Computational Methods and Function Theory
Volume 5 (2005), No. 2, 301–322

On Sparse Sets with the Green Function of the Highest Smoothness

Vladimir V. Andrievskii

(Communicated by Dmitry Khavinson)

Abstract. Let E be a regular compact subset of the real line, let $g_{\overline{\mathbb{C}}\setminus E}(z,\infty)$ be the Green function of the complement of E with respect to the extended complex plane $\overline{\mathbb{C}}$ with pole at ∞. We construct two examples of sets E of the minimum Hausdorff dimension with $g_{\overline{\mathbb{C}}\setminus E}$ satisfying the Hölder condition with $p = 1/2$ either uniformly or locally.

Keywords. Green's function, compact set, Hausdorff dimension, conformal invariants.

2000 MSC. Primary 30C10, 30C15, 41A10.

1. Introduction and main results

Let $E \subset \mathbb{R}$ be a compact set of the real line $\mathbb{R}$ with positive (logarithmic) capacity $\operatorname{cap} E > 0$. For simplicity we assume that $E \subset [-1,1]$ and $\pm 1 \in E$. We consider E as a set in the complex plane $\mathbb{C}$ and use notions of potential theory in the plane (see [14, 15]).

Let $\Omega = \overline{\mathbb{C}} \setminus E$, where $\overline{\mathbb{C}} = \{\infty\} \cup \mathbb{C}$ is the extended complex plane. Denote $g_\Omega(z) = g_\Omega(z,\infty)$, $z \in \Omega$, the Green function of Ω with pole at ∞. In what follows we assume that E is a regular set, i.e. g_Ω extends continuously to E where it takes the value 0.

The connection between the metric properties of E and the smoothness properties of g_Ω was studied by many authors. We refer the reader to [11, 5, 10, 18, 6, 2] and the many references therein for a comprehensive survey of this subject. It is of special interest to study the metric properties of E such that g_Ω satisfies the Hölder condition with $p = 1/2$

$$|g_\Omega(z_2) - g_\Omega(z_1)| \le c|z_2 - z_1|^{1/2}, \qquad z_1, z_2 \in \Omega \setminus \{\infty\}, \tag{1.1}$$

where $c > 0$ is some constant.

Received November 1, 2004, in revised form July 27, 2005.

ISSN 1617-9447/\$ 2.50

Since the monotonicity of the Green function yields

$$g_\Omega(1+r) \geq g_{\overline{\mathbb{C}}\setminus[-1,1]}(1+r) > \frac{\sqrt{r}}{2}, \qquad 0<r<1,$$

the choice of the right-hand side of (1.1) appears to be the best suited for this theory. In this regard, we discuss the properties of E whose Green's function has the "highest smoothness" (i.e. of E conforming to the condition (1.1)).

Recently Totik [17, 18] constructed two examples of a set E whose Green's function satisfies (1.1) and whose linear measure vanishes.

In this paper we analyze how sparse E can be such that it satisfies (1.1) in terms of its Hausdorff dimension $\dim E$ [13, p. 224].

First, we note that if E satisfies (1.1) then

$$\dim E \geq \frac{1}{2}\,. \tag{1.2}$$

Indeed, from (1.1) it follows immediately (for details, see [6], proof of Proposition 1.4) that for any interval $I \subset \mathbb{R}$,

$$\mu_E(|I\cap E|) \leq c_1|I|^{1/2},$$

where μ_E is the equilibrium measure of E, c_1 is a positive constant, and $|S|$ denotes the linear measure of a Borel set $S\subset\mathbb{C}$.

Hence, for any covering of E by intervals $\{I_j\}\subset\mathbb{R}$ we have

$$\sum_j |I_j|^{1/2} \geq \frac{1}{c_1}\sum_j \mu_E(|I_j\cap E|) \geq \frac{1}{c_1},$$

which proves (1.2).

Theorem 1. *There exists a regular set* $E_0\subset\mathbb{R}$ *with the following properties:*

- (i) $g_{\overline{\mathbb{C}}\setminus E_0}$ *satisfies* (1.1)*;*
- (ii) $\dim E_0 = 1/2$.

It is natural to consider the problem of how sparse the set E can be such that the following local version of (1.1) is valid:

$$g_\Omega(z) = g_\Omega(z) - g_\Omega(-1) \leq c_2|z+1|^{1/2}, \qquad z\in\Omega\setminus\{\infty\}, \tag{1.3}$$

where $c_2>0$ is a constant. The structural properties of compact sets satisfying (1.3) are discussed in [6, 2], where density of E near -1 is measured in terms of logarithmic capacity.

Theorem 2. *There exists a regular set* $E_1\subset\mathbb{R}$ *with the following properties:*

- (i) $g_{\overline{\mathbb{C}}\setminus E_1}$ *satisfies* (1.3)*;*
- (ii) $\dim E_1 = 0$.

We conclude this section with the following remark. One of the natural ways to construct sparse sets with Hölder continuous Green function is to consider (nowhere dense) Cantor-type sets (see [12, 5, 10, 16, 17], [18, Chapter 5]).

Let $\{\varepsilon_j\}$ be a sequence with $0 < \varepsilon_j < 1$. Starting from $[-1, 1]$ we first remove the middle ε_1 part of this interval. Then, in the second step, we remove the middle ε_2 part of both remaining intervals, etc. Denote the so obtained Cantor set by $\mathcal{C} = \mathcal{C}(\{\varepsilon_j\})$. According to [18, Theorem 5.1] and the reasoning in the same monograph [18, p. 48, after Corollary 5.2] the following three conditions are equivalent:

(i) $g_{\overline{\mathbb{C}}\setminus\mathcal{C}}$ satisfies (1.1);

(ii) $g_{\overline{\mathbb{C}}\setminus\mathcal{C}}$ satisfies (1.3);

(iii) $\sum_j {\varepsilon_j}^2 < \infty$.

At the same time, by [13, Theorem 10.5] each Cantor type set $\mathcal{C}(\{\varepsilon_j\})$ with the property

$$\lim_{j\to\infty} \varepsilon_j = 0$$

has Hausdorff dimension 1. Therefore, Cantor type sets cannot be used in the proof of either Theorem 1 or Theorem 2.

