the notion *configuration* for a domain bounded by a finite number of smooth curves together with a finite set of interior and boundary points taken in a certain order. Two configurations are equivalent, if there exists a conformal mapping from one domain onto the other which maps each of the specified interior and boundary points onto the corresponding points of the other configuration. The equivalence of configurations can be expressed by the equality of certain conformal invariants called *moduli* [3, p. 70]. On the other hand, as soon as a set of a sufficient number of moduli is known, any other conformal invariant of the specified configuration can be computed from them. Configurations associated to Robin capacity are characterized by one interior point (here ∞) and a certain number of boundary points (the end points of the arcs that constitute the set A). For instance, if Ω is the outer domain of a Jordan curve and if $A \neq \emptyset$ is a subarc, the configuration possesses one marked interior point and two boundary points. Such a configuration is characterized by a single modulus [3, p. 70]. One can choose the equilibrium measure $\mu(A)$, if ∞ is the marked inner point. If Ω is doubly connected and if A is one boundary curve and B the other, then there is one marked inner point and no boundary point. Such a configuration possesses two moduli [3, p. 70]. Here, if again ∞ is regarded as the specified inner point, the configuration is characterized by the conformal modulus[1] of Ω and the harmonic measure $\omega(\infty, B, \Omega)$.

In the following, the Robin capacity $\delta(A)$ is expressed in terms of the above mentioned moduli for the described configurations. This makes it possible to apply extremal points that discretize these moduli for the numerical computation of $\delta(A)$. In particular, we prove the following two theorems.[2]

Theorem 1. *Let Ω be the outer domain of a Jordan curve and $A \subset \partial\Omega$ a subarc. Then the Robin capacity $\delta(A)$ of A with respect to Ω amounts to*

$$\delta(A) = \operatorname{cap}(\partial\Omega) \cdot \sin^2\left(\frac{\pi}{2}\mu(A)\right).$$

Theorem 2. *Let M be the conformal modulus of the doubly connected domain Ω with $\infty \in \Omega$, whose boundary components are non-degenerate continua. Further, let $\omega = \omega(\infty, B, \Omega)$ be the value in ∞ of the harmonic measure in Ω with respect to the Neumann boundary B. Then*

$$\delta(A) = \left(\frac{\vartheta_2\left(\frac{\omega}{2}\right)\vartheta_3\left(\frac{\omega}{2}\right)}{\vartheta_2(0)\,\vartheta_3(0)}\right)^2 \cdot \operatorname{cap}(\partial\Omega).$$

The parameter of the above theta functions is given by

$$\tau = i\frac{\pi}{M}.$$

Note that no further assumptions on the smoothness of A and B are necessary in Theorems 1 and 2, if the conformally invariant extended version of the Robin function is considered. In particular, since A and B are non-degenerate continua in Theorem 2, $\partial\Omega$ is regular in the sense of potential theory and ω is well defined. The theta functions occurring in the above formula can be effectively computed with the help of their Fourier series: The expansion of $\vartheta_2(v) = \vartheta_2(v, \tau)$ with parameter τ and $q = e^{i\pi\tau} = e^{-\pi^2/M}$ is given by

$$\vartheta_2(v, \tau) = \sum_{k=-\infty}^{\infty} q^{(k-1/2)^2} e^{(2k-1)i\pi v}$$

[1] Throughout the following, the conformal modulus is defined as $-\log r$, if the annulus $\{r < |z| < 1\}$ is conformally equivalent to Ω.

[2] Theorem 1 and Theorem 2 are part of the author's thesis [32].

(cf. [36, p. 155]) and the theta function $\vartheta_3(v) = \vartheta_3(v, \tau)$ with parameter τ and $q = e^{i\pi\tau} = e^{-\pi^2/M}$ is represented by

$$\vartheta_3(v, \tau) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2k \cdot i\pi v}$$

(cf. [36, p. 154]).

The next step is to approximate the moduli that are related to Robin capacity by Theorems 1 and 2. This will be done with the help of extremal point systems. The classical way to discretize the equilibrium measure of a compact set E in the complex plane is to use Fekete points: a system of n points $z_{n,1}^{[F]}, \dots, z_{n,n}^{[F]}$ that maximizes

$$V_n(z_1, \dots, z_n) = \prod_{\mu=1}^{n} \prod_{\substack{\nu=1 \\ \mu\neq\nu}}^{n} |z_\mu - z_\nu|$$

among all sets $(z_k)_{k=1,\dots,n}$ of different points on E is called a system of Fekete points of order n. C. Pommerenke proved for the distribution of Fekete points on an analytic Jordan curve Γ that

$$t_{n,k} = \alpha_n + \frac{2\pi k}{n} + \frac{1}{n}\varphi\left(\alpha_n + \frac{2\pi k}{n}\right) + \mathcal{O}\left(\frac{\sqrt{\log n}}{n^2}\right), \qquad n \to \infty,$$

where $e^{it_{n,k}} = \Phi^{-1}(z_{n,k}^{[F]})$, $1 \le k \le n$, are the preimages of the Fekete points under the conformal mapping $\Phi\colon \Delta = \{|z| > 1\} \to \mathrm{Ext}(\Gamma)$ with $\Phi(z) = z + \mathcal{O}(1)$, $z \to \infty$. Further, $\alpha_n \in \mathbb{R}$ is a rotational angle independent of k, and φ is a real analytic function depending on the given analytic Jordan curve [29] (see also [6, pp. 229–241]). Considering the weaker assumption that Γ is a quasiconformal curve or arc, A. Andrievskii and H.-P. Blatt proved a discrepancy estimate for Fekete points [5], which has been generalized to the case of weighted Fekete points and weighted extremal points for Green capacity by M. Götz and E. B. Saff [14] (see also below). In the case of an analytic Jordan curve, a better discretization of the equilibrium measure is provided by an extremal point system that has been introduced by K. Menke in 1970 [20, 21]: let $z_1, \dots, z_n$, $\zeta_1, \dots, \zeta_n$ be two sets of n points each such that $z_1 \prec \zeta_1 \prec z_2 \prec \zeta_2 \prec \dots \prec z_n \prec \zeta_n$. This means, that the points z_k and ζ_k are alternating on the curve Γ. Menke defined the resultant R_n of such two sets of points by

$$R_n(z_1, \dots, z_n, \zeta_1, \dots, \zeta_n) = \prod_{\mu=1}^{n} \prod_{\nu=1}^{n} |z_\mu - \zeta_\nu|.$$

Points $z_{n,1}^{[M]}, \dots, z_{n,n}^{[M]}, \zeta_{n,1}^{[M]}, \dots, \zeta_{n,n}^{[M]}$ that maximize R_n under the above mentioned constraints are called Menke points[3]. If $e^{is_{n,k}}$, $e^{it_{n,k}}$, $k = 1, \dots, n$, are the preimages of the Menke points $z_{n,k}^{[M]} = \Phi(e^{is_{n,k}})$ and $\zeta_{n,k}^{[M]} = \Phi(e^{it_{n,k}})$ respectively under

[3]This notation has been introduced by D. Gaier.

