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On the Solution of Discrete Vekua Equations

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Abstract. We consider finite difference equations of Vekua type. Main goal of the paper is to prove a representation formula for the solution of homogeneous equations in the form of a product with one factor being a discrete holomorphic function.

Keywords. Finite difference equations, complex Vekua equation, discrete holomorphic functions, representation of solutions.

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1. Introduction

Since the creation of the theory of generalized analytic functions by Vekua [22] the Vekua equations play an important role in the theory of partial differential equations. These complex Vekua equations extend the possibilities of complex analysis and improve the connections between real and complex analysis. We mention only transformation methods which allow reduction of the solution of more general differential equations to those for analytic or generalized analytic functions with concrete application in analysis, geometry, mechanics and in the theory of quasiconformal mappings.

The Vekua equation generalizes the Cauchy-Riemann differential equations on the one hand and the concept of complex differentiability on the other. Moreover they are equivalent to certain first order systems of real partial differential equations. The Vekua equation is also called the Carleman-Bers-Vekua equation (or system) because Carleman with his investigation of general first order systems and Bers [1] with his theory of pseudo-analytic functions essentially contributed to the theory. We will strongly follow the approach of Vekua in this paper and present a finite difference method for the solution of this system. Since in the theory of finite differences all the contributions mentioned are not equivalent we shall only use the notation of Vekua equations.

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Having in mind that the theory of finite difference methods is well developed, including discrete potential theories (see e.g. [4, 16, 17] for the general theory, [2, 7, 8, 9, 16, 20] for calculations of fundamental solutions and [18, 19, 21] for estimates of fundamental solutions) and discrete function theories (see e.g. [6, 10, 13, 14, 15, 23]) it is the goal of this paper to study finite difference equations of Vekua type. Applications of discrete function theory to boundary value problems have already been considered in [11, 12, 13, 16].

We describe the general structure of the solutions. The formulae obtained are the basis of a discrete theory which is strongly related to methods in discrete function theory. Using the matrix calculus we can prove for special classes of solutions, that they can be written as a product with one factor being a discrete holomorphic function. This property is quite analogous to the continuous case.

At the discrete level some technical difficulties appear due to the product rule of differentiation for lattice functions. This problem implies that the set of discrete holomorphic functions does not form an algebra. With a special matrix product we are able to overcome these difficulties. We restrict ourselves for simplicity to Vekua equations with constant coefficients inside a bounded domain.

2. The classical form of the Vekua equation

Let $w(z)$ be a complex valued function with $z = x + iy$, the set $G \subset \mathbb{R}^2$ be a bounded domain of the complex plane and G^* be a set of isolated points with respect to G. The classical Vekua equation is a differential equation of the form

$$\partial_{\bar{z}} w + Aw + B\bar{w} = F, \tag{1}$$

where

$$\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and $F \in L_p(G)$. If $w(z)$ satisfies equation (1) at each point of $G \setminus G^*$ then this solution is called a generalized solution. If G^* is empty, then $w(z)$ is called a regular solution. Let E be the unit disc with $|z| \le 1$ and $L_{p,2}(\mathbb{R}^2)$ be a set of functions with $f(z) \in L_p(E)$ and $|z|^{-2} f(1/z) \in L_p(E)$ for $p \ge 1$. Furthermore we use the notation

$$g(z) = \begin{cases} A(z) + B(z)\dfrac{\overline{w(z)}}{w(z)} & \text{for } w(z) \neq 0,\ z \in G, \\ A(z) + B(z) & \text{for } w(z) = 0,\ z \in G. \end{cases}$$

If the coefficients A and B satisfy the conditions $A, B \in L_{p,2}(\mathbb{R}^2)$ for $p > 2$ then each generalized solution of the homogeneous equation (1) can be written in the form

$$w(z) = \Phi(z) e^{v(z)}, \tag{2}$$

where Φ is a holomorphic function in G and

$$v(z) = \frac{1}{\pi} \int_G \frac{g(\zeta)}{\zeta - z} \, d\xi d\eta \in C_{(p-2)/p}, \qquad p > 2.$$

The representation formula (2) for ω is known as the Similarity Principle. With the help of this formula many properties of analytic functions can be carried over from the classical function theory to generalized solutions of the homogeneous equation (1). In the following we look for a discretization of this important property.

3. A Solution of the homogeneous discrete problem

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the unit vectors in the space $\mathbb{R}^2$. An equidistant lattice with mesh width $h > 0$ is defined by

$$\mathbb{R}_h^2 = \{mh = (m_1 h, m_2 h) : \, m_1, m_2 \in \mathbb{Z}\}.$$

Furthermore we introduce the set

$$K = \{(0,0), (1,0), (-1,0), (0,1), (0,-1)\}$$

and denote by $G_h = G \cap \mathbb{R}_h^2$ the discrete domain with the boundary

$$\gamma_h^- = \{mh \in \mathbb{R}_h^2 \setminus G_h : \, \exists \, k \in K \text{ with } mh + kh \in G_h\}.$$

We consider complex valued functions

$$w(mh) = (w_0(mh), w_1(mh)) = (\operatorname{Re} w(mh), \operatorname{Im} w(mh))$$

and introduce for $j \in \{1, 2\}$ and $k \in \{0, 1\}$ forward differences

$$D_h^j w_k(mh) = h^{-1}(w_k(mh + he_j) - w_k(mh))$$

and backward differences

$$D_h^{-j} w_k(mh) = h^{-1}(w_k(mh) - w_k(mh - he_j)).$$

In order to simplify the problem we write $A = a_1 + ia_2$ and $B = b_1 + ib_2$, and require that the coefficients a_1, a_2, b_1 and b_2 are constant in $G_h \cup \gamma_h^-$ and are equal to zero in $\mathbb{R}_h^2 \setminus (G_h \cup \gamma_h^-)$. Using the group homeomorphism between complex numbers $\alpha + i\beta$ and matrices $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ we can approximate equation (1) by the linear system of equations

$$(3) \qquad \frac{1}{2} \begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix} \begin{pmatrix} w_0 \\ -w_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

for all $mh \in G_h$. We remark that the difference operator

$$D^{1h} = \begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}$$

has the important property

$$D^{1h}D^{2h} = \begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix} \begin{pmatrix} D_h^1 & D_h^2 \\ -D_h^{-2} & D_h^{-1} \end{pmatrix} = \begin{pmatrix} \Delta_h & 0 \\ 0 & \Delta_h \end{pmatrix}$$

where

$$\Delta_h = D_h^1 D_h^{-1} + D_h^2 D_h^{-2}$$

is the discrete Laplacian with

$$\Delta_h u(mh) = \sum_{k\in K} c_k u(mh - kh), \qquad c_k = \begin{cases} \dfrac{1}{h^2} & \text{for } k \in K,\ k \neq (0,0), \\ \dfrac{-4}{h^2} & \text{for } k = (0,0). \end{cases}$$

A function $w(mh)$ with $D^{1h}D^{2h}w(mh) = 0$ is called *discrete harmonic*. If the property $D^{1h}w(mh) = 0$ is fulfilled then $w(mh)$ is said to be *discrete holomorphic*. More details about these operators and functions are presented in [3, 5, 8, 10, 16, 23].

In order to describe the solution of the system (2) we look for a discretization of the exponential function.

Lemma 3.1. *In the one-dimensional case we have*

$$D_h^1(1+h)^{m_1} = (1+h)^{m_1}, \qquad D_h^{-1}(1-h)^{-m_1} = (1-h)^{-m_1},$$
$$D_h^1(1-h)^{m_1} = -(1-h)^{m_1}, \qquad D_h^{-1}(1+h)^{-m_1} = -(1+h)^{-m_1}.$$

Proof. We analyze only the first equation. The proof for the other equations is quite similar. Based on the definition of forward differences we obtain

$$\begin{aligned} D_h^1(1+h)^{m_1} &= h^{-1}[(1+h)^{m_1+1} - (1+h)^{m_1}] \\ &= h^{-1}(1+h)^{m_1}(1+h-1) = (1+h)^{m_1}. \end{aligned}$$

■

In relation to the definition of the discrete exponential functions in Lemma 3.1 we can describe discrete versions of the trigonometric functions $\sin(x)$ and $\cos(x)$.