2. Construction of E_0

For $-1 \le a < b \le 1$ we consider two sequences of real numbers

$$\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots, \qquad x_k - x_0 = x_0 - x_{-k}$$

and

$$y_0 > y_{\pm 1} > y_{\pm 2} > \ldots, \qquad y_k = y_{-k},$$

such that

$$\begin{aligned} x_0 &= \frac{a+b}{2}, \\ y_0 &= \frac{b-a}{2} \exp\left(-\frac{2}{b-a}\right), \\ y_k &= (b - x_k) \exp\left(-\frac{1}{b-x_k}\right), \qquad k \in \mathbb{N} = \{1, 2, \ldots\}, \\ \frac{y_k}{x_k - x_{k-1}} &= \frac{1}{\pi}\left(\frac{1}{b-x_k} - \log\frac{1}{b-x_k}\right), \qquad k \in \mathbb{N}. \end{aligned}$$

We have

$$\lim_{k\to\infty} x_{-k} = a, \qquad \lim_{k\to\infty} x_k = b \qquad \lim_{k\to\infty} y_k = 0.$$

Let $z_k = x_k + iy_k$. For $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ consider vertical intervals $J_k = [x_k, z_k]$ and horizontal intervals $I_k = [x_{k-1}, x_k]$. For multiple indices we use the notation

$$\mathbf{k}(m) = k(1), k(2), \ldots, k(m),$$
$$\mathbf{k}(m) - 1 = k(1), k(2), \ldots, k(m-1), k(m) - 1,$$

where $m \in \mathbb{N}$ and $k(m) \in \mathbb{Z}$. We inductively define two sets of intervals

$$\{J_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbb{Z}^m} \qquad \text{and} \qquad \{I_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbb{Z}^m}$$

in the following way. Denote by

$$\{J_{\mathbf{k}(1)}\}_{\mathbf{k}(1)\in\mathbb{Z}} \qquad \text{and} \qquad \{I_{\mathbf{k}(1)}\}_{\mathbf{k}(1)\in\mathbb{Z}}$$

the sequences of vertical and horizontal intervals, which we obtain by the above procedure for $[a, b] = [-1, 1]$.

Next, for $m > 1$ denote by

$$\{J_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbb{Z}^m} \qquad \text{and} \qquad \{I_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbb{Z}^m}$$

the sequences of vertical and horizontal intervals, which we obtain by the above procedure for $[a, b] = I_{\mathbf{k}(m-1)}$. The endpoints of $J_{\mathbf{k}(m)}$ we denote by $x_{\mathbf{k}(m)} \in \mathbb{R}$ and $z_{\mathbf{k}(m)} \in \mathbb{C}$ respectively, so that $I_{\mathbf{k}(m)} = [x_{\mathbf{k}(m)-1}, x_{\mathbf{k}(m)}]$. Since

$$D_0 = \{z = x + iy : |x| < 1, y > 0\} \setminus \left(\bigcup_{m\in\mathbb{N}} \bigcup_{\mathbf{k}(m)\in\mathbb{Z}^m} J_{\mathbf{k}(m)} \right)$$

is a simply connected domain, by the Riemann Mapping Theorem there exists a conformal mapping ϕ_0 of D_0 onto the upper half plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$.

We interpret the boundary of D_0 in terms of Carathéodory's theory of prime ends (see [13]). Let $P(D_0)$ denote the set of all prime ends of D_0. For a prime end $Z \in P(D_0)$ denote its impression by $|Z|$. By our construction, all prime ends of D_0 are of the first kind, i.e. $|Z|$ is a singleton for any $Z \in P(D_0)$. For the homeomorphism between $D_0 \cup P(D_0)$ and $\overline{\mathbb{H}}$ we preserve the same notation ϕ_0. We denote by $\psi_0 = \phi_0^{-1}$ the inverse homeomorphism. We identify the prime end $\psi_0(w), w \in \mathbb{R}$ with its impression when no confusion can arise. If $z \in \partial D_0$ is the impression of only one prime end it will also cause no confusion if we use the same letter z to designate the prime end and its impression. For example, we write $\infty, -1, z_{\mathbf{k}(m)}, 1$ for prime ends with impressions at those points.

To define ϕ_0 uniquely we normalize it by the boundary conditions

$$\phi_0(\infty) = \infty, \qquad \phi_0(-1) = -1, \qquad \phi_0(1) = 1.$$

Each point of $J_{\mathbf{k}(m)} \setminus \{z_{\mathbf{k}(m)}\}$ is the impression of two prime ends and $z_{\mathbf{k}(m)}$ is the impression of exactly one prime end. Moreover,

$$\phi_0\big(\{Z \in P(D_0) : |Z| \in J_{\mathbf{k}(m)} \setminus \{x_{\mathbf{k}(m)}\}\}\big)$$

is an open subinterval of $(-1,1)$ which we denote by $J'_{\mathbf{k}(m)} = (\xi^-_{\mathbf{k}(m)}, \xi^+_{\mathbf{k}(m)})$. Let $\xi_{\mathbf{k}(m)} = \phi_0(z_{\mathbf{k}(m)})$.

In Section 5 we show that the compact set

$$E_0 = [-1,1] \setminus \left(\bigcup_{m\in\mathbb{N}} \bigcup_{\mathbf{k}(m)\in\mathbb{Z}^m} J'_{\mathbf{k}(m)} \right)$$

satisfies the conditions of Theorem 1. The crucial fact is that for $w \in \overline{\mathbb{H}} \cap \Omega_0$,

$$g_{\Omega_0}(w) = \frac{\pi}{2} \operatorname{Im} \psi_0(w), \tag{2.1}$$

where $\Omega_0 = \overline{\mathbb{C}} \setminus E_0$.

In order to prove (2.1), consider the function

$$h(w) = \begin{cases} \dfrac{\pi}{2} \operatorname{Im} \psi_0(w) & \text{if } w \in \overline{\mathbb{H}} \cap \Omega_0, \\ \dfrac{\pi}{2} \operatorname{Im} \psi_0(\overline{w}) & \text{if } w \in \overline{\mathbb{C}} \setminus \mathbb{H}. \end{cases}$$

It is continuous in $\Omega_0 \setminus \{\infty\}$ and, according to the distortion properties of ψ_0, the difference

$$h(w) - \log|w|$$

is bounded in the neighborhood of ∞.

The function h is harmonic in $\mathbb{C} \setminus \mathbb{R}$. In order to prove that h coincides with g_{Ω_0} it is sufficient to show that h is harmonic in some neighborhood of each

$$\xi \in (\mathbb{R} \setminus E_0) \setminus \left(\bigcup_{m\in\mathbb{N}} \bigcup_{\mathbf{k}(m)\in\mathbb{Z}^m} \xi_{\mathbf{k}(m)} \right).$$

Let $\varepsilon = \varepsilon(\xi) > 0$ be such that

$$[\xi - \varepsilon, \xi + \varepsilon] \subset (\mathbb{R} \setminus E_0) \setminus \left(\bigcup_{m\in\mathbb{N}} \bigcup_{\mathbf{k}(m)\in\mathbb{Z}^m} \xi_{\mathbf{k}(m)} \right).$$

Since all derivatives of ψ_0 can be extended continuously to $[\xi - \varepsilon, \xi + \varepsilon]$, it is enough to show that for $k = 1, 2$, $j = 0, 1, 2$, $j \le k$ and $w = u + iv$ we have

$$\lim_{\substack{w\to\xi \\ \operatorname{Im} w > 0}} \frac{\partial^k h(w)}{\partial u^j \partial v^{k-j}} = \lim_{\substack{w\to\xi \\ \operatorname{Im} w < 0}} \frac{\partial^k h(w)}{\partial u^j \partial v^{k-j}},$$

which can be easily done.