the conformal mapping $\Phi\colon \Delta \to \mathrm{Ext}(\Gamma)$ with $\Phi(z) = z + O(1)$, then

$$\begin{aligned} s_{n,k} &= \alpha_n + \frac{2k-1}{n}\pi + \mathcal{O}(r^n)\,, \qquad n \to \infty, \\ t_{n,k} &= \alpha_n + \frac{2k}{n}\pi + \mathcal{O}(r^n)\,, \qquad n \to \infty, \end{aligned} \tag{1}$$

for some $0 < r < 1$ with a rotational angle $\alpha_n = 2\pi - t_{n,n} = \mathcal{O}(r^n)$ independent of k [20, 21, 22, 23, 24]. The maximum R_n of the nth resultant possesses the asymptotic representation

$$R_n = 2^n \cdot \mathrm{cap}(\partial\Omega)^{n^2} \cdot (1 + \mathcal{O}(r^n))\,, \qquad n \to \infty, \tag{2}$$

with $0 < r < 1$, which makes $2^{-1/n} \cdot R_n^{1/n^2}$ a geometrically fast converging approximation to the logarithmic capacity of $\partial\Omega$. For Fekete points on an analytic Jordan curve, $V_n^{1/n(n-1)}$ approaches $\mathrm{cap}(\partial\Omega)$ only with order $\mathcal{O}(1/n)$, $n \to \infty$. Moreover, Menke presents a method which allows approximation of the conformal map Φ with $\Phi(z) = z + \mathcal{O}(1), z \to \infty$ from Δ onto Ω geometrically fast on compact subsets of Ω. The approximating functions are of the form $zp(1/z)$, where p is a polynomial that maps the $2n$th roots of unity onto certain values derived from the Menke points of order n [20, 21, 22, 24]. This good distribution of Menke points can be used to discretize the equilibrium measure μ of $\partial\Omega$ in geometrical order. In [34] it is established, that any family of analytic maps f_n which are univalent in an annulus about the origin and map $\partial\mathbb{D}$ onto the analytic Jordan curve $\partial\Omega$ preserving the orientation and interpolating the Menke points of order n in the $2n$th roots of unity provide an approximation to μ with an error estimate of geometric rate of convergence. More precisely, $1/2\pi|(f_n^{-1})'(z)|\,|dz|$ approximates $d\mu(z)$ uniformly on $\partial\Omega$ with a geometrically decaying error bound. In this article, we will construct such approximations f_n with the help of trigonometric interpolation. This makes it possible to approximate the Robin capacity of the subarc of an analytic Jordan curve geometrically fast. In the following, it is often more convenient to label the extremal points in their *natural* order $w_{n,k}^{[M]}$, $k = 1, \dots, 2n$, with $w_{n,2\ell-1}^{[M]} = z_{n,\ell}^{[M]}$ and $w_{n,2\ell}^{[M]} = \zeta_{n,\ell}^{[M]}$, $\ell = 1, \dots, n$. We will prove the following result.

Theorem 3. *Let Ω be an analytically bounded Jordan domain containing ∞, parametrized over $[0, 2\pi[$ by the analytic map γ with $\gamma'(t) \neq 0$, $t \in [0, 2\pi[$. Let A be a subarc of $\partial\Omega$ with starting point $\gamma(a)$ and ending point $\gamma(b)$, $0 \le a < b < 2\pi$. Moreover, let R_n be the maximum of the resultant of order n. Further, let $\mu_n(A)$ be defined by*

$$\mu_n(A) = \frac{1}{2\pi}\left|p_n^{-1}(b) - p_n^{-1}(a)\right|,$$

where $p_n(t) = S_n(t) + t$, $t \in [0, 2\pi[$ and

$$S_n(t) = \frac{1}{2}a_0 + \sum_{k=1}^{n-1}(a_k \cos kt + b_k \sin kt) + \frac{1}{2}a_n \cos nt$$

is the trigonometric interpolation polynomial mapping equidistant points $\pi k/n$ *onto* $\gamma^{-1}(w_{n,k}^{[M]})$, $k = 1, \ldots, 2n$. *Then, there exist constants* $C > 0$ *and* $0 < r < 1$ *such that for* $n \in \mathbb{N}$

$$\left|\delta(A) - 2^{-1/n} R_n^{1/n^2} \cdot \sin^2\Big(\frac{\pi}{2}\mu_n(A)\Big)\right| \le C r^n.$$

For the computation of $\delta(A)/\operatorname{cap}(\partial\Omega)$, where $\delta(A)$ is the Robin capacity of one boundary component A with respect to a doubly connected domain Ω, the conformal modulus of Ω and the harmonic measure of ∞ with respect to the Neumann boundary have to be determined. To this end, a discretization for the doubly connected situation is required. By conformal invariance, the latter is related to the so called hyperbolic situation, where a compact set $E \subset \mathbb{D}$ is considered. Both point systems introduced above possess an analogue in the hyperbolic situation: in 1947, M. Tsuji introduced a system of n points $z_{n,1}^{[T]}, \ldots, z_{n,n}^{[T]}$ that maximizes

$$V_n^h(z_1, \ldots, z_n) = \prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{n} \prod_{\nu=1}^{n} [z_\mu, z_\nu],$$

among all sets $(z_k)_{k=1,\ldots,n}$ of different points on E, where

$$[z, \zeta] = \left|\frac{z - \zeta}{1 - z\bar{\zeta}}\right|.$$

These points are called Tsuji points of order n. Let Φ denote a conformal map from $\{r < |z| < 1\}$ onto Ω with $r = e^{-\operatorname{mod}(\Omega)}$, where $\partial\mathbb{D}$ corresponds to itself. In 1985 Menke proved the estimate

$$\left|t_{n,k} - \alpha_n - \frac{2\pi k}{n}\right| \le L \cdot \frac{(\log n)^{3/2}}{n}, \qquad 1 \le k \le n,$$

for the distribution of Tsuji points on an analytic Jordan curve $E = \Gamma$, where $\Phi(re^{it_{n,k}}) = z_{n,k}^{[T]}$, $\alpha_n = t_{n,n} - 2\pi$ and $L > 0$ independent of n and k (cf. [25]). The rotational angles α_n drop out if the respective rotation of the unit circle is pre-composed to Φ for each $n \in \mathbb{N}$. For the weaker assumption that Γ is a quasiconformal curve or arc, a discrepancy estimate for Tsuji points is contained in the more general estimate provided by Theorem 3 in [14], which has been obtained by Götz and Saff in the context of weighted extremal points both for logarithmic and Green capacity. Taking advantage of the distribution of Tsuji points, Menke developed approximation techniques for the hyperbolic capacity and the conformal map Φ [26]. Moreover, he introduced another system of extremal points [27] whose images under Φ approach rotated roots of unity geometrically fast [35]. We will refer to this point system as hyperbolic Menke points: to define it, we again consider two sets $z_1, \ldots, z_n$ and $\zeta_1, \ldots, \zeta_n$ of n points each on the analytic Jordan curve Γ with $z_1 \prec \zeta_1 \prec z_2 \prec \zeta_2 \prec \ldots \prec z_n \prec \zeta_n$. A point system of this type $z_{n,1}^{[H]}, \ldots, z_{n,n}^{[H]}, \zeta_{n,1}^{[H]}, \ldots, \zeta_{n,n}^{[H]}$ that maximizes the (hyperbolic) resultant R_n^h

$$R_n^h(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n) = \prod_{\mu=1}^{n} \prod_{\nu=1}^{n} [z_\mu, \zeta_\nu]$$

is called Menke points of order n. As in the parabolic case, Menke points approximate the distribution of the images of equidistributed points on $|z| = e^{-\operatorname{mod}(\Omega)}$ under a conformal mapping from $\{e^{-\operatorname{mod}(\Omega)} < |z| < 1\}$ onto Ω with a geometrically decaying error bound, and $2^{-1/n} \cdot (R_n^h)^{1/n^2}$ approaches $e^{-\operatorname{mod}(\Omega)}$ geometrically fast [35]. Thus, they are better suited for approximation of the conformal modulus and the harmonic measure of the doubly connected domain Ω as soon as $\Gamma = \partial\Omega$ is an analytic Jordan curve.