Lemma 3.2. *The discrete functions*

$$\begin{aligned} \cos_h^+(m_1h) &= \frac{1}{2}[(1+ih)^{m_1} + (1-ih)^{m_1}], \\ \sin_h^+(m_1h) &= \frac{1}{2i}[(1+ih)^{m_1} - (1-ih)^{m_1}] \end{aligned}$$

have the properties

$$\begin{aligned} D_h^1 D_h^1 \cos_h^+(m_1h) &= -D_h^1 \sin_h^+(m_1h) = -\cos_h^+(m_1h), \\ D_h^1 D_h^1 \sin_h^+(m_1h) &= D_h^1 \cos_h^+(m_1h) = -\sin_h^+(m_1h). \end{aligned}$$

For the functions

$$\cos_h^-(m_1h) = \frac{1}{2}[(1-ih)^{-m_1} + (1+ih)^{-m_1}],$$
$$\sin_h^-(m_1h) = \frac{1}{2i}[(1-ih)^{-m_1} - (1+ih)^{-m_1}]$$

we obtain

$$D_h^{-1}D_h^{-1}\cos_h^-(m_1h) = -D_h^{-1}\sin_h^-(m_1h) = -\cos_h^-(m_1h),$$
$$D_h^{-1}D_h^{-1}\sin_h^-(m_1h) = D_h^{-1}\cos_h^-(m_1h) = -\sin_h^-(m_1h).$$

The proof of Lemma 3.2 is again a straightforward calculation by using the definition of forward and backward differences.

Considering the properties obtained in Lemma 3.1 we are able to calculate a special solution of the homogeneous system (2) as follows.

Lemma 3.3. *The system*

$$(4)\qquad \frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix}\begin{pmatrix} w_0 \\ -w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has the special solution

$$w_0(mh) = (1+2(a_1+b_1)h)^{-m_1}(1+2(a_2+b_2)h)^{-m_2},$$
$$w_1(mh) = (1-2(a_1-b_1)h)^{m_1}(1-2(a_2-b_2)h)^{m_2}.$$

Proof. We investigate the first summand in (3) with respect to w_0 and obtain

$$\begin{aligned}\frac{1}{2}D_h^{-1}w_0(mh) &= \frac{1}{2h}[(1+2(a_1+b_1)h)^{-m_1} - (1+2(a_1+b_1)h)^{-m_1+1}] \\ &\quad \cdot(1+2(a_2+b_2)h)^{-m_2} \\ &= \frac{1}{2h}(1+2(a_1+b_1)h)^{-m_1}(1+2(a_2+b_2)h)^{-m_2} \\ &\quad \cdot[1-1-2(a_1+b_1)h] \\ &= -(a_1+b_1)w_0(mh).\end{aligned}$$

Similarly we get

$$-\frac{1}{2}D_h^2 w_1(mh) = (a_2-b_2)w_1(mh),$$
$$\frac{1}{2}D_h^{-2}w_0(mh) = -(a_2+b_2)w_0(mh),$$
$$\frac{1}{2}D_h^1 w_1(mh) = -(a_1-b_1)w_1(mh).$$

■

In order to build up a theory which is based on discrete analytic functions we look now for solutions of the homogeneous discrete Vekua equations that can be split into two factors. At first we look at the case $b_1 = b_2 = 0$.

3.1. The homogeneous discrete Vekua equations in case $b_1 = b_2 = 0$. In the following we restrict attention to the space of functions $(w_0(mh), w_1(mh))$, which are solutions of the system

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix} = \begin{pmatrix} -a_1 & a_2 \\ -a_2 & -a_1 \end{pmatrix}\begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix}. \tag{5}$$

Because we added one column, the space of functions considered is a subspace of the space of functions $w(mh)$ which solve the homogeneous system (3) in case $b_1 = b_2 = 0$. A simple calculation shows that the special solution in Lemma 3.3 does not belong to this subspace. We underline that we start with the system (3) in the case $b_1 = b_2 = 0$ and add two equations in the following sense: if we write the system (4) as four equations it is easy to see that for $h \to 0$ the equations which are related to the first and fourth matrix element approximate the same differential equation. We arrive at the same situation if we look at the difference equations which are related to the second and third matrix element. From this point of view the system (4) is well-defined.

In order to get more compact formulae we will write in the future (m_1, m_2) instead of (m_1h, m_2h) in the arguments of the functions considered. This notation includes all necessary information and is not to be understood as restriction to the mesh width $h = 1$. Using the matrix representation in (4) we can prove our main theorem below.

Theorem 3.1. *Let $w(mh)$ be an arbitrary solution of the problem (4) and $u(mh)$ be a solution of the problem*

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} u_0(m_1+1, m_2+1) & -u_1(m_1+1, m_2+1) \\ u_1(m_1, m_2) & u_0(m_1, m_2) \end{pmatrix}$$
$$= \begin{pmatrix} a_1u_0(m_1,m_2+1)-a_2u_1(m_1,m_2+1) & -a_2u_0(m_1,m_2+1)-a_1u_1(m_1,m_2+1) \\ a_2u_0(m_1+1,m_2)+a_1u_1(m_1+1,m_2) & a_1u_0(m_1+1,m_2)-a_2u_1(m_1+1,m_2) \end{pmatrix}.$$

Then we obtain for all $mh \in G_h$

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\left[\begin{pmatrix} u_0(m_1+1, m_2+1) & -u_1(m_1+1, m_2+1) \\ u_1(m_1, m_2) & u_0(m_1, m_2) \end{pmatrix}\right.$$
$$\left.\cdot\begin{pmatrix} w_0(m_1, m_2) & -w_1(m_1, m_2) \\ w_1(m_1, m_2) & w_0(m_1, m_2) \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. In the first matrix element on the left-hand side we add the two summands

$$\begin{aligned}
S_1 &= \frac{1}{2}D_h^{-1}[u_0(m_1+1,m_2+1)w_0(m_1,m_2) - u_1(m_1+1,m_2+1)w_1(m_1,m_2)] \\
&= \frac{1}{2h}u_0(m_1+1,m_2+1)w_0(m_1,m_2) - \frac{1}{2h}u_1(m_1+1,m_2+1)w_1(m_1,m_2) \\
&\quad -\frac{1}{2h}u_0(m_1,m_2+1)w_0(m_1-1,m_2) + \frac{1}{2h}u_1(m_1,m_2+1)w_1(m_1-1,m_2) \\
&\quad +\frac{1}{2h}u_0(m_1,m_2+1)w_0(m_1,m_2) - \frac{1}{2h}u_1(m_1,m_2+1)w_1(m_1,m_2) \\
&\quad -\frac{1}{2h}u_0(m_1,m_2+1)w_0(m_1,m_2) + \frac{1}{2h}u_1(m_1,m_2+1)w_1(m_1,m_2) \\
&= \frac{1}{2}u_0(m_1,m_2+1)[D_h^{-1}w_0(m_1,m_2)] \\
&\quad -\frac{1}{2}u_1(m_1,m_2+1)[D_h^{-1}w_1(m_1,m_2)] \\
&\quad +\frac{1}{2}w_0(m_1,m_2)[D_h^{-1}u_0(m_1+1,m_2+1)] \\
&\quad -\frac{1}{2}w_1(m_1,m_2)[D_h^{-1}u_1(m_1+1,m_2+1)]
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= -\frac{1}{2}D_h^2[u_1(m_1,m_2)w_0(m_1,m_2) + u_0(m_1,m_2)w_1(m_1,m_2)] \\
&= \frac{1}{2}u_1(m_1,m_2+1)[-D_h^2w_0(m_1,m_2)] + \frac{1}{2}u_0(m_1,m_2+1)[-D_h^2w_1(m_1,m_2)] \\
&\quad +\frac{1}{2}w_0(m_1,m_2)[-D_h^2u_1(m_1,m_2)] + \frac{1}{2}w_1(m_1,m_2)[-D_h^2u_0(m_1,m_2)]
\end{aligned}$$