The boundary behavior of ψ_0 and (2.1) imply the regularity of E_0.

3. Construction of E_1

We start with two sequences of real numbers

$$1 = x_0 > x_1 > x_2 > \ldots > -1 \qquad \text{and} \qquad 4 = y_0 > y_1 > y_2 > \ldots > 0$$

such that

$$\begin{aligned} y_k &= (x_k+1)^2, \qquad k \in \mathbb{N}, \\ \lim_{k\to\infty} x_k &= -1, \\ \lim_{k\to\infty} y_k &= 0, \\ \frac{y_k}{x_{k-1}-x_k} &\geq \frac{2}{\pi}\log\frac{1}{x_{k-1}-x_k}, \\ x_{k-1}-x_k &< \frac{1}{2}, \qquad k \in \mathbb{N}. \end{aligned}$$

Starting with the set of intervals

$$I_k = [x_{k-1}, x_k], \qquad J_k = [x_k, x_k + iy_k] = [x_k, z_k], \quad k = k(1) \in \mathbb{N},$$

we construct the sets of intervals $\{I_{\mathbf{k}(m)}\}$ and $\{J_{\mathbf{k}(m)}\}$ in the following manner. Let for $m \geq 2$, intervals $\{I_{\mathbf{k}(m-1)}\}$ and $\{J_{\mathbf{k}(m-1)}\}$ be constructed, and let

$$(A_{\mathbf{k}(m-1)})^2 = \exp\left(m^2 + \pi \sum_{j=1}^{m-1} \frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|}\right).$$

We define $\delta_{\mathbf{k}(m-1)} > 0$ such that

$$\frac{|J_{\mathbf{k}(m-1)}|}{\delta_{\mathbf{k}(m-1)}} \geq \frac{4m}{\pi}\log\frac{A_{\mathbf{k}(m-1)}}{\delta_{\mathbf{k}(m-1)}}. \tag{3.1}$$

Next, we select a finite number of points

$$x_{\mathbf{k}(m-1)-1} = x_{\mathbf{k}(m-1),0} > x_{\mathbf{k}(m-1),1} > \ldots > x_{\mathbf{k}(m-1),K(m)} = x_{\mathbf{k}(m-1)}$$

such that for any $1 \leq k(m) \leq K(m)$,

$$\frac{1}{2}\delta_{\mathbf{k}(m-1)} \leq x_{\mathbf{k}(m-1),k(m)-1} - x_{\mathbf{k}(m-1),k(m)} \leq \delta_{\mathbf{k}(m-1)}. \tag{3.2}$$

Let

$$\begin{aligned} y_{\mathbf{k}(m)} &= \frac{1}{2}y_{\mathbf{k}(m-1)}, \\ z_{\mathbf{k}(m)} &= x_{\mathbf{k}(m)} + iy_{\mathbf{k}(m)}, \qquad 0 \leq k(m) \leq K(m), \\ J_{\mathbf{k}(m)} &= [x_{\mathbf{k}(m)}, z_{\mathbf{k}(m)}], \qquad 0 \leq k(m) \leq K(m), \\ I_{\mathbf{k}(m)} &= [x_{\mathbf{k}(m)}, x_{\mathbf{k}(m)-1}], \qquad 1 \leq k(m) \leq K(m). \end{aligned}$$

Denote by ϕ_1 a conformal mapping of the simply connected domain

$$D_1 = \{z = x + iy : |x| < 1, y > 0\} \setminus \left(\bigcup_{m \in \mathbb{N}} \bigcup_{\substack{1 \le k(j) \le K(j) \\ 1 \le j \le m}} J_{\mathbf{k}(m)} \right),$$

where $K(1) = \infty$, onto $\mathbb{H}$.

Let $P(D_1)$ be the set of all prime ends of D_1. The reasoning about the structure of $P(D_0)$ from Section 2 applies to $P(D_1)$.

We extend ϕ_1 to the homeomorphism $\phi_1 : D_1 \cup P(D_1) \to \overline{\mathbb{H}}$ and denote the inverse mapping by $\psi_1 = \phi_1^{-1}$. Sometimes, for simplicity, we identify $\psi_1(w), w \in \mathbb{R}$, with the impression of $\psi_1(w)$.

We normalize ϕ_1 by the boundary conditions

$$\phi_1(\infty) = \infty, \qquad \phi_1(-1) = -1, \qquad \phi_1(1) = 1.$$

For $1 \le k(j) \le K(j), 1 \le j \le m-1$ and $1 \le k(m) \le K(m) - 1$ define intervals

$$J'_{\mathbf{k}(m)} = (\xi^-_{\mathbf{k}(m)}, \xi^+_{\mathbf{k}(m)}) = \phi_1(\{Z \in P(D_1) : |Z| \in J_{\mathbf{k}(m)} \setminus \{x_{\mathbf{k}(m)}\}\})$$

and points $\xi_{\mathbf{k}(m)} = \phi_1(z_{\mathbf{k}(m)})$.

In Section 6, we show that a compact set

$$E_1 = [-1, 1] \setminus \left(\bigcup_{m \in \mathbb{N}} \bigcup_{\substack{1 \le k(j) \le K(j) \\ 1 \le j \le m-1 \\ 1 \le k(m) \le K(m)-1}} J'_{\mathbf{k}(m)} \right)$$

satisfies the conditions of Theorem 2. The basic idea is to apply formula

$$g_{\Omega_1}(w) = \frac{\pi}{2} \operatorname{Im} \psi_1(w), \qquad w \in \overline{\mathbb{H}} \cap \Omega_1, \tag{3.3}$$

where $\Omega_1 = \overline{\mathbb{C}} \setminus E_1$, whose proof is the same as the proof of (2.1).

The boundary behavior of ψ_1 and (3.3) imply the regularity of E_1.

4. Moduli of some families of curves

Our analysis of the behavior of the Green function involves auxiliary conformal mappings and it is based on the application of the notion of the module $m(\Gamma)$ of a family of curves Γ. We briefly recall the *L-definition* of the module of families of curves. A non-negative Borel measurable function $\rho(z)$, $z \in \mathbb{C}$ (*metric* for short) is called *admissible* (in the L-definition of $m(\Gamma)$) for the family Γ if for all locally rectifiable $\gamma \in \Gamma$

$$\int_\gamma \rho(z)|dz| \ge 1 \tag{4.1}$$

holds.