From now on we do not assume that $\partial\mathbb{D}$ is a boundary component of Ω: let B be one boundary component of the doubly connected domain Ω with $\infty \in \Omega$ and let $G(z, \zeta)$ be the Green function in the outer domain of the other boundary component A with pole at ζ. In particular, if A coincides with the unit circle, $G(z, \zeta)$ is given by

$$G(z, \zeta) = \begin{cases} \log \left| \dfrac{1 - z\bar{\zeta}}{z - \zeta} \right| & , \zeta \neq \infty, \\ \log |z| & , \zeta = \infty. \end{cases}$$

The situation considered represents a special case of a plane condenser. The classical definition of a plane condenser can be found in [7]. Moreover, its relation to Green energy is pointed out there, which is the basis of the discretization techniques considered below. H. Kloke generalized the Tsuji point process to plane condensers by considering extremal points $z_{n,1}^{[K]}, \ldots, z_{n,n}^{[K]}$ which minimize

$$V_n^G(z_1, \ldots, z_n) = \sum_{\substack{\mu=1 \\ \mu \neq \nu}}^{n} \sum_{\nu=1}^{n} G(z_\mu, z_\nu),$$

among all sets $(z_k)_{k=1,\ldots,n}$ of different points on B [16] (see also [17]). The numbers $V_n^G = V_n^G(z_{n,1}^{[K]}, \ldots, z_{n,n}^{[K]})$ provide a discretization of the conformal modulus. In particular, Kloke proved

$$0 < \operatorname{mod}(\Omega) - \frac{1}{n(n-1)} V_n^G \leq \frac{C}{n} + 2\frac{\log n}{n}$$

with $C > 0$ independent of $n \geq 2$. Estimates for classical extremal points related to both conducting plates A and B of a condenser can, for instance, be found in [13]. To obtain geometrically fast converging approximations, we extend Menke's construction to condensers. From now on, let B be an analytic Jordan curve. We consider $2n$ points $z_1 \prec \zeta_1 \prec z_2 \prec \zeta_2 \prec \ldots \prec z_n \prec \zeta_n$ on B. A point system

$z_{n,1}^{[C]}, \dots, z_{n,n}^{[C]}, \zeta_{n,1}^{[C]}, \dots, \zeta_{n,n}^{[C]}$ that minimizes

$$J_n^c(z_1, \dots, z_n, \zeta_1, \dots, \zeta_n) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} G\left(z_\mu, \zeta_\nu\right)$$

is called Menke points for condensers of order n on B. Again, we will also use the notation $w_{n,k}^{[C]}$, $k = 1, \dots, 2n$, with $w_{n,2\ell-1}^{[C]} = z_{n,\ell}^{[C]}$ and $w_{n,2\ell}^{[C]} = \zeta_{n,\ell}^{[C]}$, $\ell = 1, \dots, n$. Let J_n^c denote the minimum $J_n^c = J_n^c(z_{n,1}^{[C]}, \dots, z_{n,n}^{[C]}, \zeta_{n,1}^{[C]}, \dots, \zeta_{n,n}^{[C]})$. The quantity

$$I_n^c = \frac{1}{n^2} J_n^c + \frac{\log 2}{n} \tag{3}$$

approaches $\operatorname{mod}(\Omega)$ geometrically fast. In particular, we will prove the following result.

Theorem 4. *Let the boundary component B of the doubly connected domain Ω be an analytic Jordan curve and the boundary component A be a non-degenerate continuum. Further, let M be the conformal modulus of Ω. Then there are constants C and ρ with $C > 0$ and $0 < \rho < 1$ such that*

$$|I_n^c - M| \le C\rho^n.$$

Here and in the following, the weak assumptions on A correspond to the assumption that the Green function G in the external domain of A is known. If fast converging numerical approximations to G are desired, further assumptions on A are necessary. Moreover, we will show that

$$\omega_n = \frac{1}{2n \operatorname{mod}(\Omega)} \sum_{\mu=1}^{2n} G(w_{n,\mu}^{[C]}, \infty) \tag{4}$$

converges to $\omega(\infty, B, \Omega)$ geometrically fast.

Theorem 5. *Let the boundary component B of the doubly connected domain Ω be an analytic Jordan curve and the boundary component A be a non-degenerate continuum. For ω_n defined as above and $\omega = \omega(\infty, B, \Omega)$, there are constants C and ρ with $C > 0$ and $0 < \rho < 1$ such that*

$$|\omega_n - \omega| \le C\rho^n.$$

If A is the unit circle, (4) reduces to the convenient formula

$$\omega_n = \frac{1}{2n \operatorname{mod}(\Omega)} \log \prod_{\mu=1}^{2n} \left| w_{n,\mu}^{[C]} \right|.$$

The corresponding approximation result for $\delta(A)/\operatorname{cap}(\partial\Omega)$ can be derived with the help of Theorem 2.

Theorem 6. *Let the boundary component B of the doubly connected domain Ω be an analytic Jordan curve and the boundary component A be a non-degenerate continuum. Further, let ω_n and I_n^c be defined as above. Moreover, define*

$$\eta_n(A) = \left(\frac{\vartheta_2\left(\frac{\omega_n}{2}\right) \vartheta_3\left(\frac{\omega_n}{2}\right)}{\vartheta_2(0)\, \vartheta_3(0)} \right)^2,$$

where the parameter of the above theta functions is given by

$$\tau = i\frac{\pi}{I_n^c}.$$

Then there are constants C and ρ with $C > 0$ and $0 < \rho < 1$ such that

$$\left| \eta_n(A) - \frac{\delta(A)}{\operatorname{cap}(\partial\Omega)} \right| \leq C\rho^n.$$

2. Representation of Robin capacity by moduli

We now prove the representation theorems for $\delta(A)$ in terms of certain conformal moduli. Let Ω be the outer domain of the Jordan curve Γ and μ the equilibrium measure on Γ.

Proof of Theorem 1. Let A_0 be the subarc of the unit circle that corresponds to $A \subset \partial\Omega$ by a conformal map $\Phi\colon \Delta \to \Omega$ fixing ∞ and let α denote the length of A_0. Duren and Pfaltzgraff [9] have shown that

$$\delta(A_0) = \sin^2\left(\frac{\alpha}{4}\right)$$

for the Robin capacity of a subarc A_0 of the unit circle of length α with respect to the outer domain Δ of the unit circle. From $\mu(A_0) = \alpha/2\pi$, $\operatorname{cap}(\partial\Delta) = 1$ and from the invariance of both the equilibrium measure and $\delta(A)/\operatorname{cap}(\partial\Omega)$ under all conformal mappings of Δ fixing ∞, we conclude

$$\frac{\delta(A)}{\operatorname{cap}(\partial\Omega)} = \frac{\delta(A_0)}{\operatorname{cap}(\partial\Delta)} = \sin^2\left(\frac{\pi}{2}\mu(A_0)\right) = \sin^2\left(\frac{\pi}{2}\mu(A)\right).$$

∎

Next, we consider a doubly connected domain Ω, one boundary component A of which is the Dirichlet and the other component B the Neumann boundary.