and obtain

$$\begin{aligned}
S_1 + S_2 &= \frac{1}{2}u_0(m_1,m_2+1)[D_h^{-1}w_0(m_1,m_2) - D_h^2w_1(m_1,m_2)] \\
&\quad +\frac{1}{2}u_1(m_1,m_2+1)[-D_h^{-1}w_1(m_1,m_2) - D_h^2w_0(m_1,m_2)] \\
&\quad +\frac{1}{2}w_0(m_1,m_2)[D_h^{-1}u_0(m_1+1,m_2+1) - D_h^2u_1(m_1,m_2)] \\
&\quad +\frac{1}{2}w_1(m_1,m_2)[-D_h^{-1}u_1(m_1+1,m_2+1) - D_h^2u_0(m_1,m_2)] \\
&= u_0(m_1,m_2+1)[-a_1w_0(m_1,m_2) + a_2w_1(m_1,m_2)] \\
&\quad +u_1(m_1,m_2+1)[a_1w_1(m_1,m_2) + a_2w_0(m_1,m_2)] \\
&\quad +w_0(m_1,m_2)[a_1u_0(m_1,m_2+1) - a_2u_1(m_1,m_2+1)] \\
&\quad +w_1(m_1,m_2)[-a_2u_0(m_1,m_2+1) - a_1u_1(m_1,m_2+1)] \\
&= 0.
\end{aligned}$$

By the same way we get for the second matrix element

$$\frac{1}{2}D_h^{-1}[-u_0(m_1+1,m_2+1)w_1(m_1,m_2)-u_1(m_1+1,m_2+1)w_0(m_1,m_2)]$$
$$-\frac{1}{2}D_h^2[-u_1(m_1,m_2)w_1(m_1,m_2)+u_0(m_1,m_2)w_0(m_1,m_2)]=0.$$

Finally we obtain for the last two matrix elements

$$\frac{1}{2}D_h^{-2}[u_0(m_1+1,m_2+1)w_0(m_1,m_2)-u_1(m_1+1,m_2+1)w_1(m_1,m_2)]$$
$$+\frac{1}{2}D_h^1[u_1(m_1,m_2)w_0(m_1,m_2)+u_0(m_1,m_2)w_1(m_1,m_2)]=0$$

as well as

$$\frac{1}{2}D_h^{-2}[-u_0(m_1+1,m_2+1)w_1(m_1,m_2)-u_1(m_1+1,m_2+1)w_0(m_1,m_2)]$$
$$+\frac{1}{2}D_h^1[-u_1(m_1,m_2)w_1(m_1,m_2)+u_0(m_1,m_2)w_0(m_1,m_2)]=0.$$

■

We can identify the product of the matrices with respect to $u(mh)$ and $w(mh)$ with the matrix representation of a discrete holomorphic function $\Phi(mh)$. If we can show the existence of a solution $u(mh)$ of the matrix equation in Theorem 3.1 and if it is possible to invert the matrix of $u(mh)$ then we can describe the solution of the problem (4) in a form that is quite similar to the representation in the classical case which was described in Section 2.

Further we remark that only with the special discretization of the Cauchy-Riemann operator is it possible to overcome the difficulties that appear from the product rule

$$D_h^1(u_i(m_1,m_2)v_k(m_1,m_2))$$
$$=u_i(m_1+1,m_2)D_h^1v_k(m_1,m_2)+v_k(m_1,m_2)D_h^1u_i(m_1,m_2),\quad i,k\in\{0,1\},$$

for the operator D_h^1 (and also for the other difference operators) in form of the factor $u_i(m_1+1,m_2)$ in the neighbourhood of the mesh point mh. More explicitly it is difficult to collect all necessary terms in the proof of Theorem 3.1.

In order to construct a solution $u(mh)$ we make a special "ansatz" which is based on the definition of the discrete sinus and cosinus function in Lemma 3.2.

Lemma 3.4. *For unknowns α, β, γ and δ with $\alpha^2+\beta^2\neq 0$ and $\gamma^2+\delta^2\neq 0$ the substitutions*

$$u_0(m_1,m_2)=\frac{1}{2}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2}+\frac{1}{2}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}$$
$$u_1(m_1,m_2)=\frac{1}{2i}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2}-\frac{1}{2i}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}$$

give a non-trivial solution of the problem

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} u_0(m_1+1,m_2+1) & -u_1(m_1+1,m_2+1) \\ u_1(m_1,m_2) & u_0(m_1,m_2) \end{pmatrix}$$
$$= \begin{pmatrix} a_1u_0(m_1,m_2+1)-a_2u_1(m_1,m_2+1) & -a_2u_0(m_1,m_2+1)-a_1u_1(m_1,m_2+1) \\ a_2u_0(m_1+1,m_2)+a_1u_1(m_1+1,m_2) & a_1u_0(m_1+1,m_2)-a_2u_1(m_1+1,m_2) \end{pmatrix}.$$

Proof. We can write the first matrix element in the form

$$\begin{aligned}
&\frac{1}{2}D_h^{-1}u_0(m_1+1,m_2+1) - \frac{1}{2}D_h^2u_1(m_1,m_2) \\
&\quad = \frac{1}{2}h^{-1}\Big[\frac{1}{2}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2+1}(\alpha-i\beta-1) \\
&\qquad + \frac{1}{2}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2+1}(\alpha+i\beta-1) \\
&\qquad + \frac{1}{2i}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2+1}\left(\frac{\gamma-i\delta}{\gamma^2+\delta^2}-1\right) \\
&\qquad - \frac{1}{2i}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2+1}\left(\frac{\gamma+i\delta}{\gamma^2+\delta^2}-1\right)\Big] \\
&\quad = \frac{1}{2}u_0(m_1,m_2+1)\left(\frac{\alpha-1}{h} - \frac{\delta}{h(\gamma^2+\delta^2)}\right) \\
&\qquad + \frac{1}{2}u_1(m_1,m_2+1)\left(\frac{\beta-1}{h} + \frac{\gamma}{h(\gamma^2+\delta^2)}\right).
\end{aligned}$$

Consequently the unknowns α, β, γ and δ have to satisfy the equations

$$2ha_1 = \alpha - 1 - \frac{\delta}{\gamma^2+\delta^2}, \qquad -2ha_2 = \beta - 1 + \frac{\gamma}{\gamma^2+\delta^2}. \tag{6}$$

For the second matrix element we obtain

$$\begin{aligned}
&-\frac{1}{2}D_h^{-1}u_1(m_1+1,m_2+1) - \frac{1}{2}D_h^2u_0(m_1,m_2) \\
&\quad = \frac{1}{2}u_0(m_1,m_2+1)\left(\frac{\gamma}{h(\gamma^2+\delta^2)} + \frac{\beta-1}{h}\right) \\
&\qquad + \frac{1}{2}u_1(m_1,m_2+1)\left(\frac{1-\alpha}{h} + \frac{\delta}{h(\gamma^2+\delta^2)}\right),
\end{aligned}$$

where the unknowns α, β, γ and δ have to satisfy the same equations as in the above case. We consider now the third matrix element. From

$$\begin{aligned}
&\frac{1}{2}D_h^{-2}u_0(m_1+1,m_2+1)+\frac{1}{2}D_h^1u_1(m_1,m_2)\\
&\quad=\frac{1}{2}u_0(m_1+1,m_2)\left(\frac{\gamma-1}{h}-\frac{\beta}{h(\alpha^2+\beta^2)}\right)\\
&\qquad+\frac{1}{2}u_1(m_1+1,m_2)\left(\frac{-\alpha}{h(\alpha^2+\beta^2)}+\frac{1-\delta}{h}\right)
\end{aligned}$$

we conclude

$$2ha_2=\gamma-1-\frac{\beta}{\alpha^2+\beta^2},\qquad -2ha_1=\delta-1+\frac{\alpha}{\alpha^2+\beta^2}. \tag{7}$$