Then, the quantity

$$m(\Gamma) = \inf_{\rho} \int_{\mathbb{C}} \rho^2(z)\, dm(z), \tag{4.2}$$

where $dm(z)$ is the two-dimensional Lebesgue measure on $\mathbb{C}$ and infimum is taken over all admissible metrics ρ, is called the *module* of the family Γ.

In the sequel we refer to [1, 9] for the basic properties of the module (such as conformal invariance, comparison principle, composition laws, etc.). We will use these properties without further citation.

Special families of separating curves play an extremely useful role. Let $D \subset \mathbb{C}$ be a simply connected domain and let $P(D)$ be the set of all prime ends of D. We say that a crosscut γ of D (i.e. locally rectifiable Jordan arc $\gamma \in D$ with endpoints on ∂D) separates $Z_1, \ldots, Z_m \in P(D)$ and $a_1, \ldots, a_n \in D$ from $\mathcal{Z}_1, \ldots, \mathcal{Z}_l \in P(D)$ and $b_1, \ldots, b_k \in D$ if it divides D into two domains D' and D'' such that $a_1, \ldots, a_n \in D'$, $b_1, \ldots, b_k \in D''$, $Z_1, \ldots, Z_m$ are adjacent for D' and $\mathcal{Z}_1, \ldots, \mathcal{Z}_l$ are adjacent for D''. The term *adjacent* means that in the domain and subdomain the prime end can be defined by the same null chain of crosscuts. We denote by

$$\Gamma(Z_1, \ldots, Z_m, a_1, \ldots, a_n; \mathcal{Z}_1, \ldots, \mathcal{Z}_l, b_1, \ldots, b_k; D)$$

the family of all crosscuts separating $Z_1, \ldots, Z_m \in P(D)$ and $a_1, \ldots, a_n \in D$ from $\mathcal{Z}_1, \ldots, \mathcal{Z}_l \in P(D)$ and $b_1, \ldots, b_k \in D$.

Next, we discuss some auxiliary estimates for the module of families of crosscuts which we need in Sections 5 and 6.

Let $w_0, w_1 \in \mathbb{R}$ and $w_2 \in \overline{\mathbb{H}}$ be distinct points and let $\Gamma = \Gamma(w_0, w_1; w_2, \infty; \mathbb{H})$. Then

$$m(\Gamma) \geq \frac{1}{\pi} \log \left| \frac{w_2 - w_0}{w_1 - w_0} \right|. \tag{4.3}$$

Indeed, if $|w_1 - w_0| \geq |w_2 - w_0|$, then (4.3) follows from the non-negativity of the module. If $|w_1 - w_0| < |w_2 - w_0|$, we compare Γ with the family Γ_1 of all crosscuts of the annular sector

$$G = \{w \in \mathbb{H} : |w_1 - w_0| < |w - w_0| < |w_2 - w_0|\}$$

which join the radial parts of ∂G. We obtain

$$m(\Gamma) \geq m(\Gamma_1) = \frac{1}{\pi} \log \left| \frac{w_2 - w_0}{w_1 - w_0} \right|.$$

Let $S = \{\zeta_1, \zeta_2, \ldots\}$ be an at most countable set of points $\zeta_j \in \mathbb{C}$, and let the family of curves Γ be such that

$$\gamma \cap S \neq \emptyset, \qquad \gamma \in \Gamma.$$

Then

$$m(\Gamma) = 0. \tag{4.4}$$

Indeed, for any $j \in \mathbb{N}$ and $0 < \varepsilon < 1$ consider metrics

$$\rho_{j,\varepsilon}(\zeta) = \begin{cases} \left(|\zeta - \zeta_j| \log \dfrac{1}{|\zeta - \zeta_j|} \right)^{-1} & \text{if } |\zeta - \zeta_j| < \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

$$\rho_\varepsilon(\zeta) = \sup_j \{2^{-j} \rho_{j,\varepsilon}(\zeta)\}, \qquad \zeta \in \mathbb{C}.$$

Since

$$\int_\gamma \rho_\varepsilon(\zeta) |\, d\zeta| = \infty > 1, \qquad \gamma \in \Gamma,$$

by (4.2) we obtain

$$m(\Gamma) \le \int_{\mathbb{C}} {\rho_\varepsilon}^2 \, dm \le \sum_j 2^{-2j} \int_{\mathbb{C}} {\rho_{j,\varepsilon}}^2 \, dm \le \pi \left(\log \frac{1}{\varepsilon} \right)^{-1}.$$

Letting $\varepsilon \to 0$ gives (4.4).

Next, we cite a result of Jenkins and Oikawa [7] concerning Ahlfors' fundamental inequalities.

Lemma 1 ([7]). *For* $0 < r_1 < r_2 < \infty$, *let*

$$Q = Q(r_1, r_2) := \{re^{i\theta} : \, r_1 < r < r_2, \, -\theta_1(r) < \theta < \theta_2(r)\},$$

where the functions θ_j, $j = 1, 2$, *have finite total variation* V_j *on* $[r_1, r_2]$ *and satisfy*

$$0 < \theta_0 \le \theta_j(r) \le \pi.$$

Then, for the module of the family $\Gamma = \Gamma(Q)$ *of all arcs separating in* Q *its boundary circular components, we have*

$$m(\Gamma) \le \int_{r_1}^{r_2} \frac{dr}{(\theta_1(r) + \theta_2(r))\, r} + \frac{\pi}{{\theta_0}^2}(V_1 + V_2). \tag{4.5}$$

Let us mention two consequences of Lemma 1 which prove to be useful in Sections 5 and 6.

In what follows we use the convention that $c, c_1, \ldots$ denote positive constants, different in different cases.

For $x > 0$ we consider the functions

$$f_j(x) = \begin{cases} xe^{-1/x} & \text{if } j = 0, \\ x^2 & \text{if } j = 1. \end{cases}$$

For $j = 0, 1$, $x_0 \in \mathbb{R}$ and $0 < r < R \le 2$ consider the quadrilateral

$$Q_j(x_0, r, R) = \{z = x + iy : \, r < |z - x_0| < R, \, x > x_0, \, y > f_j(x - x_0)\}. \tag{4.6}$$

Applying (4.5) to a suitable rotation of $Q_j(x_0,r,R)$, for the module of the family Γ of all crosscuts of $Q_j(x_0,r,R)$ separating its boundary circular components we have

$$
\begin{aligned}
m(\Gamma) &\le \int_r^R \frac{d\rho}{g_j(\rho)} + c_1\\
&= \frac{2}{\pi}\int_r^R \frac{d\rho}{\rho} + \frac{2}{\pi}\int_r^R \frac{\frac{\pi}{2}\rho - g_j(\rho)}{g_j(\rho)\rho}\,d\rho + c_1\\
&\le \frac{2}{\pi}\log\frac{R}{r} + c_2\int_0^2 \frac{f_j(\rho)}{\rho^2}\,d\rho + c_1\\
&\le \frac{2}{\pi}\log\frac{R}{r} + c_3,
\end{aligned}
\tag{4.7}
$$

where $g_j(\rho) := |\{z \in Q_j(x_0,r,R) : \, |z-x_0| = \rho\}|$.