Proof of Theorem 2. Let Ψ_1 be a conformal map of Ω onto the annulus

$$\mathcal{A} = \left\{ z \in \mathbb{C} : e^{-M} < |z| < 1 \right\}$$

with $M = \operatorname{mod}(\Omega)$ such that the Dirichlet boundary A corresponds to the unit circle and such that $\Psi_1(\infty) > 0$. Due to

$$\omega\left(z, \left\{|\zeta| = e^{-M}\right\}, \mathcal{A}\right) = -\frac{1}{M}\log|z|$$

and the conformal invariance of the harmonic measure, one obtains

$$\Psi_1(\infty) = e^{-\omega M},$$

where $\omega = \omega(\infty, B, \Omega)$. The annulus $\mathcal{A}$ can be mapped by a conformal mapping Ψ_2 onto the outer domain $\mathcal{R} = \widehat{\mathbb{C}} \setminus E$ with $E = [-1, \beta] \cup [\alpha, 1]$ and $-1 < \beta < \alpha < 1$ such that $\Psi_1(\infty) = e^{-\omega M}$ is mapped again to infinity and that the image of the unit circle coincides with $[\alpha, 1]$. The construction of Ψ_2, which has been carried out by N. I. Achieser[4] in 1932 [1], yields

$$\alpha = 2\,\mathrm{sn}^2(\omega M) - 1, \tag{5}$$

where $\tau = i\pi/M$ is the parameter of the elliptic sine with primitive periods $4M$ and $4\pi i$. The composition $\Psi = \Psi_2 \circ \Psi_1$ is a conformal map from Ω onto $\mathcal{R}$ with $\Psi(\infty) = \infty$, where the Dirichlet boundary of Ω corresponds to the interval $[\alpha, 1]$.

The symmetry of the Green function G in the complement of $[\alpha, 1]$ with pole at ∞ enforces $\partial G/\partial y(x) = 0$ for $x \in \mathbb{R} \setminus [\alpha, 1]$. Thus, G coincides with the Robin function in $\mathcal{R}$ with Dirichlet boundary $[\alpha, 1]$. Consequently, we obtain

$$\delta([\alpha, 1]) = \mathrm{cap}([\alpha, 1]) = \frac{1-\alpha}{4}$$

(see [19, p. 172]).

The conformal invariance of $\delta(A)/\,\mathrm{cap}(\partial\Omega)$ implies

$$\frac{\delta(A)}{\mathrm{cap}(\partial\Omega)} = \frac{\delta([\alpha, 1])}{\mathrm{cap}(E)} = \frac{1-\alpha}{4} \cdot \frac{1}{\mathrm{cap}(E)}. \tag{6}$$

Inserting $\alpha = 2\,\mathrm{sn}^2(\omega M) - 1$ leads to

$$\frac{1-\alpha}{4} = \frac{1 - \mathrm{sn}^2(\omega M)}{2} = \frac{1}{2}\,\mathrm{cn}^2(\omega M).$$

On the other hand, Achieser showed[5] that

$$\mathrm{cap}(E) = \frac{1}{2}\left(\frac{\vartheta_4(0)\,\vartheta_3(0)}{\vartheta_4\left(\frac{\omega}{2}\right)\vartheta_3\left(\frac{\omega}{2}\right)}\right)^2 \tag{7}$$

for the logarithmic capacity of the union $E = [-1, \beta] \cup [\alpha, 1]$ (cf. [2, p. 317]).

[4] Achieser considers the outer domain of the two intervals $[-1, \tilde{\alpha}]$ and $[\tilde{\beta}, 1]$ with $-1 < \tilde{\alpha} < \tilde{\beta} < 1$. The situation that is considered here can be obtained from Achieser's setting by an application of the transformation $z \mapsto -z$.

[5] Achieser uses Jacobi's notation, where $\Theta_1(\omega M) = \vartheta_3(\omega/2)$ and $\Theta(\omega M) = \vartheta_4(\omega/2)$ [36, p. 305].

Inserting (7) in (6) yields

$$\frac{\delta(A)}{\operatorname{cap}(\partial\Omega)} = 2\left(\frac{\vartheta_3\left(\frac{\omega}{2}\right)\vartheta_4\left(\frac{\omega}{2}\right)}{\vartheta_3(0)\,\vartheta_4(0)}\right)^2 \cdot \frac{1}{2}\operatorname{cn}^2(\omega M)$$
$$= \left(\frac{\vartheta_3\left(\frac{\omega}{2}\right)\vartheta_4\left(\frac{\omega}{2}\right)}{\vartheta_3(0)\,\vartheta_4(0)}\,\frac{\vartheta_2\left(\frac{\omega}{2}\right)\vartheta_4(0)}{\vartheta_4\left(\frac{\omega}{2}\right)\vartheta_2(0)}\right)^2$$
$$= \left(\frac{\vartheta_2\left(\frac{\omega}{2}\right)\vartheta_3\left(\frac{\omega}{2}\right)}{\vartheta_2(0)\,\vartheta_3(0)}\right)^2$$

and the theorem is proven. ■

3. Extremal point methods

In this section the extremal point methods presented in the introduction are applied to numerically determine the equilibrium measure on an analytic Jordan curve (Subsection 3.1), the harmonic measure in a doubly connected domain of one boundary component and the conformal modulus of a doubly connected domain (Subsection 3.2). For the configurations considered, this, combined with Theorems 1 and 2, finally yields approximations to $\delta(A)/\operatorname{cap}(\partial\Omega)$, converging geometrically fast.

3.1. Subarc of a Jordan Domain. The following two lemmas contain some basic properties of the approximation of holomorphic functions in an annulus with the help of trigonometric sums. Let f be holomorphic in the annulus $\mathcal{A}_\rho = \{\rho < |z| < 1/\rho\}$ with $0 < \rho < 1$. Then, the map $f(e^{i\xi})$ is holomorphic for $\xi \in \mathcal{S}_\rho = \mathbb{R} \times i]\log\rho, -\log\rho[$. The trigonometric interpolation polynomial

$$T_n(\xi, f) = \sum_{k=-n+1}^{n-1} \hat{c}_k e^{ik\xi} + \hat{c}_n \cos n\xi \tag{8}$$

with

$$\hat{c}_k = \frac{1}{2n}\sum_{\ell=1}^{2n} f\left(e^{i\pi\ell/n}\right) e^{-i\pi k\ell/n}, \qquad k = -n, \dots, n, \tag{9}$$

maps $2n$ equidistant points $\pi k/n$, $k = 1, \dots, 2n$, onto $f(e^{i\pi k/n})$ (see e.g. [15, pp. 43–53]). If, in addition, f assumes real values on the unit circle, we have $\hat{c}_k = \overline{\hat{c}_{-k}}$ for $-n \le k \le n$ and thus, we can write

$$T_n(\xi, f) = \frac{1}{2}a_0 + \sum_{k=1}^{n-1}(a_k \cos k\xi + b_k \sin k\xi) + \frac{1}{2}a_n \cos n\xi$$

with $a_k = 2\operatorname{Re}\hat{c}_k$ and $b_k = -2\operatorname{Im}\hat{c}_k$.