For the last matrix element we write

$$\begin{aligned}
&-\frac{1}{2}D_h^{-2}u_1(m_1+1,m_2+1)+\frac{1}{2}D_h^1u_0(m_1,m_2)\\
&\quad=\frac{1}{2}u_1(m_1+1,m_2)\left(\frac{1-\gamma}{h}+\frac{\beta}{h(\alpha^2+\beta^2)}\right)\\
&\qquad+\frac{1}{2}u_0(m_1+1,m_2)\left(\frac{-\alpha}{h(\alpha^2+\beta^2)}+\frac{1-\delta}{h}\right)
\end{aligned}$$

and obtain again the equations (6). ■

With the equations in (5) and (6) we have found four equations for the unknowns α, β, γ and δ in the expressions for $u_0(m_1,m_2)$ and $u_1(m_1,m_2)$ in Lemma 3.4. We are now interested in the solution of this system of equations. More efficient and in our case sufficient is the determination of the expressions $\alpha+i\beta$, $\alpha-i\beta$, $\gamma+i\delta$ and $\gamma-i\delta$, because otherwise we would obtain equations of higher order. In the following two lemmata we present a method of calculating these four complex expressions. We simplify the notation by substituting

$$s_1=1+2a_1h,\qquad s_2=1+2a_2h,\qquad s_3=1-2a_2h,\qquad s_4=1-2a_1h.$$

Lemma 3.5. *The system* (5)–(6) *can be written in the form*

$$\begin{aligned}
\alpha+i\beta&=s_1-i\frac{1}{\gamma-i\delta}+is_3 &\qquad \alpha-i\beta&=s_1+i\frac{1}{\gamma+i\delta}-is_3\\
\gamma-i\delta&=s_2+i\frac{1}{\alpha+i\beta}-is_4 &\qquad \gamma+i\delta&=s_2-i\frac{1}{\alpha-i\beta}+is_4.
\end{aligned}$$

Proof. From (5)–(6) and the above substitution it follows that

$$\alpha=s_1+\frac{\delta}{\gamma^2+\delta^2},\quad \beta=s_3-\frac{\gamma}{\gamma^2+\delta^2},\quad \gamma=s_2+\frac{\beta}{\alpha^2+\beta^2},\quad \delta=s_4-\frac{\alpha}{\alpha^2+\beta^2}.$$

In order to prove the assertion we start with these representations and calculate the four expressions on the left-hand side of the above equations. ■

Lemma 3.6. *The numbers $\alpha+i\beta$, $\alpha-i\beta$, $\gamma+i\delta$ and $\gamma-i\delta$ can be obtained as a solution of the system*

$$\left[(\alpha+i\beta)-\left(\frac{1}{s_4+is_2}+\frac{s_1+is_3}{2}\right)\right]^2 = \frac{{s_4}^2-{s_2}^2}{({s_4}^2+{s_2}^2)^2}+\frac{{s_1}^2-{s_3}^2}{4}-\frac{2is_2s_4}{({s_4}^2+{s_2}^2)^2}+\frac{2is_1s_3}{4}$$

$$\left[(\alpha-i\beta)-\left(\frac{1}{s_4-is_2}+\frac{s_1-is_3}{2}\right)\right]^2 = \frac{{s_4}^2-{s_2}^2}{({s_4}^2+{s_2}^2)^2}+\frac{{s_1}^2-{s_3}^2}{4}+\frac{2is_2s_4}{({s_4}^2+{s_2}^2)^2}-\frac{2is_1s_3}{4}$$

$$\left[(\gamma+i\delta)-\left(\frac{1}{s_3+is_1}+\frac{s_2+is_4}{2}\right)\right]^2 = \frac{{s_3}^2-{s_1}^2}{({s_3}^2+{s_1}^2)^2}+\frac{{s_2}^2-{s_4}^2}{4}-\frac{2is_1s_3}{({s_3}^2+{s_1}^2)^2}+\frac{2is_2s_4}{4}$$

$$\left[(\gamma-i\delta)-\left(\frac{1}{s_3-is_1}+\frac{s_2-is_4}{2}\right)\right]^2 = \frac{{s_3}^2-{s_1}^2}{({s_3}^2+{s_1}^2)^2}+\frac{{s_2}^2-{s_4}^2}{4}+\frac{2is_1s_3}{({s_3}^2+{s_1}^2)^2}-\frac{2is_2s_4}{4}.$$

Proof. A straightforward calculation shows that the equations in Lemma 3.5 are satisfied. We only show that the first and second equation of the assertion do not contradict each other, because the same parameters α and β are chosen. Similarly the last two equations can be studied. We remark that the right-hand side depends only on the mesh width h and on the parameters a_1 and a_2 which are substituted in s_1, s_2, s_3 and s_4. In order to get a short notation we write $u+iv$ for the right-hand side of the first equation. Then the right-hand side of the second equation is $u-iv$. We use the trigonometric form to calculate the root and obtain for $k=0,1$:

$$\alpha+i\beta = \frac{s_4}{{s_4}^2+{s_2}^2}+\frac{s_1}{2}+\sqrt{r}\cos\frac{\varphi+k\pi}{2} + i\left(-\frac{s_2}{{s_4}^2+{s_2}^2}+\frac{s_3}{2}+\sqrt{r}\sin\frac{\varphi+k\pi}{2}\right)$$

$$\alpha-i\beta = \frac{s_4}{{s_4}^2+{s_2}^2}+\frac{s_1}{2}+\sqrt{r}\cos\frac{\varphi+k\pi}{2} + i\left(\frac{s_2}{{s_4}^2+{s_2}^2}-\frac{s_3}{2}-\sqrt{r}\sin\frac{\varphi+k\pi}{2}\right).$$

In the case $k=0$ we have no contradictions. ∎

Consequently we have shown that a solution $u(mh)$ of the system in Theorem 3.1 exists. In the following theorem we define

$$\begin{aligned}&\begin{pmatrix}\Phi_1(m_1,m_2) & \Phi_2(m_1,m_2)\\ \Phi_3(m_1,m_2) & \Phi_4(m_1,m_2)\end{pmatrix}\\ &\quad=\begin{pmatrix}u_0(m_1{+}1,m_2{+}1) & -u_1(m_1{+}1,m_2{+}1)\\ u_1(m_1,m_2) & u_0(m_1,m_2)\end{pmatrix}\begin{pmatrix}w_0(m_1,m_2) & -w_1(m_1,m_2)\\ w_1(m_1,m_2) & w_0(m_1,m_2)\end{pmatrix}\end{aligned}$$

and invert the matrix with respect to $u(mh)$.