Further, for $x_0, x_1 \in \mathbb{R}$ such that $x_0 < x_1 < x_0+1$, $0 < r < x_1 - x_0$ and $j = 0,1$ consider the quadrilateral

$$
\begin{aligned}
Q_j^*(x_0,x_1,r) = \{z = x+iy : \, r < |z - (x_1 + if_j(x_1-x_0))| < x_1 - x_0,\\
y > f_j(x-x_0)\}.
\end{aligned}
\tag{4.8}
$$

According to (4.5) the module of the family Γ of all crosscuts of $Q_j^*(x_0,x_1,r)$ separating its boundary circular components satisfies the inequality

$$
m(\Gamma) \le \frac{1}{\pi}\log\frac{x_1-x_0}{r} + c_4. \tag{4.9}
$$

Further, let $\xi_1 < \xi_2 < \xi_3 < \xi_4$, and let $\Gamma = \Gamma(\xi_2,\xi_3;\xi_1,\xi_4;\mathbb{H})$. Then

$$
m(\Gamma) \le \frac{1}{\pi}\log\frac{\xi_4-\xi_1}{\xi_3-\xi_2} + c_5. \tag{4.10}
$$

Indeed, define the metric

$$
\rho(w) = \begin{cases} \left(\pi\left|w - \dfrac{\xi_3+\xi_2}{2}\right|\right)^{-1} & \text{if } \dfrac{\xi_3-\xi_2}{2e^\pi} \le \left|w - \dfrac{\xi_3+\xi_2}{2}\right| \le e^\pi(\xi_4-\xi_1),\\ 0 & \text{otherwise.}\end{cases}
$$

Standard calculation yields (4.1) (for details, see [3, p. 349]). Hence, according to (4.2) we have

$$
m(\Gamma) \le \int_{\mathbb{C}} \rho^2\,dm = \frac{1}{\pi}\log 2\frac{\xi_4-\xi_1}{\xi_3-\xi_2} + 2,
$$

which is the desired estimate (4.10).

We complete this section by recalling the Lavrentiev Distortion Theorem. Let $Z, \mathcal{Z} \in P(D_j)$, where $j = 0,1$ is fixed and let $z_0 = 2i$, $\phi_j(Z) = w$, $\phi_j(\mathcal{Z}) = \tau$. We say that a crosscut $\gamma \in D_j$ joins Z to $\mathcal{Z}$ if $\phi_j(\gamma)$ joins w to τ.

Consider the *relative distance* between Z and $\mathcal{Z}$, i.e.

$$\rho(Z,\mathcal{Z},D_j)=\min_{k=1,2}\rho_k(Z,\mathcal{Z},D_j),$$

where

$$\rho_k(Z,\mathcal{Z},D_j)=\inf|\gamma|,$$

and infimum is taken over all crosscuts γ of D_j joining Z to $\mathcal{Z}$ for $k=1$, and all crosscuts γ of D_j separating z_0 from Z and $\mathcal{Z}$ for $k=2$.

According to [8]

$$|w-\tau|\le c_6\rho(Z,\mathcal{Z},D_j)^{1/2} \tag{4.11}$$

for any $w,\tau\in[-1,1]$.

5. Proof of Theorem 1

We have divided the proof into a sequence of lemmas and remarks.

We begin with some technical facts concerning inequalities from above for some special families of crosscuts of D_0. In what follows we use the convention that $\Gamma,\Gamma_1,\ldots$ denote families of curves, different in different circumstances.

Lemma 2. *Let* $Z\in P(D_0),|Z|=z=x+iy$ *be such that* $y>0$ *if* $x=-1$ *and* $y=(x+1)\exp(-1/(x+1))$ *if* $x>-1$. *Let* $|z+1|\le 2$, $z_0=2i$. *Then, for the module of* $\Gamma=\Gamma(-1,Z;z_0,\infty;D_0)$ *we have*

$$m(\Gamma)\le\frac{2}{\pi}\log\frac{1}{|z+1|}+c_1.$$

Proof. Consider the quadrilateral $Q_0=Q_0(-1,|z+1|,2)$ defined by (4.6). Denote by Γ_1 the family of all crosscuts of Q_0 which separate the circular arcs of ∂Q_0. Let ρ_1 be an admissible metric for Γ_1 which is also extremal in the sense that

$$m(\Gamma_1)=\int_{\mathbb{C}}{\rho_1}^2\,dm. \tag{5.1}$$

We claim that the metric

$$\rho(\zeta)=\max_{j=1,2,3}\rho_j(\zeta),\qquad \zeta\in\mathbb{C}, \tag{5.2}$$

where

$$\rho_2(\zeta)=\begin{cases}\dfrac{2}{|z+1|} & \text{if } |\zeta+1|<2|z+1|,\\ 0 & \text{otherwise,}\end{cases}$$

$$\rho_3(\zeta)=\begin{cases}1 & \text{if } |\operatorname{Re}\zeta|<1,\ 0<\operatorname{Im}\zeta<3,\\ 0 & \text{otherwise,}\end{cases}$$

is admissible for Γ. The proof of this fact separates naturally into three parts.

If $\gamma \in \Gamma$ and $\gamma \cap \{\zeta : |\zeta+1| = |z+1|\} \neq \emptyset$, then

$$|\gamma \cap \{\zeta : |\zeta+1| \leq 2|z+1|\}| \geq \frac{|z+1|}{2}.$$

Therefore,

$$\int_\gamma \rho\,|d\zeta| \geq \int_\gamma \rho_2\,|d\zeta| \geq 1. \tag{5.3}$$

If $\gamma \in \Gamma$ and $\gamma \cap \{\zeta : |\zeta+1| \geq 2\} \neq \emptyset$, then

$$|\gamma \cap \{\zeta = x+iy : |x| \leq 1, 0 < y < 3\}| \geq 1,$$

i.e.

$$\int_\gamma \rho\,|d\zeta| \geq \int_\gamma \rho_3\,|d\zeta| \geq 1. \tag{5.4}$$

If $\gamma \in \Gamma$ and $\gamma \subset \{\zeta : |z+1| < |\zeta+1| < 2\}$, then

$$\int_\gamma \rho\,|d\zeta| \geq \int_\gamma \rho_1\,|d\zeta| \geq 1. \tag{5.5}$$

Comparing (5.3)-(5.5) we have (4.1).