Lemma 7. *For every $\tilde{\rho}$ with $\rho < \tilde{\rho} < 1$ there are constants C and ρ with $C > 0$ and $0 < r < 1$ such that*

$$\left|f\left(e^{i\xi}\right) - T_n(\xi, f)\right| < Cr^n, \qquad \xi \in \mathcal{S}_{\tilde{\rho}}.$$

Proof. As f is holomorphic in $\mathcal{A}_\rho$ it possesses a convergent Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k, \qquad z \in \mathcal{A}_\rho.$$

Expressing the error $\hat{c}_k - c_k$ of the discrete Fourier coefficients $\hat{c}_k$ as defined in (9) for $-n \le k \le n$ in terms of the coefficients c_k, $k \in \mathbb{Z}$, and applying Cauchy's estimate leads to

$$|\hat{c}_k - c_k| \le \max_{|z|\in\{R,1/R\}} |f(z)| \cdot \frac{R^k + R^{-k}}{R^{2n} - 1}, \qquad -n \le k \le n, \tag{10}$$

for every $1 < R < 1/\rho$ (see [15, pp. 31]). Now, for given $\tilde{\rho}$, define $\tilde{R} = 1/\tilde{\rho}$ and choose $\tilde{R} < R < 1/\rho$. For $\xi \in \mathcal{S}_{\tilde{\rho}}$, we obtain

$$\begin{aligned}
\left|f\left(e^{i\xi}\right) - T_n(\xi, f)\right| &= \left|\sum_{|k|>n} c_k e^{ik\xi} + \sum_{|k|=n}\left(c_k - \frac{\hat{c}_k}{2}\right)e^{ik\xi} + \sum_{|k|<n}\left(c_k - \hat{c}_k\right)e^{ik\xi}\right| \\
&\le \max_{|z|\in\{R,1/R\}} |f(z)| \cdot \left[2\sum_{k=n+1}^{\infty}\left(\frac{\tilde{R}}{R}\right)^k\right. \\
&\qquad \left. + \left(\frac{\tilde{R}}{R}\right)^n + \frac{R^n + R^{-n}}{R^{2n}-1}\tilde{R}^n + 2\sum_{k=0}^{n-1}\frac{R^k + R^{-k}}{R^{2n}-1}\tilde{R}^k\right] \\
&< \max_{|z|\in\{R,1/R\}} |f(z)| \cdot C_R \left(\frac{\tilde{R}}{R}\right)^n
\end{aligned}$$

with a constant C_R independent of n. To obtain this result, we have considered

$$|e^{ik\xi}| \le \tilde{R}^k, \qquad k \in \mathbb{Z},$$

for $\xi \in \mathcal{S}_{\tilde{\rho}}$. The statement of the lemma follows now with $r = \tilde{R}/R$ and $C = C_R \cdot \max_{|z|\in\{R,1/R\}} |f(z)|$. ■

Lemma 8 asserts that the statement of Lemma 7 remains true even if the interpolation data are slightly distorted: to show this we consider interpolation data $s_{n,k} \in \mathbb{C}$, $1 \le k \le 2n$, and the trigonometric interpolation polynomial

$$S_n(\xi) = \sum_{k=-n+1}^{n-1} \tilde{c}_k e^{ik\xi} + \tilde{c}_n \cos n\xi \tag{11}$$

with

$$\tilde{c}_k = \frac{1}{2n} \sum_{\ell=1}^{2n} s_{n,\ell} e^{-i\pi k\ell/n}, \qquad k = -n, \dots, n, \tag{12}$$

mapping $\pi k/n$, $k = 1, \dots, 2n$, onto $s_{n,k}$, $1 \le k \le 2n$. If again the interpolation data $s_{n,k}$ are real numbers, we can write

$$S_n(\xi) = \frac{1}{2}\tilde{a}_0 + \sum_{k=1}^{n-1}(\tilde{a}_k \cos k\xi + \tilde{b}_k \sin k\xi) + \frac{1}{2}\tilde{a}_n \cos n\xi$$

with $\tilde{a}_k = 2\operatorname{Re}\tilde{c}_k$ and $\tilde{b}_k = -2\operatorname{Im}\tilde{c}_k$.

Lemma 8. *Choose ρ with $0 < \rho < 1$ such that f is holomorphic in A_ρ. Further, let $s_{n,k} \in \mathbb{C}$ be given such that*

$$\left| f\left(e^{i\pi k/n}\right) - s_{n,k} \right| < \tilde{C}\tilde{r}^n, \qquad 1 \le k \le 2n,$$

with constants $\tilde{C}$ and $\tilde{r}$ with $\tilde{C} > 0$ and $0 < \tilde{r} < 1$. If $S_n(\xi)$ is defined as above, then there exist ρ_0 with $\rho < \rho_0 < 1$ and constants C and r with $C > 0$ and $0 < r < 1$ independent of n such that

$$\left| f\left(e^{i\xi}\right) - S_n(\xi) \right| < Cr^n, \qquad \xi \in \mathcal{S}_{\rho_0}.$$

Proof. The hypothesis on the interpolation data $s_{n,k}$, $1 \le k \le 2n$, expression (9) of the discrete Fourier coefficients $\hat{c}_k$ and (12) of the disturbed coefficients $\tilde{c}_k$ imply immediately

$$|\hat{c}_k - \tilde{c}_k| < \tilde{C}\tilde{r}^n, \qquad -n \le k \le n.$$

If one chooses ρ_0 such that $\max\{\rho, \tilde{r}\} < \rho_0 < 1$, one obtains from (8) and (11)

$$|S_n(\xi) - T_n(\xi, f)| < \tilde{C}\tilde{r}^n \left(1 + 2\sum_{k=1}^{n-1} \rho_0^{-k} + \rho_0^{-n}\right) \le C_1 \cdot \left(\frac{\tilde{r}}{\rho_0}\right)^n$$

for $\xi \in S_{\rho_0}$ with $0 < \tilde{r}/\rho_0 < 1$ according to our choice of these constants. The statement of the lemma follows with Lemma 7 and the triangle inequality. ■

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $\Phi\colon \Delta \to \Omega$ be a conformal mapping normalized by $\Phi(z) = z + \mathcal{O}(1)$, $z \to \infty$. If $\gamma\colon [0, 2\pi[\to \Gamma$ is an appropriate parametrization of the analytic Jordan curve $\Gamma = \partial\Omega$ and if g is defined by $g(e^{it}) = \gamma(t)$, $t \in [0, 2\pi[$, then there is ρ with $0 < \rho < 1$ such that g extends to a conformal map of the annulus $\mathcal{A}_\rho = \{\rho < |z| < 1/\rho\}$ mapping the unit circle onto Γ and preserving orientation. If $0 \le s_{n,1} < \ldots < s_{n,2n} < 2\pi$ are the preimages of the Menke points of order n under the parametrization γ, there exist $\alpha_n \in \mathbb{R}$, a constants $\tilde{C}_1$ and $\tilde{r}$ with $\tilde{C}_1 > 0$ and $0 < \tilde{r} < 1$ such that

$$\left| \gamma^{-1} \circ \Phi\left(e^{i(\alpha_n + \pi k/n)}\right) - s_{n,k} \right| \le \tilde{C}_1\tilde{r}^n, \qquad 1 \le k \le 2n,$$

due to Menke's result on the distribution of the extremal points (1) and the uniform continuity of the mapping $\gamma^{-1} \circ \Phi$ on the unit circle. From $\alpha_n = \mathcal{O}(\tilde{r}^n)$ (see the explanations to (1)), we deduce the existence of $\tilde{C}_2 > 0$ so that

$$\left|\gamma^{-1} \circ \Phi\left(e^{i\pi k/n}\right) - s_{n,k}\right| \leq \tilde{C}_2 \tilde{r}^n, \qquad 1 \leq k \leq 2n.$$

Lemma 8 says that the trigonometric interpolation polynomial S_n mapping the points $\pi k/n$ onto $s_{n,k} - \pi k/n$ for $1 \leq k \leq 2n$ provides a sequence of functions $p_n(\xi) = S_n(\xi) + \xi$ which converges uniformly to $\gamma^{-1} \circ \Phi(e^{i\xi})$ on the strip $\mathcal{S}_{\sqrt{\rho}} = \mathbb{R} \times i]1/2 \log \rho, -1/2 \log \rho[$. Note that, due to the Reflection Principle, Φ possesses a univalent continuation to $\mathcal{A}_\rho$. Thus, the sequence (f_n) with

$$f_n\left(e^{i\xi}\right) = \gamma \circ p_n(\xi)$$

converges uniformly to Φ. By Hurwitz' Theorem, there exists $n_0 \in \mathbb{N}$ such that all mappings f_n with $n \geq n_0$ are univalent in $\mathcal{A}_{\rho^{1/3}}$. Moreover, f_n maps the $2n$th roots of unity onto the corresponding Menke points. Considering the representation $e^{ip_n^{-1}(t)} = f_n^{-1} \circ \gamma(t)$, $t \in [0, 2\pi[$, we conclude that

$$\begin{aligned}\mu_n(A) &= \frac{1}{2\pi}\left|p_n^{-1}(b) - p_n^{-1}(a)\right| = \frac{1}{2\pi}\int_a^b \left(p_n^{-1}\right)'(t)\,dt \\ &= \frac{1}{2\pi}\int_a^b \left|\left(f_n^{-1} \circ \gamma\right)'(t)\right|\,dt = \frac{1}{2\pi}\int_A \left|\left(f_n^{-1}\right)'(z)\right|\,|dz|.\end{aligned}$$