Theorem 3.2. *If the mesh width h is sufficiently small such that $1+4a_1a_2h^2 \neq 0$ then any solution of the problem (4) can be written in the form*

$$\begin{aligned}&\begin{pmatrix}w_0(m_1,m_2) & -w_1(m_1,m_2)\\ w_1(m_1,m_2) & w_0(m_1,m_2)\end{pmatrix}\\ &\quad=\frac{1}{\det(u(m_1,m_2))}\begin{pmatrix}u_0(m_1,m_2) & u_1(m_1+1,m_2+1)\\ -u_1(m_1,m_2) & u_0(m_1+1,m_2+1)\end{pmatrix}\\ &\qquad\cdot\begin{pmatrix}\Phi_1(m_1,m_2) & \Phi_2(m_1,m_2)\\ \Phi_3(m_1,m_2) & \Phi_4(m_1,m_2)\end{pmatrix}\end{aligned}\tag{8}$$

with

$$\begin{aligned}\det(u(m_1,m_2)) &= u_0(m_1,m_2)u_0(m_1+1,m_2+1)\\ &\quad+u_1(m_1,m_2)u_1(m_1+1,m_2+1).\end{aligned}$$

The matrix elements $u_0(mh)$ and $u_1(mh)$ can be calculated by using Lemma 3.4 and Lemma 3.6 and the matrix $\Phi(mh)$ has the property

$$\frac{1}{2}\begin{pmatrix}D_h^{-1} & -D_h^2\\ D_h^{-2} & D_h^1\end{pmatrix}\begin{pmatrix}\Phi_1(m_1,m_2) & \Phi_2(m_1,m_2)\\ \Phi_3(m_1,m_2) & \Phi_4(m_1,m_2)\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\qquad \textit{for all } mh\in G_h.$$

Proof. The assertion immediately follows from Theorem 3.1, if we can prove that the matrix

$$\begin{pmatrix}u_0(m_1+1,m_2+1) & -u_1(m_1+1,m_2+1)\\ u_1(m_1,m_2) & u_0(m_1,m_2)\end{pmatrix}$$

is invertible. Using u_0, u_1 from Lemma 3.4 the determinant has the structure

$$\begin{aligned}
&u_0(m_1+1,m_2+1)u_0(m_1,m_2)+u_1(m_1+1,m_2+1)u_1(m_1,m_2)\\
&\quad=\frac{1}{4}(\alpha-i\beta)^{2m_1+1}(\gamma+i\delta)^{2m_2+1}\\
&\qquad+\frac{1}{4}(\alpha-i\beta)^{m_1+1}(\alpha+i\beta)^{m_1}(\gamma+i\delta)^{m_2+1}(\gamma-i\delta)^{m_2}\\
&\qquad+\frac{1}{4}(\alpha-i\beta)^{m_1}(\alpha+i\beta)^{m_1+1}(\gamma+i\delta)^{m_2}(\gamma-i\delta)^{m_2+1}\\
&\qquad+\frac{1}{4}(\alpha+i\beta)^{2m_1+1}(\gamma-i\delta)^{2m_2+1}\\
&\qquad-\frac{1}{4}(\alpha-i\beta)^{2m_1+1}(\gamma+i\delta)^{2m_2+1}\\
&\qquad+\frac{1}{4}(\alpha-i\beta)^{m_1+1}(\alpha+i\beta)^{m_1}(\gamma+i\delta)^{m_2+1}(\gamma-i\delta)^{m_2}\\
&\qquad+\frac{1}{4}(\alpha-i\beta)^{m_1}(\alpha+i\beta)^{m_1+1}(\gamma+i\delta)^{m_2}(\gamma-i\delta)^{m_2+1}\\
&\qquad-\frac{1}{4}(\alpha+i\beta)^{2m_1+1}(\gamma-i\delta)^{2m_2+1}\\
&\quad=(\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}\left(\frac{1}{2}(\alpha-i\beta)(\gamma+i\delta)+\frac{1}{2}(\alpha+i\beta)(\gamma-i\delta)\right).
\end{aligned}$$

We now analyze the case in which

$$\frac{1}{2}(\alpha-i\beta)(\gamma+i\delta)+\frac{1}{2}(\alpha+i\beta)(\gamma-i\delta)=0.$$

This equation is equal to $\alpha\gamma+\beta\delta=0$. In the following we write this condition as a condition on the coefficients a_1 and a_2. Using the notation as in the proof of Lemma 3.5 we obtain

$$\begin{aligned}
\alpha\gamma&=\left(s_1+\frac{\delta}{\gamma^2+\delta^2}\right)\left(s_2+\frac{\beta}{\alpha^2+\beta^2}\right)\\
&=s_1s_2+s_1\frac{\beta}{\alpha^2+\beta^2}+s_2\frac{\delta}{\gamma^2+\delta^2}+\frac{\beta\delta}{(\alpha^2+\beta^2)(\gamma^2+\delta^2)},\\
\beta\delta&=s_3s_4-s_3\frac{\alpha}{\alpha^2+\beta^2}-s_4\frac{\gamma}{\gamma^2+\delta^2}+\frac{\alpha\gamma}{(\alpha^2+\beta^2)(\gamma^2+\delta^2)}.
\end{aligned}$$

From $\alpha\gamma=-\beta\delta$ it follows that

$$s_1s_2+s_3s_4=s_3\frac{\alpha}{\alpha^2+\beta^2}+s_4\frac{\gamma}{\gamma^2+\delta^2}-s_1\frac{\beta}{\alpha^2+\beta^2}-s_2\frac{\delta}{\gamma^2+\delta^2}.$$

With the help of the relations

$$s_3\alpha - s_1\beta = s_3\left(s_1 + \frac{\delta}{\gamma^2+\delta^2}\right) - s_1\left(s_3 - \frac{\gamma}{\gamma^2+\delta^2}\right) = \frac{s_3\delta + s_1\gamma}{\gamma^2+\delta^2},$$
$$s_4\gamma - s_2\delta = s_4\left(s_2 + \frac{\beta}{\alpha^2+\beta^2}\right) - s_2\left(s_4 - \frac{\alpha}{\alpha^2+\beta^2}\right) = \frac{s_4\beta + s_2\alpha}{\alpha^2+\beta^2}$$

we get

$$s_1s_2 + s_3s_4 = \frac{s_3\delta + s_1\gamma + s_4\beta + s_2\alpha}{(\alpha^2+\beta^2)(\gamma^2+\delta^2)}.$$

For the expression in the above numerator we have

$$\delta s_3 + \gamma s_1 = \delta\left(\beta + \frac{\gamma}{\gamma^2+\delta^2}\right) + \gamma\left(\alpha - \frac{\delta}{\gamma^2+\delta^2}\right) = \beta\delta + \alpha\gamma = 0,$$
$$\beta s_4 + \alpha s_2 = \beta\left(\delta + \frac{\alpha}{\alpha^2+\beta^2}\right) + \alpha\left(\gamma - \frac{\beta}{\alpha^2+\beta^2}\right) = \beta\delta + \alpha\gamma = 0$$

and consequently $s_1s_2 + s_3s_4 = 0$. This equation can be also written in the form

$$0 = s_1s_2 + s_3s_4 = (1+2a_1h)(1+2a_2h) + (1-2a_2h)(1-2a_1h) = 2 + 8a_1a_2h^2.$$

■

We remark that the matrix elements $\Phi_1(m_1, m_2)$, $\Phi_2(m_1, m_2)$, $\Phi_3(m_1, m_2)$ and $\Phi_4(m_1, m_2)$ are not independent. We describe the relation between these elements by using (7). Obviously the following equations are satisfied:

$$\begin{aligned} & w_0(m_1, m_2)\det(u(m_1, m_2)) \\ & \quad = u_0(m_1, m_2)\Phi_1(m_1, m_2) + u_1(m_1+1, m_2+1)\Phi_3(m_1, m_2) \\ & \quad = -u_1(m_1, m_2)\Phi_2(m_1, m_2) + u_0(m_1+1, m_2+1)\Phi_4(m_1, m_2), \end{aligned} \tag{9}$$

$$\begin{aligned} & w_1(m_1, m_2)\det(u(m_1, m_2)) \\ & \quad = -u_1(m_1, m_2)\Phi_1(m_1, m_2) + u_0(m_1+1, m_2+1)\Phi_3(m_1, m_2) \\ & \quad = -u_0(m_1, m_2)\Phi_2(m_1, m_2) - u_1(m_1+1, m_2+1)\Phi_4(m_1, m_2). \end{aligned} \tag{10}$$