Hence, by (4.7) and (5.1) we obtain

$$m(\Gamma) \leq \sum_{j=1}^{3} \int_{\mathbb{C}} {\rho_j}^2\,dm \leq \frac{2}{\pi} \log \frac{1}{|z+1|} + c_1.$$

■

The proof of Lemma 2 demonstrates a typical approach to deriving estimates from above for the module of families of crosscuts that we frequently use below. Informally the proof can be described as follows.

First, we extract the "main subfamily"

$$\Gamma_1^* = \{\gamma \in \Gamma : \gamma \subset \{\zeta : |z+1| < |\zeta+1| < 2\}\}$$

of Γ. According to the comparison principle $m(\Gamma_1^*) \leq m(\Gamma_1)$.

The module $m(\Gamma_1)$, in its turn, can be estimated from above by applying the Ahlfors' fundamental inequality (see Lemma 1). We do this implicitly by constructing the metric $\rho \geq \rho_1$.

The family $\Gamma \setminus \Gamma_1^*$ can be represented in the form

$$\Gamma \setminus \Gamma_1^* = \Gamma_2 \cup \Gamma_3,$$

where

$$\Gamma_2 = \{\gamma \in \Gamma : \gamma \cap \{\zeta : |\zeta+1| = |z+1|\} \neq \emptyset\},$$
$$\Gamma_3 = \{\gamma \in \Gamma : \gamma \cap \{\zeta : |\zeta+1| \geq 2\} \neq \emptyset\},$$

and $m(\Gamma_2)$ as well as $m(\Gamma_3)$ are bounded by absolute constants. Since we are not interested in sharp values of these constants we relatively simply construct metrics ρ_2 and ρ_3 which are admissible for Γ_2 and Γ_3 respectively, and such that

$$m(\Gamma_k) \leq \int_{\mathbb{C}} {\rho_k}^2 \, dm \leq c, \qquad k = 2, 3.$$

At last, defining ρ as in (5.2) we implicitly have

$$m(\Gamma) \leq m(\Gamma_1^*) + m(\Gamma_2) + m(\Gamma_3),$$

which is a composition law for families of curves.

Let $X^{\pm}_{\mathbf{k}(j)} = \psi_0(\xi^{\pm}_{\mathbf{k}(j)})$, that is $|X^{\pm}_{\mathbf{k}(j)}| = x_{\mathbf{k}(j)}$. For convenience, let

$$X^{+}_{\mathbf{k}(0)-1} = x_{\mathbf{k}(0)-1} = -1, \qquad X^{-}_{\mathbf{k}(0)} = x_{\mathbf{k}(0)} = 1.$$

From two prime ends $X^{+}_{\mathbf{k}(j-1)-1}$ and $X^{-}_{\mathbf{k}(j-1)}$ we choose the closest to the interval $I_{\mathbf{k}(j)}$ (in the sense of distance between $I_{\mathbf{k}(j)}$ and impressions of these two prime ends) and denote it by $\mathcal{Z}'_{\mathbf{k}(j)}$. We denote by $\mathcal{Z}''_{\mathbf{k}(j)}$ the other prime end. Let $d_{\mathbf{k}(j)} = |x_{\mathbf{k}(j)} - |\mathcal{Z}'_{\mathbf{k}(j)}||$.

Remark 1. The following result may be proved in much the same way as Lemma 2. Let $Z \in P(D_0)$ satisfy one of the following conditions:

(a) $\phi_0(Z) = \xi$, $\xi^{-}_{\mathbf{k}(j)} < \xi \leq \xi_{\mathbf{k}(j)}$,

(b) $Z = z_{\mathbf{k}(j),k(j+1)}$ for some $k(j+1) \in \mathbb{Z}$.

Let $z = |Z|, |z - x_{\mathbf{k}(j)}| \leq |I_{\mathbf{k}(j)}|$ and $\Gamma = \Gamma(X^{-}_{\mathbf{k}(j)}, Z; X^{+}_{\mathbf{k}(j)-1}, \infty; D_0)$. Then

$$m(\Gamma) \leq \frac{2}{\pi} \log \frac{|I_{\mathbf{k}(j)}|}{|z - x_{\mathbf{k}(j)}|} + c_2. \tag{5.6}$$

Lemma 3. *For the module of* $\Gamma = \Gamma(X^{-}_{k}, z_k; k/|k|, \infty; D_0)$*, where* $k = k(1) \in \mathbb{Z}$*,* $0/|0| = 1$*, we have*

$$m(\Gamma) \leq \frac{1}{\pi} \log \frac{\left| x_k - \frac{k}{|k|} \right|}{|I_k|} + c_3.$$

Proof. There is no loss of generality in assuming that $-k \in \{0\} \cup \mathbb{N}$, i.e. $x_k \leq 0$. Consider the quadrilateral $Q_0^* = Q_0^*(-1, x_k, |z_k - z_{k-1}|)$ defined by (4.8). Denote by Γ_1 the family of all crosscuts of Q_0^* which separate the circular arcs of ∂Q_0^*. Let ρ_1 be an admissible metric for Γ_1 which is also extremal in the sense that

$$m(\Gamma_1) = \int_{\mathbb{C}} {\rho_1}^2 \, dm \leq \frac{1}{\pi} \log \frac{x_k + 1}{|I_k|} + c_4 \tag{5.7}$$

(cf. (4.9)). The same reasoning as in the proof of Lemma 2 shows that the metric

$$\rho(\zeta) = \max_{j=1,2,3} \rho_j(\zeta), \qquad \zeta \in \mathbb{C},$$

where

$$\rho_2(\zeta) = \begin{cases} \dfrac{2}{|I_k|} & \text{if } |\zeta - z_k| < 2|z_k - z_{k-1}|, \\ 0 & \text{otherwise,} \end{cases}$$

$$\rho_3(\zeta) = \begin{cases} \dfrac{2}{(x_k+1)} & \text{if } |\zeta - z_k| \le 2(x_k+1), \\ 0 & \text{otherwise,} \end{cases}$$

is admissible for Γ. Therefore, by (4.2) and (5.7) we obtain

$$m(\Gamma) \le \sum_{j=1}^{3} \int_{\mathbb{C}} {\rho_j}^2 \, dm \le \frac{1}{\pi} \log \frac{x_k+1}{|I_k|} + c_5.$$