Theorem 4 in [34] says that this expression provides a geometrically fast converging approximation to $\mu(A)$. On the other hand, the asymptotic representation (2) assures that $2^{-1/n} R_n^{1/n^2}$ approaches $\operatorname{cap}(\partial\Omega)$ geometrically fast. Considering the representation of $\delta(A)$ in terms of $\operatorname{cap}(\partial\Omega)$ and $\mu(A)$ provided by Theorem 1, the statement of the theorem follows now by an elementary estimate. ■

3.2. Jordan curve in a doubly connected domain. The conformal modulus $M = \operatorname{mod}(\Omega)$ of a doubly connected domain Ω with boundary components A and B (and $\infty \in \Omega$) can be computed via the solution of the following minimal energy problem: let $G(z, \zeta)$ be the Green function in the external domain $\operatorname{Ext}(A)$ of A with pole at ζ. If $\mathcal{M}(B)$ denotes the set of all Borel measures in $\mathbb{C}$ with support in B and total mass 1, then

$$\operatorname{mod}(\Omega) = I^c(\mu^*),$$

where

$$I^c(\mu^*) = \min_{\mu \in \mathcal{M}(B)} I^c(\mu) = \min_{\mu \in \mathcal{M}(B)} \int_B \int_B G(z, \zeta)\,d\mu(z) d\mu(\zeta)$$

(see e.g. [19, p. 175] or [30, pp. 123–137]). We define

$$I_n^c = \frac{1}{n^2} J_n^c + \frac{\log 2}{n}$$

as approximation to $I^c(\mu^*) = \operatorname{mod}(\Omega)$.

For the proof of Theorem 4, the same method will be applied as in [34] and [35]. Thus, the description below is reduced to the essentials.

Proof of Theorem 4. Let (f_n) be a sequence of analytic functions that interpolate the extremal points $w_{n,1}^{[C]}, \ldots, w_{n,2n}^{[C]}$ of order n in the $2n$th roots of unity and that are univalent in an annulus $\mathcal{A}_{\rho_1} = \{\rho_1 < |z| < 1/\rho_1\}$ for some $0 < \rho_1 < 1$. Such a sequence exists due to Lemma 3 in [35]. Then,

$$\begin{aligned} I_n^c = \frac{1}{n^2} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} G\big(f_n\left(e^{i\pi(2\mu-1)/n}\right), f_n\left(e^{i2\pi\nu/n}\right)\big) \\ + \frac{1}{n^2} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \log\left|e^{i\pi(2\mu-1)/n} - e^{i2\pi\nu/n}\right|. \end{aligned} \tag{13}$$

If Ψ denotes a conformal map from the outer domain of A onto $\Delta = \{|z| > 1\}$, fixing ∞, then

$$G(z,\zeta) = \begin{cases} \log\left|\dfrac{1-\Psi(z)\overline{\Psi(\zeta)}}{\Psi(z)-\Psi(\zeta)}\right| & \text{for } \zeta \neq \infty, \\ \log|\Psi(z)| & \text{for } \zeta = \infty. \end{cases} \tag{14}$$

For a function f that is univalent in $\mathcal{A}_{\rho_1}$ and that maps $\partial\mathbb{D}$ onto B, we define

$$F_f(z,\zeta) = \begin{cases} \log\dfrac{\Psi\circ f(z) - \Psi\circ f(\zeta)}{z-\zeta} & \text{for } z \neq \zeta, \\ \log(\Psi\circ f)'(z) & \text{for } z = \zeta, \end{cases}$$

and

$$H_f(z,\zeta) = \log\Big(1 - \Psi\circ f(z)\cdot\overline{\Psi\circ f(\zeta)}\Big).$$

We assume that ρ_1 has been chosen sufficiently large, such that the mapping F_f is analytic in z and ζ throughout $\mathcal{A}_{\rho_1}$, and H_f is analytic for $z \in \mathcal{A}_{\rho_1}$ and antiholomorphic for $\zeta \in \mathcal{A}_{\rho_1}$. Thus, there exist Grunsky-type expansions

$$F_f(z,\zeta) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} b_{k,\ell}(\Psi\circ f)\, z^k \zeta^\ell$$

and

$$H_f(z,\zeta) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} d_{k,\ell}(\Psi\circ f)\, z^k \bar{\zeta}^\ell.$$

Inserting these representations in (13) yields

$$
\begin{aligned}
I_n^c &= -\frac{1}{n^2}\,\mathrm{Re}\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} F_{f_n}\left(e^{i\pi(2\mu-1)/n}, e^{i2\pi\nu/n}\right)\\
&\quad+\frac{1}{n^2}\,\mathrm{Re}\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} H_{f_n}\left(e^{i\pi(2\mu-1)/n}, e^{i2\pi\nu/n}\right)\\
&= -\frac{1}{n^2}\,\mathrm{Re}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} b_{k,\ell}(\Psi\circ f_n)\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} e^{i\pi(2\mu-1)k/n}e^{i2\pi\ell\nu/n}\\
&\quad+\frac{1}{n^2}\,\mathrm{Re}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} d_{k,\ell}(\Psi\circ f_n)\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} e^{i\pi(2\mu-1)k/n}e^{-i2\pi\ell\nu/n}\\
&= \mathrm{Re}\left(d_{0,0}(\Psi\circ f_n) - b_{0,0}(\Psi\circ f_n)\right) + E_n(\Psi\circ f_n)
\end{aligned}
\tag{15}
$$

with

$$
E_n(\Psi\circ f_n) = \mathrm{Re}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}(-1)^k\left(d_{nk,n\ell}(\Psi\circ f_n) - b_{nk,n\ell}(\Psi\circ f_n)\right).
$$

As in [34] and [35] we have profited from the fact that the geometric sums over μ and ν vanish as soon as n does not divide k or ℓ.