From these equations it follows that

$$\begin{aligned} & \Phi_4(m_1, m_2) \\ & \quad = \frac{\det(u(m_1, m_2))\Phi_1(m_1, m_2)}{{u_0}^2(m_1+1, m_2+1) + {u_1}^2(m_1+1, m_2+1)} \\ & \quad\quad + \frac{u_1(m_1,m_2)u_0(m_1+1,m_2+1) - u_0(m_1,m_2)u_1(m_1+1,m_2+1)}{{u_0}^2(m_1+1,m_2+1) + {u_1}^2(m_1+1,m_2+1)}\Phi_2(m_1, m_2) \end{aligned}$$

and

$$\Phi_3(m_1,m_2)$$
$$= \frac{1}{u_1(m_1+1,m_2+1)}\Bigg[-u_1(m_1,m_2)\Phi_2(m_1,m_2) - u_0(m_1,m_2)\Phi_1(m_1,m_2)$$
$$+u_0(m_1+1,m_2+1)\left(\frac{\det(u(m_1,m_2))\Phi_1(m_1,m_2)}{u_0^2(m_1+1,m_2+1)+u_1^2(m_1+1,m_2+1)}\right.$$
$$\left.+\frac{u_1(m_1,m_2)u_0(m_1+1,m_2+1)-u_0(m_1,m_2)u_1(m_1+1,m_2+1)}{{u_0}^2(m_1+1,m_2+1)+{u_1}^2(m_1+1,m_2+1)}\Phi_2(m_1,m_2)\right)\Bigg]$$

if

$$\begin{aligned}{u_0}^2(m_1+1,m_2+1)+{u_1}^2(m_1+1,m_2+1) &\neq 0,\\ u_1(m_1+1,m_2+1) &\neq 0.\end{aligned}$$

These conditions are satisfied because in the case

$${u_0}^2(m_1+1,m_2+1)+{u_1}^2(m_1+1,m_2+1)=0$$

we obtain by using the expressions for u_0, u_1 in Lemma 3.4 the relation

$$(\alpha^2+\beta^2)^{m_1+1}(\gamma^2+\delta^2)^{m_2+1}=0.$$

This is exactly the case which we do not consider because we are interested in a non-trivial solution of the system of equations in Lemma 3.4. We now look at the second condition. This condition can be replaced by $u_0(m_1+1,m_2+1)\neq 0$ if we eliminate $\Phi_3(m_1,m_2)$ in equation (9) instead of equation (8). One of these two expressions must be different from zero because otherwise

$$0={u_0}^2(m_1+1,m_2+1)+{u_1}^2(m_1+1,m_2+1)=(\alpha^2+\beta^2)^{m_1+1}(\gamma^2+\delta^2)^{m_2+1}$$

and we end up with the first condition.

Example. We present now a short example in order to show how the theoretical results can be used for numerical calculations. We set $a_1=1$ and $a_2=3$. Furthermore we restrict attention to the case $h=0.125$, $m_1=1$ and $m_2=2$. We assume that the point $(m_1h,m_2h)=(0.125,0.25)$ is an inner mesh point of a bounded domain G_h and we have $D^{1h}\Phi(mh)=0$ for all $mh\in G_h$. As solution of the equations in Lemma 3.6 we obtain with the MAPLE program

$$\begin{aligned}\alpha &= 1.264652496, & \beta &= -0.4080319774,\\ \gamma &= 1.518929861, & \delta &= 0.0338222383.\end{aligned}$$

Using again

$$\begin{aligned}u_0(m_1,m_2) &= \frac{1}{2}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2}+\frac{1}{2}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}\\ u_1(m_1,m_2) &= \frac{1}{2i}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2}-\frac{1}{2i}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}\end{aligned}$$

we can prove that the system of equations in Lemma 3.4 is satisfied. Based on Theorem 3.2 and the relations between $\Phi_1(m_1,m_2)$, $\Phi_2(m_1,m_2)$, $\Phi_3(m_1,m_2)$ and $\Phi_4(m_1,m_2)$ we get

$$\begin{aligned} w_0(1,2) &= 0.124437639\Phi_1(1,2) - 0.1029098652\Phi_2(1,2),\\ w_1(1,2) &= -0.124437639\Phi_2(1,2) - 0.1029098653\Phi_1(1,2). \end{aligned}$$

Finally we prove that the system (4) is satisfied. For this we use the identities

$$\begin{aligned} &D_h^{-1}\Phi_1(1,2) - D_h^2\Phi_3(1,2) = 0, \qquad D_h^{-1}\Phi_2(1,2) - D_h^2\Phi_4(1,2) = 0,\\ &D_h^{-2}\Phi_1(1,2) + D_h^1\Phi_3(1,2) = 0, \qquad D_h^{-2}\Phi_2(1,2) + D_h^1\Phi_4(1,2) = 0 \end{aligned}$$

and once more the relations between the four components of the variable Φ. In the following table the results on the left-hand side of (4) are presented for all four matrix elements:

$$\begin{aligned} &-0.4332\Phi_1(1,2) - 0.2704\Phi_2(1,2) - 0.34\cdot 10^{-10}\Phi_2(1,3) - 0.24\cdot 10^{-10}\Phi_1(1,3),\\ &\ \ 0.2704\Phi_1(1,2) - 0.4332\Phi_2(1,2) - 0.60\cdot 10^{-10}\Phi_1(1,3) + 0.33\cdot 10^{-10}\Phi_2(1,3),\\ &-0.2704\Phi_1(1,2) + 0.4332\Phi_2(1,2) + 0.20\cdot 10^{-9}\Phi_2(2,2) + 0.91\cdot 10^{-10}\Phi_1(2,2),\\ &-0.4332\Phi_1(1,2) - 0.2704\Phi_2(1,2) - 0.12\cdot 10^{-9}\Phi_1(2,2) + 0.26\cdot 10^{-9}\Phi_2(2,2). \end{aligned}$$

On the right-hand side we obtain in each matrix element the first two summands of the results on the left-hand side. There is only a difference in the ninth decimal place.

3.2. Convergence of the discrete solution in the case $b_1 = b_2 = 0$. Using the same expressions for u_0 and u_1 as in Lemma 3.4 we can write

$$\begin{aligned} u_0(m_1,m_2) &= \frac{1}{2}(\alpha - i\beta)^{m_1}(\gamma + i\delta)^{m_2} + \frac{1}{2}(\alpha + i\beta)^{m_1}(\gamma - i\delta)^{m_2}\\ &= \frac{1}{2i}\left((\alpha+i\beta)^{m_1} - (\alpha-i\beta)^{m_1}\right)\cdot\frac{1}{2i}\left((\gamma+i\delta)^{m_2} - (\gamma-i\delta)^{m_2}\right)\\ &\quad + \frac{1}{2}\left((\alpha+i\beta)^{m_1} + (\alpha-i\beta)^{m_1}\right)\cdot\frac{1}{2}\left((\gamma+i\delta)^{m_2} + (\gamma-i\delta)^{m_2}\right),\\ u_1(m_1,m_2) &= -\frac{1}{2i}\left((\alpha+i\beta)^{m_1} - (\alpha-i\beta)^{m_1}\right)\cdot\frac{1}{2}\left((\gamma+i\delta)^{m_2} + (\gamma-i\delta)^{m_2}\right)\\ &\quad + \frac{1}{2}\left((\alpha+i\beta)^{m_1} + (\alpha-i\beta)^{m_1}\right)\cdot\frac{1}{2i}\left((\gamma+i\delta)^{m_2} - (\gamma-i\delta)^{m_2}\right). \end{aligned}$$