■

Remark 2. The same reasoning as above can be used to prove the following assertion. Let $j > 1$. For the module of the family $\Gamma = \Gamma(X^-_{\mathbf{k}(j)}, z_{\mathbf{k}(j)}; \mathcal{Z}'_{\mathbf{k}(j)}, \infty; D_0)$ we have

$$m(\Gamma) \le \frac{1}{\pi} \log \frac{d_{\mathbf{k}(j)}}{|I_{\mathbf{k}(j)}|} + c_6.$$

Lemma 4. *For the module of the family* $\Gamma = \Gamma(X^+_{\mathbf{k}(j)-1}, X^-_{\mathbf{k}(j)}; z_{\mathbf{k}(j)}, \infty; D_0)$, $j \in \mathbb{N}$, *we have*

$$m(\Gamma) \le \frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + c_7. \tag{5.8}$$

Proof. Since the metric

$$\rho(\zeta) = \begin{cases} \dfrac{1}{|I_{\mathbf{k}(j)}|} & \text{if } \zeta \in \{\zeta = x + iy : x_{\mathbf{k}(j)-1} \le x \le x_{\mathbf{k}(j)},\, 0 \le y \le y_{\mathbf{k}(j)}\} \\ & \qquad \cup \{\zeta : |\zeta - z_{\mathbf{k}(j)}| \le 2|z_{\mathbf{k}(j)-1} - z_{\mathbf{k}(j)}|\}, \\ 0 & \text{otherwise,} \end{cases}$$

is admissible for Γ, by (4.2) we obtain

$$m(\Gamma) \le \int_{\mathbb{C}} \rho^2 \, dm \le \frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + c_7.$$

■

Remark 3. The estimate (5.8) is sharp in the following sense. Comparing the family $\Gamma = \Gamma(X^+_{\mathbf{k}(j)-1}, X^-_{\mathbf{k}(j)}; z_{\mathbf{k}(j)}, z_{\mathbf{k}(j)-1}; D_0), j \in \mathbb{N}$, with the family of line intervals

$$\Gamma_1 = \{\gamma_r = [x_{\mathbf{k}(j)-1} + ir, x_{\mathbf{k}(j)} + ir] : |I_{\mathbf{k}(j)}| < r < |J_{\mathbf{k}(j)}| - |I_{\mathbf{k}(j)}|\},$$

we have

$$m(\Gamma) \ge m(\Gamma_1) = \frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} - 2.$$

Lemma 5. *For* $\Gamma = \Gamma(X^+_{\mathbf{k}(m)-1}, X^-_{\mathbf{k}(m)}; z_0, \infty; D_0), m \in \mathbb{N}$*, we have*

$$m(\Gamma) \leq \frac{2}{\pi} \log \frac{1}{|J_{\mathbf{k}(m)}|} + c_8.$$

Proof. Our proof starts with the observation that

$$\Gamma \subset \bigcup_{j=1}^{m} \Big(\Gamma(X^+_{\mathbf{k}(j)-1}, X^-_{\mathbf{k}(j)}; z_{\mathbf{k}(j)}, \infty; D_0) \cup \Gamma(X^-_{\mathbf{k}(j)}, z_{\mathbf{k}(j)}; \mathcal{Z}'_{\mathbf{k}(j)}, \infty; D_0)$$
$$\cup \Gamma(z_{\mathbf{k}(j)}, \mathcal{Z}'_{\mathbf{k}(j)}; \mathcal{Z}''_{\mathbf{k}(j)}, \infty; D_0) \Big) \cup \Gamma(-1, 1; z_0, \infty; D_0) \cup \Gamma_1,$$

where

$$\Gamma_1 = \{\gamma \in \Gamma : \overline{\gamma} \cap \cup_{j=1}^m \{(X^+_{\mathbf{k}(j)-1}, X^-_{\mathbf{k}(j)}; z_{\mathbf{k}(j)}, \mathcal{Z}'_{\mathbf{k}(j)}, \mathcal{Z}''_{\mathbf{k}(j)}\} \neq \emptyset\},$$

and $m(\Gamma_1) = 0$ by (4.4).

Clearly, for $\Gamma_2 = \Gamma(-1, 1; z_0, \infty; D_0)$ the inequality

$$m(\Gamma_2) \leq 2 \tag{5.9}$$

holds.

Indeed, since the metric

$$\rho(\zeta) = \begin{cases} \dfrac{1}{2} & \text{if } |\operatorname{Re}\zeta| \leq 1, 0 < \operatorname{Im}\zeta < 4, \\ 0 & \text{otherwise,} \end{cases}$$

is admissible for Γ_2, according to (4.2) we have

$$m(\Gamma_2) \leq \int_{\mathbb{C}} \rho^2 \, dm = 2.$$

Due to subadditivity of the module of a family of curves and Lemma 2, Remarks 1 and 2 as well as Lemma 4 and (5.9) we obtain

$$\begin{aligned} m(\Gamma) &\leq \sum_{j=1}^{m} \left(\frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + \frac{1}{\pi} \log \frac{d_{\mathbf{k}(j)}}{|I_{\mathbf{k}(j)}|} + \frac{2}{\pi} \log \frac{|I_{\mathbf{k}(j-1)}|}{d_{\mathbf{k}(j)}} + c_9 \right) + 2 \\ &= \sum_{j=1}^{m} \left(\frac{2}{\pi} \log \frac{|J_{\mathbf{k}(j-1)}|}{|J_{\mathbf{k}(j)}|} + \frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + \frac{2}{\pi} \log \frac{|I_{\mathbf{k}(j-1)}|}{|J_{\mathbf{k}(j-1)}|} \right. \\ &\qquad \left. + \frac{1}{\pi} \log \frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + \frac{1}{\pi} \log \frac{|J_{\mathbf{k}(j)}|}{d_{\mathbf{k}(j)}} + c_9 \right) + 2 \\ &\leq \frac{2}{\pi} \log \frac{|J_{\mathbf{k}(0)}|}{|J_{\mathbf{k}(m)}|} + \frac{1}{\pi} \log \frac{|J_{\mathbf{k}(m)}|}{|I_{\mathbf{k}(m)}|} + c_{10} \\ &\quad + \sum_{j=1}^{m} \left(\frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + \frac{1}{\pi} \log \frac{|J_{\mathbf{k}(j)}|}{d_{\mathbf{k}(j)}} \right), \end{aligned}$$

where $|I_{\mathbf{k}(0)}| = |J_{\mathbf{k}(0)}| = 2$.

