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Maximin Polynomials and Inverse Balayage

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Abstract. For a bounded domain G in the complex plane, we focus on the problem of maximizing the minimum on the boundary ∂G of (weighted) polynomials of degree n having all zeros in a set $D \subset G$. For arbitrary unit measures μ on ∂G and weight $w := \exp\{U^\mu\}$, the n-th root asymptotics of

$$\sup_{p_n} \inf_{z\in\partial G} |p_n(z)w^n(z)|$$

is considered and related to the existence and construction of an inverse balayage of μ on $\overline{D}$, i.e. of a measure such that μ is its balayage when sweeping to ∂G.

Keywords. Logarithmic potential, weighted polynomial, equilibrium distribution, capacity, balayage, inverse balayage.

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1. Introduction

Suppose G is a bounded domain in the complex plane $\mathbb{C}$. Borodin [2] considers the extremal problem of maximizing the quantity

$$\min_{z\in\partial G} |p_n(z)|$$

with respect to the class $\mathcal{P}_n(G)$ of monic polynomials p_n of exact degree n having all zeros in G. There exist $p_n^* \in \mathcal{P}_n(G)$ such that

$$m_n := \min_{z\in\partial G} |p_n^*(z)| = \max_{p_n\in\mathcal{P}_n(G)} \min_{z\in\partial G} |p_n(z)|. \tag{1}$$

These polynomials are, hence, counterparts of the classical Chebychev polynomials, which are commonly introduced by exchanging the role of the maximum and minimum in (1). Using deep results from complex analysis to extend previous work of Faber [3], Borodin shows that

$$\lim_{n\to\infty} \sqrt[n]{m_n} = \operatorname{cap}(\overline{G}), \tag{2}$$

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where $\operatorname{cap}(\overline{G})$ denotes the logarithmic capacity of the closure $\overline{G}$. We start by giving a potential theoretic interpretation of this phenomenon: for a (finite Borel-) measure μ on $\mathbb{C}$ with compact support, consider the logarithmic potential of μ, given by

$$U^{\mu}(z) = \int \frac{1}{|z-t|}\, d\mu(t), \qquad z \in \mathbb{C}.$$

If p is any monic polynomial of degree k, denote by ν_p the normalized zero counting measure associating with each zero of p the mass $1/k$, taking into account multiplicities. Then

$$U^{\nu_p}(z) = -\log\left(|p(z)|^{1/k}\right), \qquad z \in \mathbb{C}.$$

With these notions, (2) states that

$$-\log\operatorname{cap}(\overline{G}) + \log\left({m_n}^{1/n}\right) = \sup_{p_n \in \mathcal{P}_n(G)} \min_{z \in \partial G} \left\{U^{\mu_{\overline{G}}}(z) - U^{\nu_{p_n}}(z)\right\} =: \delta_n(\mu_{\overline{G}}, \partial G, G)$$

tends to zero as n tends to ∞. Here, $\mu_{\overline{G}}$ denotes the Robin equilibrium distribution of $\overline{G}$ (see below), which is a unit measure on ∂G having constant potential $U^{\mu_{\overline{G}}} = -\log\operatorname{cap}(\overline{G})$ on $\overline{G}$.

Now, generalizing the above notions to arbitrary non-empty sets A and B, respectively, let $\mathcal{P}_n(B)$ stand for the set of monic polynomials of exact degree n with all zeros in B and, for unit measures μ, define

$$\delta_n(\mu, A, B) := \sup_{p_n \in \mathcal{P}_n(B)} \inf_{z \in A} \left\{U^{\mu}(z) - U^{\nu_{p_n}}(z)\right\}.$$

In this paper we focus on the case when μ is a unit measure on $A = \partial G$ and $B = D \subset G$. We will show that

$$\lim_{n\to\infty} \delta_n(\mu, \partial G, G) = 0$$

(see Theorem 1) and give an interpretation in terms of weighted polynomial approximation (Corollary 2). Moreover, we will relate the question whether $\delta_n(\mu, \partial G, D) \to 0$ to the problem of the existence (Theorem 3 and 5) and construction (Corollary 4) of an inverse balayage of μ, i.e. of a unit measure ν on $\overline{D}$ such that μ is the balayage of ν when sweeping to the boundary of G.

2. Formulation of the results

Throughout this paper, we will assume that G is a bounded domain in the complex plane and μ is a unit measure on ∂G. Moreover, $\partial_\infty G$ denotes the boundary of the unbounded component of $\mathbb{C} \setminus \overline{G}$.