From the properties

$$\begin{aligned} \frac{1}{2i}\left((\alpha+i\beta)^{m_1} - (\alpha-i\beta)^{m_1}\right) &= \frac{\sqrt{\alpha^2+\beta^2}^{\,m_1}}{2i}\left(e^{im_1\arctan\frac{\beta}{\alpha}} - e^{-im_1\arctan\frac{\beta}{\alpha}}\right)\\ &= \sqrt{\alpha^2+\beta^2}^{\,m_1}\sin\left(m_1\arctan\frac{\beta}{\alpha}\right),\\ \frac{1}{2}\left((\alpha+i\beta)^{m_1} + (\alpha-i\beta)^{m_1}\right) &= \sqrt{\alpha^2+\beta^2}^{\,m_1}\cos\left(m_1\arctan\frac{\beta}{\alpha}\right) \end{aligned}$$

it follows that

$$u_0(m_1,m_2) = \sqrt{\alpha^2+\beta^2}^{m_1}\sqrt{\gamma^2+\delta^2}^{m_2}\cos\left(m_1\arctan\frac{\beta}{\alpha} - m_2\arctan\frac{\delta}{\gamma}\right),$$

$$u_1(m_1,m_2) = -\sqrt{\alpha^2+\beta^2}^{m_1}\sqrt{\gamma^2+\delta^2}^{m_2}\sin\left(m_1\arctan\frac{\beta}{\alpha} - m_2\arctan\frac{\delta}{\gamma}\right).$$

Using the representation (7) for the solution $w(mh)$ of (4) we obtain

$$\begin{aligned} w_0(m_1,m_2) = & \frac{\cos\left(m_1\arctan\frac{\beta}{\alpha} - m_2\arctan\frac{\delta}{\gamma}\right)\Phi_1(m_1,m_2)}{\sqrt{\alpha^2+\beta^2}^{m_1+1}\sqrt{\gamma^2+\delta^2}^{m_2+1}\cos\left(\arctan\frac{\beta}{\alpha} - \arctan\frac{\delta}{\gamma}\right)} \\ & - \frac{\sin\left((m_1+1)\arctan\frac{\beta}{\alpha} - (m_2+1)\arctan\frac{\delta}{\gamma}\right)\Phi_3(m_1,m_2)}{\sqrt{\alpha^2+\beta^2}^{m_1}\sqrt{\gamma^2+\delta^2}^{m_2}\cos\left(\arctan\frac{\beta}{\alpha} - \arctan\frac{\delta}{\gamma}\right)}, \\ w_1(m_1,m_2) = & \frac{\sin\left(m_1\arctan\frac{\beta}{\alpha} - m_2\arctan\frac{\delta}{\gamma}\right)\Phi_1(m_1,m_2)}{\sqrt{\alpha^2+\beta^2}^{m_1+1}\sqrt{\gamma^2+\delta^2}^{m_2+1}\cos\left(\arctan\frac{\beta}{\alpha} - \arctan\frac{\delta}{\gamma}\right)} \\ & + \frac{\cos\left((m_1+1)\arctan\frac{\beta}{\alpha} - (m_2+1)\arctan\frac{\delta}{\gamma}\right)\Phi_3(m_1,m_2)}{\sqrt{\alpha^2+\beta^2}^{m_1}\sqrt{\gamma^2+\delta^2}^{m_2}\cos\left(\arctan\frac{\beta}{\alpha} - \arctan\frac{\delta}{\gamma}\right)}. \end{aligned}$$

We now consider the case $h \to 0$ and show the next steps only for $w_0(mh)$. Obviously we can write

$$\begin{aligned} & \lim_{h\to 0} w_0(m_1,m_2) \\ & = \frac{\cos\left(x_1\lim_{h\to 0}\frac{\arctan\frac{\beta}{\alpha}}{h} - x_2\lim_{h\to 0}\frac{\arctan\frac{\delta}{\gamma}}{h}\right)\Phi_1(x_1,x_2)}{e^{x_1\lim_{h\to 0}\frac{\ln(\alpha^2+\beta^2)}{2h}}e^{x_2\lim_{h\to 0}\frac{\ln(\gamma^2+\delta^2)}{2h}}\lim_{h\to 0}\cos\left(\arctan\frac{\beta}{\alpha} - \arctan\frac{\delta}{\gamma}\right)} \\ & - \frac{\sin\left(x_1\lim_{h\to 0}\frac{\arctan\frac{\beta}{\alpha}}{h} - x_2\lim_{h\to 0}\frac{\arctan\frac{\delta}{\gamma}}{h}\right)\Phi_3(x_1,x_2)}{e^{x_1\lim_{h\to 0}\frac{\ln(\alpha^2+\beta^2)}{2h}}e^{x_2\lim_{h\to 0}\frac{\ln(\gamma^2+\delta^2)}{2h}}\lim_{h\to 0}\cos\left(\arctan\frac{\beta}{\alpha} - \arctan\frac{\delta}{\gamma}\right)} \end{aligned}$$

so that we have to investigate five limits on the right-hand side of this equation.

$$L1 = \lim_{h\to 0}\cos\left(\arctan\frac{\beta}{\alpha} - \arctan\frac{\delta}{\gamma}\right).$$

Using the equations for $\alpha + i\beta$ and $\alpha - i\beta$ in Lemma 3.6 we get

$$\lim_{h\to 0}\alpha = 1, \qquad \lim_{h\to 0}\beta = 0, \qquad \lim_{h\to 0}\arctan\frac{\beta}{\alpha} = 0.$$

Similarly we prove

$$\lim_{h\to 0}\gamma = 1, \qquad \lim_{h\to 0}\delta = 0, \qquad \lim_{h\to 0}\arctan\frac{\delta}{\gamma} = 0.$$

Consequently we have $L1 = 1$.

Next we consider

$$L2 = \lim_{h\to 0}\frac{\arctan\frac{\beta}{\alpha}}{h}, \qquad L3 = \lim_{h\to 0}\frac{\arctan\frac{\delta}{\gamma}}{h}.$$

Using the rule of l'Hospital and the properties $\lim_{h\to 0}\alpha = 1$ and $\lim_{h\to 0}\beta = 0$ we obtain $L2 = \lim_{h\to 0}\frac{d\beta}{dh}$. In the same way we get $L3 = \lim_{h\to 0}\frac{d\delta}{dh}$.

Finally, we have

$$L4 = \lim_{h\to 0}\frac{\ln(\alpha^2+\beta^2)}{2h}, \qquad L5 = \lim_{h\to 0}\frac{\ln(\gamma^2+\delta^2)}{2h}.$$

Using again the rule of l'Hospital and the properties of α and β we finally obtain $L4 = \lim_{h\to 0}\frac{d\alpha}{dh}$ and in the same way $L5 = \lim_{h\to 0}\frac{d\gamma}{dh}$.

Considering the equations in Lemma 3.6 it is possible to prove that the limits $L2$, $L3$, $L4$ and $L5$ exist and that they are finite. From Lemma 3.5 we conclude

$$\begin{aligned}
L4 + iL2 &= 2a_1 + i\frac{L5 - iL3}{(\lim_{h\to 0}\gamma - i\lim_{h\to 0}\delta)^2} - 2ia_2,\\
L4 - iL2 &= 2a_1 - i\frac{L5 + iL3}{(\lim_{h\to 0}\gamma + i\lim_{h\to 0}\delta)^2} + 2ia_2,\\
L5 - iL3 &= 2a_2 - i\frac{L4 + iL2}{(\lim_{h\to 0}\alpha + i\lim_{h\to 0}\beta)^2} + 2ia_1,\\
L5 + iL3 &= 2a_2 + i\frac{L4 - iL2}{(\lim_{h\to 0}\alpha - i\lim_{h\to 0}\beta)^2} - 2ia_1.
\end{aligned}$$

From these four equations we get the relations

$$L5 = L2 + 2a_2, \qquad L4 = L3 + 2a_1.$$

Using all this information we obtain

$$\begin{aligned}
\lim_{h\to 0} w_0(m_1,m_2) &= \frac{\cos(x_1L2 - x_2\,L3)\Phi_1(x_1,x_2)}{e^{x_1(L3+2a_1)}e^{x_2(L2+2a_2)}}\\
&\quad - \frac{\sin(x_1L2 - x_2L3)\Phi_3(x_1,x_2)}{e^{x_1(L3+2a_1)}e^{x_2(L2+2a_2)}},\\
\lim_{h\to 0} w_1(m_1,m_2) &= \frac{\sin(x_1L2 - x_2L3)\Phi_1(x_1,x_2)}{e^{x_1(L3+2a_1)}e^{x_2(L2+2a_2)}}\\
&\quad + \frac{\cos(x_1L2 - x_2L3)\Phi_3(x_1,x_2)}{e^{x_1(L3+2a_1)}e^{x_2(L2+2a_2)}}.
\end{aligned}$$