In the same way, we can compute the discrete energy

$$
\begin{aligned}
I_{n,f^*}^c &= \frac{1}{n^2}\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} G\left(f^*\left(e^{i\pi(2\mu-1)/n}\right), f^*\left(e^{i2\pi\nu/n}\right)\right) + \frac{\log 2}{n}\\
&= \mathrm{Re}\left(d_{0,0}(\Psi\circ f^*) - b_{0,0}(\Psi\circ f^*)\right) + E_n(\Psi\circ f^*)
\end{aligned}
$$

of a conformal map f^* mapping $\{1<|z|<e^{I^c(\mu^*)}\}$ onto Ω such that $\{|z|=1\}$ corresponds to B, which has been continued to $\mathcal{A}_{\rho_1}$. The minimality of I_n^c implies

$$
I_n^c - \mathrm{Re}(d_{0,0}(\Psi\circ f^*) - b_{0,0}(\Psi\circ f^*)) \le E_n(\Psi\circ f^*). \tag{16}
$$

On the other hand, the extremal measure μ^* can be recovered from its Green potential

$$
u^*(z) = \int_B G(z,\zeta)\,d\mu^*(\zeta)
$$

via

$$
d\mu^*(z) = \frac{1}{2\pi}\frac{\partial u^*}{\partial n}(z)\,|dz| \tag{17}
$$

(see [30]). The potential u^* solves the boundary value problem $\Delta u^* = 0$ in Ω, $u^*=0$ on A and $u^* = I^c(\mu^*)$ on B. From (17), we conclude

$$
I^c(\mu^*) = \frac{1}{4\pi^2}\int_B\int_B G(z,\zeta)\,\frac{\partial u^*}{\partial n}(z)\frac{\partial u^*}{\partial n}(\zeta)\,|dz||d\zeta|.
$$

Parametrizing B over $[0,2\pi[$ by $f^*(e^{it})$, where f^* is a conformal map from the annulus $\{1<|z|<e^{I^c(\mu^*)}\}$ onto Ω, with $\{|z|=1\}$ corresponding to B, leads to

$$I^c(\mu^*) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G\left(f^*(e^{is}), f^*(e^{it})\right) \frac{\partial u^*\circ f^*}{\partial n}(e^{is})\frac{\partial u^*\circ f^*}{\partial n}(e^{it})\, ds\, dt. \tag{18}$$

Now, $u^*\circ f^*$ satisfies the boundary value problem $\Delta(u^*\circ f^*)=0$ in the annulus $\{1<|z|<e^{I^c(\mu^*)}\}$, $u^*=0$ on $\{|z|=e^{I^c(\mu^*)}\}$ and $u^*=I^c(\mu^*)$ on $\{|z|=1\}$. Consequently, $u^*\circ f^* = I^c(\mu^*)-\log|z|$, $z\in\{1<|z|<e^{I^c(\mu^*)}\}$, and (18) reduces to

$$I^c(\mu^*) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G\left(f^*(e^{is}), f^*(e^{it})\right)\, ds\, dt.$$

Considering

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\left|e^{is}-e^{it}\right|\, ds\, dt = 0$$

and substituting (14), one obtains

$$\begin{aligned}
I^c(\mu^*) &= -\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\log\left|\frac{\Psi\circ f^*\left(e^{is}\right)-\Psi\circ f^*\left(e^{it}\right)}{e^{is}-e^{it}}\right| ds\, dt \\
&\quad +\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\log\left|1-\Psi\circ f^*\left(e^{is}\right)\overline{\Psi\circ f^*\left(e^{it}\right)}\right| ds\, dt \\
&= -\frac{1}{4\pi^2}\,\mathrm{Re}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} b_{k,\ell}(\Psi\circ f^*)\cdot\int_0^{2\pi}\int_0^{2\pi} e^{iks}e^{i\ell t}\, ds\, dt \\
&\quad +\frac{1}{4\pi^2}\,\mathrm{Re}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} d_{k,\ell}(\Psi\circ f^*)\cdot\int_0^{2\pi}\int_0^{2\pi} e^{iks}e^{-i\ell t}\, ds\, dt \\
&= \mathrm{Re}\left(d_{0,0}(\Psi\circ f^*)-b_{0,0}(\Psi\circ f^*)\right).
\end{aligned} \tag{19}$$

Next, we compare this with $I^c(\mu_n)$ for the measure $\mu_n\in\mathcal{M}(B)$ with

$$d\mu_n(z) = \frac{1}{2\pi}\left|\left(f_n^{-1}\right)'\right|\,|dz|.$$

Parametrizing B over $[0,2\pi[$ by $f_n(e^{it})$ leads to

$$I^c(\mu_n) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G\left(f_n(e^{is}), f_n(e^{it})\right)\, ds\, dt,$$

and consequently, proceeding exactly as in the derivation of (19),

$$I^c(\mu_n) = \mathrm{Re}\left(d_{0,0}(\Psi\circ f_n)-b_{0,0}(\Psi\circ f_n)\right) = I_n^c - E_n(\Psi\circ f_n)\,,$$

where also (15) has been considered. The minimality of $I^c(\mu^*)$ implies

$$I_n^c - I^c(\mu^*) \geq E_n(\Psi\circ f_n).$$

This combined with (16) and (19) yields

$$E_n(\Psi\circ f_n) \leq I_n^c - I^c(\mu^*) \leq E_n(\Psi\circ f^*).$$

Finally, consider that $\Psi \circ f^* \in \mathcal{F}_{\rho_1}(\Psi \circ B)$ and $\Psi \circ f_n \in \mathcal{F}_{\rho_1}(\Psi \circ B)$, where $\mathcal{F}_{\rho_1}(\Psi \circ B)$ denotes the family of all functions that are univalent in $\mathcal{A}_{\rho_1}$ and map $\partial\mathbb{D}$ onto $\Psi \circ B$ such that orientation is preserved. For all functions g in this class, $|E_n(\Psi \circ g)|$ has been estimated uniformly in [35]. In particular, to estimate $\sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} (-1)^k b_{nk,n\ell}(g)$ for $g \in \mathcal{F}_{\rho_1}(\Psi \circ B)$, a generalization of the classical Grunsky inequalities to functions univalent in an annulus by R. Kühnau has been applied [18]. One obtains for any r with $\rho_1 < r < 1$ a constant C that is independent of $n \in \mathbb{N}$ such that

$$|E_n(\Psi \circ g)| \le C r^n$$

for all $g \in \mathcal{F}_{\rho_1}(\Psi \circ B)$. An explicit representation of $C = C(r)$ can be found in [35]. This completes the proof of the theorem, since $I^c(\mu^*) = \operatorname{mod}(\Omega)$. ∎

Now we prove Theorem 5.

Proof of Theorem 5. First observe that the harmonic measure $\omega(z, B, \Omega)$ coincides with $(\operatorname{mod}(\Omega))^{-1} \cdot u^*(z)$, where

$$u^*(z) = \int_B G(z, \zeta)\, d\mu^*(\zeta)$$

is the Green potential of the equilibrium measure. Considering the symmetry of G, one obtains

$$\omega = \omega(\infty, B, \Omega) = \frac{1}{\operatorname{mod}(\Omega)} \int_B G(\zeta, \infty)\, d\mu^*(\zeta).$$

We claim that

$$\omega_n = \frac{1}{2n \operatorname{mod}(\Omega)} \sum_{\mu=1}^{2n} G\big(w_{n,\mu}^{[C]}, \infty\big),$$

where $w_{n,\mu}^{[C]}$, $\mu = 1, \ldots, 2n$, are the Menke points for condensers of order n on B, is an $\mathcal{O}(r^n)$ approximation to ω with a suitable r with $0 < r < 1$: let the sequence (f_n) be defined as in the proof of Theorem 4. Moreover, let Ψ be a conformal map from the external domain of A onto Δ fixing ∞. Then

$$\omega = \frac{1}{\operatorname{mod}(\Omega)} \int_B \log|\Psi(\zeta)|\, d\mu^*(\zeta) = \frac{1}{2\pi \operatorname{mod}(\Omega)} \int_0^{2\pi} \log\left|\Psi \circ f_n^*\left(e^{it}\right)\right| dt, \tag{20}$$

where f_n^* is a conformal map from $\{1 < |z| < e^{I^c(\mu^*)}\}$ onto Ω such that $\{|z| = 1\}$ corresponds to B and $f_n^*(1) = f_n(1)$. Again we have used $\partial(u^* \circ f_n^*)/\partial n(e^{it}) = 1$ as in the proof of Theorem 4. We claim that ω is geometrically fast approximated by the quantities

$$\tilde{\omega}_n = \frac{1}{\operatorname{mod}(\Omega)} \int_B \log|\Psi(\zeta)|\, d\mu_n(\zeta) \tag{21}$$

with

$$d\mu_n(z) = \frac{1}{2\pi} \left|\left(f_n^{-1}\right)'(z)\right| |dz|.$$

Taking f_n^* as parametric representation of B leads to

$$\tilde{\omega}_n = \frac{1}{2\pi \operatorname{mod}(\Omega)} \int_0^{2\pi} \log \left|\Psi \circ f_n^*\left(e^{it}\right)\right| \cdot \left|\left(f_n^{-1} \circ f_n^*\right)'\left(e^{it}\right)\right| \, dt.$$