Theorem 1. *For each unit measure μ on ∂G,*

$$\lim_{n\to\infty} \delta_n(\mu, \partial G, G) = 0.$$

This result can be related to the weighted equilibrium problem, which we may state as follows: suppose E is a compact set in the complex plane and w an upper semi-continuous, positive (or admissible, see [7, I.1.1]) function on E. There exists a unit measure μ_w on E having minimal weighted logarithmic energy

$$\begin{aligned} I_w(\mu_w) &= \iint \log \frac{1}{|z-t|w(t)w(z)}\, d\mu_w(z) d\mu_w(t) \\ &= \iint \log \frac{1}{|z-t|}\, d\mu_w(z)\, d\mu_w(t) - 2\int Q(z)\, d\mu_w(z) \end{aligned} \tag{3}$$

with respect to all unit measures on E. Here, $Q(z) := -\log w(z)$, $z \in E$. Moreover, there is a constant F_w such that

$$U^{\mu_w}(z) + Q(z) \geq F_w$$

quasi-everywhere (see below) on E and

$$U^{\mu_w}(z) + Q(z) \leq F_w$$

on the support of μ_w.

The number $V_E := I_1(\mu_1)$ associated with $w = 1$ is referred to as the Robin constant of E and $\mu_E := \mu_1$ as the equilibrium distribution of E; $\operatorname{cap}(E) := \exp(-V_E)$ is called the capacity of E. A relation is said to hold quasi-everywhere, if it holds everywhere except for a set of zero capacity [7, I.1]. If E is of positive capacity, as is the case for $E = \partial G$, then F_w is finite and the extremal measure μ_w is unique [7, I.1.3].

We are now in position to formulate an extension of Borodin's result to arbitrary unit measures on the boundary of G.

Corollary 2. *Let w be an upper semi-continuous, positive function on ∂G. Suppose that*

$$U^{\mu_w}(z) + Q(z) = F_w \tag{4}$$

for all $z \in \partial G$. Then

$$\lim_{n\to\infty} \left(\sup_{p_n \in \mathcal{P}_n(G)} \inf_{z\in\partial G} |p_n(z) w^n(z)| \right)^{1/n} = \exp(-F_w). \tag{5}$$

Relation (4) means that the solution to the (generalized) Dirichlet problem with boundary values $-Q(z)$ can be constructed by means of Gauss' classical approach. Conditions sufficient for (4) may be found, for instance, in [4].

The relations (1) or (5) can also be interpreted as the classical Chebychev problem applied to reciprocals of polynomials with poles constrained to G. They are, therefore, in analogy to the theory of classical weighted Chebychev polynomials [7, III.3].

We will now turn to the notion of inverse balayage.

Definition. For a non-empty set $D \subset G$ let $\mathrm{Bal}^{-1}(\mu, D)$ denote the collection of all inverse balayages to μ on D, i.e. unit measures ν on D with the property that

$$U^{\mu}(z) = U^{\nu}(z), \qquad z \in \mathbb{C} \setminus \overline{G}. \tag{6}$$

That is to say, an inverse balayage is a measure which is more densely concentrated (namely, on D) but generating the same electrostatic potential outside $\overline{G}$.

For simplicity we will from now on impose a regularity condition on G, namely, that $\partial_\infty G = \partial G$. Moreover, it is in some sense natural to consider only measures μ with finite logarithmic energy $I_1(\mu)$.

Theorem 3. *Let $\emptyset \neq D \subset G$ be arbitrary. Suppose $\partial G = \partial_\infty G$ and that μ is a unit measure on ∂G with finite energy. Then*

$$\delta_n(\mu, \partial G, D) \to 0 \qquad \textit{implies} \qquad \mathrm{Bal}^{-1}(\mu, \overline{D}) \neq \emptyset. \tag{7}$$

An approach to construct such an inverse balayage on $\overline{D}$ is given in the following corollary. There and in what follows, weak-star convergence of a sequence of measures μ_n to μ means that

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu$$

for all continuous functions f with compact support.

Corollary 4. *Suppose $\delta_n(\mu, \partial G, D) \to 0$, where μ is a unit measure on ∂G. Let $w := \exp(U^\mu)$ on ∂G. There exist polynomials $p_n^* \in \mathcal{P}_n(D)$ that are asymptotically extremal in the sense of*

$$\lim_{n\to\infty} \left(\inf_{z\in\partial G} |p_n^*(z)\, w^n(z)| \right)^{1/n} = 1. \tag{8}$$

Moreover, if $\partial G = \partial_\infty G$ and μ has finite energy, then each weak-star limit point of the sequence $\{\nu_{p_n^*}\}$ associated with polynomials $p_n^* \in \mathcal{P}_n(D)$ satisfying (8) is an inverse balayage from $\mathrm{Bal}^{-1}(\mu, \overline{D})$.