We show that these limits are solutions of the homogeneous differential equations: we substitute

$$W_0(x_1,x_2) = \lim_{h\to 0} w_0(m_1,m_2), \qquad W_1(x_1,x_2) = \lim_{h\to 0} w_1(m_1,m_2).$$

If

$$\frac{\partial\Phi_1}{\partial x_1} - \frac{\partial\Phi_3}{\partial x_2} = 0, \qquad \frac{\partial\Phi_1}{\partial x_2} + \frac{\partial\Phi_3}{\partial x_1} = 0$$

we obtain for the real part

$$\frac{1}{2}\left(\frac{\partial W_0}{\partial x_1} - \frac{\partial W_1}{\partial x_2}\right) = -\frac{L2}{2}W_1 - \frac{L4}{2}W_0 + \frac{L3}{2}W_0 + \frac{L5}{2}W_1 = a_2W_1 - a_1W_0$$

and for the imaginary part

$$\frac{1}{2}\left(\frac{\partial W_0}{\partial x_2} + \frac{\partial W_1}{\partial x_1}\right) = \frac{L3}{2}W_1 - \frac{L5}{2}W_0 + \frac{L2}{2}W_0 - \frac{L4}{2}W_1 = -a_1W_1 - a_2W_0.$$

The right-hand sides of these equations are obviously the real and imaginary part of the product $(-a_1 - ia_2)(W_0 + iW_1)$ so that we have indeed found a solution of the homogeneous differential equation, where one factor of this solution is a holomorphic function.

We will now consider the homogeneous discrete Vekua equations for $b_1^2 + b_2^2 \neq 0$. At the moment it is enough to consider the case $a_1 = a_2 = 0$ because the system (3) is linear and we can construct a solution of this problem if we separately solve the problems in the cases $a_1 = a_2 = 0$ and $b_1 = b_2 = 0$.

3.3. The homogeneous discrete Vekua equations in case $a_1 = a_2 = 0$. We look for a system of four equations similar to the system (4). If we write this system in the form

$$\begin{aligned} &\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} w_0(m_1h,m_2h) & -w_1(m_1h,m_2h) \\ w_1(m_1h,m_2h) & w_0(m_1h,m_2h) \end{pmatrix} \\ &\quad = \begin{pmatrix} -b_1 & b_2 \\ -b_2 & -b_1 \end{pmatrix}\begin{pmatrix} w_0(m_1h,m_2h) & w_1(m_1h,m_2h) \\ -w_1(m_1h,m_2h) & w_0(m_1h,m_2h) \end{pmatrix} \end{aligned} \tag{11}$$

we can show that the equations which are related to the first and fourth matrix element approximate the same differential equation for $h \to 0$. We are in the same situation if we consider the equations which are related to the second and third matrix elements.

Compared with the system (4) we investigate now a system in which the matrix for $w(mh)$ on the right-hand side has another structure. We are interested in reducing this problem to the same situation as in Theorem 3.1 because our aim is to find a product in which one factor is a discrete holomorphic function. From the "ansatz"

$$\begin{pmatrix} -b_1 & b_2 \\ -b_2 & -b_1 \end{pmatrix} \begin{pmatrix} w_0(m_1h,m_2h) & w_1(m_1h,m_2h) \\ -w_1(m_1h,m_2h) & w_0(m_1h,m_2h) \end{pmatrix}$$
$$= \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} w_0(m_1h,m_2h) & -w_1(m_1h,m_2h) \\ w_1(m_1h,m_2h) & w_0(m_1h,m_2h) \end{pmatrix}$$

we conclude in each mesh point $mh \in G_h$

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} -b_1 & b_2 \\ -b_2 & -b_1 \end{pmatrix} \begin{pmatrix} w_0 & w_1 \\ -w_1 & w_0 \end{pmatrix} \begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix}^{-1}$$
$$= \frac{1}{{w_0}^2 + {w_1}^2} \begin{pmatrix} -b_1({w_0}^2 - {w_1}^2) - 2b_2 w_0 w_1 & -2b_1 w_0 w_1 + b_2({w_0}^2 - {w_1}^2) \\ -b_2({w_0}^2 - {w_1}^2) + 2b_1 w_0 w_1 & -2b_2 w_0 w_1 - b_1({w_0}^2 - {w_1}^2) \end{pmatrix}.$$

We remark that

$$\frac{{w_0}^2 - {w_1}^2}{{w_0}^2 + {w_1}^2} \qquad \text{and} \qquad \frac{2w_0w_1}{{w_0}^2 + {w_1}^2}$$

are the real and imaginary parts of $\bar{w}/w$ in the function $g(z)$ in Section 2. Using this idea we can prove the following theorem in a manner similar to Theorem 3.1.

Theorem 3.3. *Let $w(mh)$ be an arbitrary solution of the problem (10) and $u(mh)$ be a solution of the problem*

$$\frac{1}{2} \begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix} \begin{pmatrix} u_0(m_1+1,m_2+1) & -u_1(m_1+1,m_2+1) \\ u_1(m_1,m_2) & u_0(m_1,m_2) \end{pmatrix} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$$

with

$$U_1 = \frac{1}{{w_0}^2 + {w_1}^2} \Big((b_1({w_0}^2 - {w_1}^2) + 2b_2 w_0 w_1) u_0(m_1, m_2+1)$$
$$+ (2b_1 w_0 w_1 - b_2({w_0}^2 - {w_1}^2)) u_1(m_1, m_2+1) \Big),$$
$$U_2 = \frac{1}{{w_0}^2 + {w_1}^2} \Big((2b_1 w_0 w_1 - b_2({w_0}^2 - {w_1}^2)) u_0(m_1, m_2+1)$$
$$+ (-b_1({w_0}^2 - {w_1}^2) - 2b_2 w_0 w_1) u_1(m_1, m_2+1) \Big),$$
$$U_3 = \frac{1}{{w_0}^2 + {w_1}^2} \Big((b_2({w_0}^2 - {w_1}^2) - 2b_1 w_0 w_1) u_0(m_1+1, m_2)$$
$$+ (2b_2 w_0 w_1 + b_1({w_0}^2 - {w_1}^2)) u_1(m_1+1, m_2) \Big),$$
$$U_4 = \frac{1}{{w_0}^2 + {w_1}^2} \Big((2b_2 w_0 w_1 + b_1({w_0}^2 - {w_1}^2)) u_0(m_1+1, m_2)$$
$$+ (-b_2({w_0}^2 - {w_1}^2) + 2b_1 w_0 w_1) u_1(m_1+1, m_2) \Big).$$

Then we obtain

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\left[\begin{pmatrix} u_0(m_1+1,m_2+1) & -u_1(m_1+1,m_2+1) \\ u_1(m_1,m_2) & u_0(m_1,m_2) \end{pmatrix} \cdot \begin{pmatrix} w_0(m_1,m_2) & -w_1(m_1,m_2) \\ w_1(m_1,m_2) & w_0(m_1,m_2) \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The results in this article can be considered as a first step towards building up a discrete theory. Further work should be related to the inhomogeneous Vekua equation and to the question of characterizing which systems of difference equations can be transformed into the discrete Vekua equations. The last aspect corresponds to a discrete version of the Beltrami equation.

The main theorem of this article can also be proved in the case of quaternion valued functions. Elements of a discrete function theory in higher dimensions are described already in [13] such that an extension of the theory presented here seems to be possible.

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