This, together with (20), implies

$$|\omega - \tilde{\omega}_n| \le \frac{1}{2\pi \operatorname{mod}(\Omega)} \int_0^{2\pi} \log \left|\Psi \circ f_n^*\left(e^{it}\right)\right| \cdot \left|\left|g_n'\left(e^{it}\right)\right| - 1\right| \, dt, \tag{22}$$

where $g_n = f_n^{-1} \circ f_n^* \in \mathcal{F}_{\rho_1}(\partial\mathbb{D})$. As before, $\mathcal{F}_{\rho_1}(\partial\mathbb{D})$ denotes the family of all functions which are univalent in $\mathcal{A}_{\rho_1}$. Fix $\partial\mathbb{D}$, preserving its orientation. From Theorem 4 in [35], the existence of constants C_0 and r_1 with $C_0 > 0$ and $0 < r_1 < 1$ follows such that

$$|h_n(t) - t| \le C_0 {r_1}^n$$

for a lift h_n of g_n, i.e. a map with $g_n(e^{it}) = e^{ih_n(t)}$ and $h_n(0) = 0$. Considering $h_n'(t) = e^{it} \cdot g_n'(e^{it})/g_n(e^{it}) > 0$ and h_n being analytic, we deduce the existence of a positive constant C_1 with

$$\left|\left|g_n'(e^{it})\right| - 1\right| = \left|e^{it}\frac{g_n'(e^{it})}{g_n(e^{it})} - 1\right| = |h_n'(t) - 1| \le C_1 {r_1}^n$$

uniformly for $t \in \mathbb{R}$. Applying this estimate on (22) yields the existence of constants C_2 and r_2 with $C_2 > 0$ and $0 < r_2 < 1$ such that

$$|\omega - \tilde{\omega}_n| \le C_2 {r_2}^n, \tag{23}$$

as we have claimed. Next, we show that the deviation of $\tilde{\omega}_n$ from ω_n tends to 0 geometrically fast, if n tends to infinity: for this purpose, we pull the integral (21) back to the unit circle by the parametric representation f_n and obtain

$$\tilde{\omega}_n = \frac{1}{2\pi \operatorname{mod} \Omega} \int_0^{2\pi} \log \left|\Psi \circ f_n\left(e^{it}\right)\right| \, dt.$$

The conformal map Ψ of $\operatorname{Ext}(A)$ onto Δ maps a bounded, simply connected domain D_B containing B onto the simply connected domain $\Psi(D_B) \subset \Delta$. By possibly enlarging ρ_1, we can achieve that $f_n(\mathcal{A}_{\rho_1}) \subset D_B$ and $\Psi \circ f_n(\mathcal{A}_{\rho_1}) \subset \Psi(D_B)$. Consequently $\log \Psi \circ f_n$ can be defined as a holomorphic function throughout $\mathcal{A}_{\rho_1}$. Applying the estimate (10) to its discrete Fourier coefficient

$$\hat{c}_0 = \frac{1}{2n} \sum_{\ell=1}^{2n} \log \Psi\left(w_{n,\ell}^{[C]}\right),$$

where $w_{n,k}^{[C]} = f_n(e^{2\pi ik/n})$ are the Menke points for condensers of order n, yields for any ρ_2 with $\rho_1 < \rho_2 < 1$

$$|\hat{c}_0 - c_0| \le C_3 \cdot \frac{2{\rho_2}^{2n}}{1 - {\rho_2}^{2n}}, \tag{24}$$

with $C_3 = C_3(\rho) > 0$ and

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} \log \Psi \circ f_n\left(e^{it}\right)\, dt.$$

Obviously, we have

$$\omega_n = \frac{1}{2n \operatorname{mod}(\Omega)} \sum_{\nu=1}^{2n} G\left(\omega_{n,\nu}^{[C]}, \infty\right) = \frac{1}{\operatorname{mod}(\Omega)} \operatorname{Re} \hat{c}_0$$

and

$$\tilde{\omega}_n = \frac{1}{\operatorname{mod}(\Omega)} \operatorname{Re} c_0.$$

Combining inequalities (23) and (24) leads to the statement of the theorem. ∎

This article is concluded with a proof of Theorem 6:

Proof of Theorem 6. Theorem 4 implies the existence of constants C and r with $C > 0$ and $0 < \rho < 1$ such that

$$\left|\frac{1}{I_n^c} - \frac{1}{M}\right| \le C\rho^n.$$

For $v \in \mathbb{R}$, we deduce

$$\begin{aligned}
&\left|\vartheta_3\Big(v, i\frac{\pi}{M}\Big) - \vartheta_3\Big(v, i\frac{\pi}{I_n^c}\Big)\right| \\
&\quad = \left|\sum_{k=-\infty}^{\infty}\Big(e^{-k^2\pi^2/M} - e^{-k^2\pi^2/I_n^c}\Big) e^{2ki\pi v}\right| \\
&\quad \le \sum_{k=-\infty}^{\infty} e^{-k^2\pi^2/M}\left|1 - e^{k^2\pi^2(1/M - 1/I_n^c)}\right| \\
&\quad \le \Big(e^{N^2\pi^2 C\rho^n} - 1\Big)\cdot \sum_{k=-N}^{N} e^{-k^2\pi^2/M} + \sum_{|k|>N} e^{-k^2\pi^2/M}\cdot\Big(e^{k^2\pi^2 C\rho^n} - 1\Big) \\
&\quad \le N^2\pi^2 C\rho^n \cdot e^{N^2\pi^2 C\rho^n}\cdot \sum_{k=-\infty}^{\infty} e^{-k^2\pi^2/M} + \pi^2 C\rho^n \cdot \sum_{|k|>N} k^2 e^{-k^2\pi^2(1/M - C\rho^n)}.
\end{aligned}$$

With the choice $\rho^{-n/2} - 1 \le N^2 < \rho^{-n/2}$ and n_0 such that $1/M - C\rho^n > 1/(2M)$ for $n \ge n_0$, one obtains the existence of positive constants C_1 and C_2 such that

$$\left|\vartheta_3\Big(v, i\frac{\pi}{M}\Big) - \vartheta_3\Big(v, i\frac{\pi}{I_n^c}\Big)\right| \le C_1\rho^{n/2} + \pi^2 C\rho^n \cdot \sum_{|k|>\rho^{-n/2}} k^2 e^{-k^2\pi^2/(2M)} \le C_2\rho^{n/2},$$

where ρ and C_2 are independent of v and n. Similarly, we obtain

$$\left|\vartheta_2\Big(v, i\frac{\pi}{M}\Big) - \vartheta_2\Big(v, i\frac{\pi}{I_n^c}\Big)\right| \le C_3\rho^{n/2}.$$

Together with Theorem 5,

$$|\omega_n - \omega| \leq C_4 {\rho_2}^n,$$

the estimate stated in the theorem follows now from Theorem 2 after a straightforward computation. ∎

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Marcus Stiemer E-MAIL: `stiemer@math.uni-dortmund.de`
ADDRESS: *Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany.*