Since $\overline{D}$ is compact, such weak-star limit points exist by Helly's Selection Theorem [7, 0.1.3]. Hence, Corollary 4 proposes to find an inverse balayage by constructing near-extremal weighted maximin polynomials and considering their asymptotic zero distributions. The question if or under which conditions there are somehow canonically designated or even unique weak-star limit points is open. This may be related to the existence of so-called mother bodies (see [5],[6] and the references therein). We will also not dwell on possible quantifications of such a convergence in terms of discrepancy between μ and the balayage measures of $\nu_{p_n^*}$ when sweeping to ∂G (see [1]).

There is also a converse to Theorem 3.

Theorem 5. *Suppose $\partial G = \partial_\infty G$, μ is a unit measure on ∂G and $D \subset G$ is either a domain or a closed set. Then*

$$\mathrm{Bal}^{-1}(\mu, \overline{D}) \neq \emptyset \qquad \textit{implies} \qquad \delta_n(\mu, \partial G, D) \to 0. \tag{9}$$

3. Proofs of the results

We start by formulating two auxiliary results.

Lemma 6. *For each unit measure ν on $\overline{G}$, there exists a point $z \in \partial G$ with $U^\mu(z) \leq U^\nu(z)$. In particular, $\delta_n(\mu, \partial G, D) \leq 0$.*

Moreover, if both, μ and ν, are located on $\partial_\infty G$ and if ν has finite logarithmic energy, then $U^\mu \geq U^\nu$ on ∂G implies $\mu = \nu$.

Proof. Consider the Robin equilibrium distribution $\mu_{\overline{G}}$ of $\overline{G}$ and the corresponding Robin constant $V_{\overline{G}} = -\log \operatorname{cap}(\overline{G})$. Since $U^{\mu_{\overline{G}}} = V_{\overline{G}}$ on $\overline{G}$, it follows that

$$\int U^\mu \, d\mu_{\overline{G}} = \int U^{\mu_{\overline{G}}} \, d\mu = V_{\overline{G}} = \int U^{\mu_{\overline{G}}} \, d\nu = \int U^\nu \, d\mu_{\overline{G}}.$$

Therefore, it is impossible to have $U^\mu > U^\nu$ on $\partial G \supset \operatorname{supp}(\mu_{\overline{G}})$.

Now, if $U^\mu \geq U^\nu$ on $\partial_\infty G \supset \operatorname{supp}(\nu) \cup \operatorname{supp}(\mu)$ and $I_1(\nu) < \infty$, then $h := U^\mu - U^\nu$ is non-negative by the Principle of Domination [7, II.3.2]. But h is harmonic in the unbounded component Ω of $\overline{\mathbb{C}} \setminus \overline{G}$, and $h(\infty) = 0$, because μ and ν have the same total mass. It follows from the Minimum Principle that $U^\mu = U^\nu$ in Ω. Hence, $\mu = \nu$ by Carleson's Unicity Theorem [7, II.4.13], provided both measures have support in $\partial_\infty G$. ∎

Lemma 7. *Suppose ν is a unit measure with compact support and $\emptyset \neq D \subset G$. Then*

$$\delta_n(\nu, \partial G, D) \geq \delta_n(\nu, \partial D, D).$$

Proof. For each polynomial $p_n \in \mathcal{P}_n(D)$ the function $U^\nu - U^{\mu_{p_n}}$ is superharmonic in $\mathbb{C} \setminus \overline{D}$ and harmonic at ∞. By the Minimum Principle [7, I.2.4], it attains its infimum with respect to $\mathbb{C} \setminus \overline{D}$ on ∂D. ∎

Proof of Theorem 1. There exists an exhaustion of G by an increasing sequence of open sets $G_m \subset \overline{G_m} \subset G$, i.e. $G_m \subset G_{m+1}$ and $\cup_m G_m = G$. Moreover, it can be assumed that the complement of $\overline{G_m}$ in $\mathbb{C}$ is regular with respect to the Dirichlet problem. Then the functions

$$g_m := V_{\overline{G_m}} - U^{\mu_{\overline{G_m}}}$$

are non-negative and continuous with $g_m \geq g_{m+1}$. They are the Green functions of the unbounded component of $\mathbb{C} \setminus \overline{G_m}$ with pole at ∞.

Denote by μ_m the balayage of μ to $\overline{G_m}$ [7, II.4.7]: μ_m is the unique unit measure on $\overline{G_m}$ such that $U^{\mu_m} = U^{\mu} + c_m$ on $\overline{G_m}$, where

$$c_m = \int g_m \, d\mu. \tag{10}$$

Moreover, $U^{\mu_m} \leq U^{\mu} + c_m$ in $\mathbb{C}$.

There exists a sequence of monic polynomials $t_n = t_n^{(m)}$ of degree n with all zeros in $\overline{G_m}$ such that $U^{\mu_m} - U^{\mu_{t_n}} \to 0$ locally uniformly outside $\overline{G_m}$. Therefore, taking into account Lemma 6 and the definition of $\delta_n(\mu, \partial G, \cdot)$,

$$\begin{aligned} 0 &\geq \delta_n(\mu, \partial G, G) \geq \delta_n(\mu, \partial G, \overline{G_m}) \\ &\geq \inf_{z \in \partial G} U^{\mu}(z) - U^{\mu_{t_n}}(z) \geq \min_{z \in \partial G} U^{\mu_m}(z) - U^{\mu_{t_n}}(z) - c_m, \end{aligned}$$

so that

$$0 \geq \limsup_{n\to\infty} \delta_n(\mu, \partial G, G) \geq \liminf_{n\to\infty} \delta_n(\mu, \partial G, G) \geq -c_m.$$

We complete the proof by showing that $c_m \to 0$ as $m \to \infty$. Since for a planar domain the boundary in the Euclidean and in the fine topology is the same [7, I.6.6], we have $\operatorname{cap}(G) = \operatorname{cap}(\overline{G})$. From this it follows that $\mu_{\overline{G_m}} \to \mu_{\overline{G}}$ in the weak-star sense and $V_{\overline{G_m}} \to V_{\overline{G}}$. Thus, by the Principle of Descent [7, I.6.8],

$$U^{\mu_{\overline{G}}}(z) \leq \liminf_{m\to\infty} U^{\mu_{\overline{G_m}}}(z), \qquad z \in \mathbb{C},$$

so that $\lim_{m\to\infty} g_m(z) = \limsup_{m\to\infty} g_m(z) \leq 0$ for all $z \in \partial G$. Now, by (10) and the Monotone Convergence Theorem, $c_m \to 0$ as $m \to \infty$. ■

Proof of Corollary 2. The assertion of Corollary 2 follows from Theorem 1, taking into account that for each monic polynomial p of degree n,

$$\begin{aligned} \left(\inf_{z\in\partial G} |p(z)\, w^n(z)|\right)^{1/n} &= \inf_{z\in\partial G} |p(z)|^{1/n}\, w(z) \\ &= \inf_{z\in\partial G} \exp(U^{\mu_w}(z) - F_w - U^{\nu_p}(z)) \\ &= \exp\left(\inf_{z\in\partial G} U^{\mu_w}(z) - U^{\nu_p}(z)\right) \exp(-F_w). \end{aligned} \tag{11}$$

■

Proof of Theorem 3. Since $\delta_n(\mu, \partial G, D) \to 0$, there is a sequence of polynomials $p_n^* \in \mathcal{P}_n(D)$ such that

$$\lim_{n\to\infty} \inf_{z\in\partial G} U^{\mu}(z) - U^{\nu_{p_n^*}}(z) = 0.$$

By Helly's Selection Theorem, there exists a weak-star limit point ν^* of the corresponding sequence $\{\nu_{p_n}\}$. From the Principle of Descent,

$$U^{\mu}(z) \geq U^{\nu^*}(z), \qquad z \in \partial G.$$

Applying the second part of Lemma 6 to the balayage measure $\hat{\nu^*}$ of ν^* when sweeping to $\partial_\infty G = \partial G$, we find that ν^* is an inverse balayage to μ, supported by $\overline{D}$. In fact, $U^\mu \geq U^{\nu^*} \geq U^{\hat{\nu^*}}$ on ∂G [7, II.4.1(b)] and $I_1(\hat{\nu^*}) \leq I_1(\mu) < \infty$. ■

Proof of Corollary 4. The assertion of Corollary 4 follows by combining (11) with the arguments in the proof of Theorem 3. ■

Proof of Theorem 5. Suppose first that D is a domain. Let $\nu \in \mathrm{Bal}^{-1}(\mu, \overline{D})$ and denote by $\hat{\nu}$ the balayage of ν to ∂D. Then

$$U^\mu(z) = U^\nu(z) \geq U^{\hat{\nu}}(z)$$

for all $z \in \partial G$ (see [7, I.5.6 and II.4.1(b)]). Hence, taking into account the first part of Lemma 6 and Lemma 7,

$$0 \geq \delta_n(\mu, \partial G, D) \geq \delta_n(\hat{\nu}, \partial G, D) \geq \delta_n(\hat{\nu}, \partial D, D).$$

On the other hand, from Theorem 1,

$$\lim_{n\to\infty} \delta_n(\hat{\nu}, \partial D, D) = 0.$$

Thus, (9) is true.

If $D = \overline{D} \subset G$, then we may choose a sequence of monic polynomials t_n of degree n with zeros in D such that

$$\lim_{n\to\infty} U^\nu - U^{\nu_{t_n}} = 0$$

locally uniformly outside $\overline{D}$. This yields (9), since $U^\mu = U^\nu$ on ∂G. ■

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