From the definition of E_0 we conclude that for $j \in \mathbb{N}$

$$\frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} = \frac{1}{\pi}\left(\frac{1}{d_{\mathbf{k}(j)}} - \log\frac{1}{d_{\mathbf{k}(j)}}\right),$$
$$\frac{|J_{\mathbf{k}(j)}|}{d_{\mathbf{k}(j)}} = \exp\left(-\frac{1}{d_{\mathbf{k}(j)}}\right).$$

Hence

$$\sum_{j=1}^{m}\left(\frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + \frac{1}{\pi}\log\frac{|J_{\mathbf{k}(j)}|}{d_{\mathbf{k}(j)}}\right) = \sum_{j=1}^{m}\left(\frac{1}{\pi}\frac{1}{d_{\mathbf{k}(j)}} - \frac{1}{\pi}\log\frac{1}{d_{\mathbf{k}(j)}} - \frac{1}{\pi}\frac{1}{d_{\mathbf{k}(j)}}\right)$$
$$\le \frac{1}{\pi}\log d_{\mathbf{k}(m)},$$

and finally

$$m(\Gamma) \le \frac{2}{\pi}\log\frac{2}{|J_{\mathbf{k}(m)}|} + \frac{1}{\pi}\log\frac{|J_{\mathbf{k}(m)}|}{|I_{\mathbf{k}(m)}|}d_{\mathbf{k}(m)} + c_{10} \le \frac{2}{\pi}\log\frac{2}{|J_{\mathbf{k}(m)}|} + c_{10}.$$

■

Having disposed of these preliminary steps, we can now return to the proof of Theorem 1.

Proof of Theorem 1 (i)**.** Note that E_0 is symmetric with respect to 0 and by (4.11) the set

$$\bigcup_{m\in\mathbb{N}}\bigcup_{\mathbf{k}(m)\in\mathbb{Z}^m}\{\xi^{\pm}_{\mathbf{k}(m)}\}$$

is dense in E_0. According to [17, Lemma 1.3] and (2.1) it is enough to establish the following two inequalities:

$$y \le c_1|\xi + 1|^{1/2} \tag{5.10}$$

for any $0 < y < 1$ and $\xi = \phi_0(-1 + iy)$, and

$$y \le c_2|\xi - \xi^{\pm}_{\mathbf{k}(m)}|^{1/2} \tag{5.11}$$

for any $\xi \in (\xi^{-}_{\mathbf{k}(m)}, \xi^{+}_{\mathbf{k}(m)})$ satisfying $|\xi - \xi^{\pm}_{\mathbf{k}(m)}| \le |\xi_{\mathbf{k}(m)} - \xi^{\pm}_{\mathbf{k}(m)}|$ and $0 < y \le y_{\mathbf{k}(m)}$ such that $x_{\mathbf{k}(m)} + iy$ is the impression of the prime end $\psi_0(\xi)$.

First, we prove (5.10). Let $z_0 = 2i, w_0 = \phi_0(z_0), \Gamma = \Gamma(-1, -1 + iy; z_0, \infty; D_0)$. By Lemma 2

$$m(\Gamma) \le \frac{2}{\pi}\log\frac{1}{y} + c_3. \tag{5.12}$$

Also, for the family of crosscuts

$$\Gamma' = \phi_0(\Gamma) = \Gamma(-1, \xi; w_0, \infty; \mathbb{H})$$

by virtue of (4.3) we have

$$m(\Gamma') \geq \frac{1}{\pi} \log \frac{|w_0 + 1|}{|\xi + 1|}. \tag{5.13}$$

Comparing (5.12) and (5.13) we obtain (5.10).

Our next concern is the verification of (5.11). We prove it for the point $\xi^-_{\mathbf{k}(m)}$ (for $\xi^+_{\mathbf{k}(m)}$ the proof follows the same line of reasoning). Due to the symmetry of E_0 we can assume that $k(1) \leq 0$.

Let

$$\begin{aligned} Z &= \psi_0(\xi) = x_{\mathbf{k}(m)} + iy, \\ \Gamma &= \Gamma(X^-_{\mathbf{k}(m)}, Z; z_0, \infty; D_0), \\ \Gamma' &= \phi_0(\Gamma) = \Gamma(\xi^-_{\mathbf{k}(m)}, \xi; w_0, \infty; \mathbb{H}). \end{aligned}$$

According to (4.3)

$$m(\Gamma') \geq \frac{1}{\pi} \log \frac{|w_0 - \xi^-_{\mathbf{k}(m)}|}{|\xi - \xi^-_{\mathbf{k}(m)}|} \geq \frac{1}{\pi} \log \frac{1}{|\xi - \xi^-_{\mathbf{k}(m)}|} - c_4. \tag{5.14}$$

Our next objective is to estimate $m(\Gamma)$ from above. We begin with the following observation. Let $\Gamma_1 = \Gamma(X^-_{\mathbf{k}(m)}, Z; X^+_{\mathbf{k}(m)-1}, \infty; D_0)$. We claim that

$$m(\Gamma_1) \leq \frac{2}{\pi} \log^+ \frac{|I_{\mathbf{k}(m)}|}{y} + c_5. \tag{5.15}$$

Indeed, if $y \leq |I_{\mathbf{k}(m)}|$ then (5.15) follows from (5.6). If $y > |I_{\mathbf{k}(m)}|$ we note that the metric

$$\rho(\zeta) = \begin{cases} \dfrac{2}{|I_{\mathbf{k}(m)}|} & \text{if } |\zeta - x_{\mathbf{k}(m)}| \leq 2|I_{\mathbf{k}(m)}|,\ \operatorname{Im} \zeta \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

is admissible for Γ_1. Therefore, by (4.2)

$$m(\Gamma_1) \leq \int_{\mathbb{C}} \rho^2 \, dm = 8\pi,$$

which proves (5.15) in this case.

Since

$$\Gamma \subset \Gamma_1 \cup \Gamma(X^+_{\mathbf{k}(m)-1}, X^-_{\mathbf{k}(m)}; z_0, \infty; D_0) \cup \Gamma_2,$$

where $m(\Gamma_2) = 0$ (see (4.4)), by Lemma 5 and (5.15) we have

$$\begin{aligned} m(\Gamma) &\leq m(\Gamma_1) + m(\Gamma(X^+_{\mathbf{k}(m)-1}, X^-_{\mathbf{k}(m)}; z_0, \infty; D_0)) \\ &\leq \frac{2}{\pi} \log^+ \frac{|I_{\mathbf{k}(m)}|}{y} + \frac{2}{\pi} \log \frac{1}{|J_{\mathbf{k}(m)}|} + c_6 \qquad (5.16) \\ &\leq \frac{2}{\pi} \log \frac{1}{y} + c_6. \end{aligned}$$

Comparing (5.14) and (5.16) we have (5.11). This completes the proof of (i). ■

Proof of Theorem 1 (ii)**.** We begin with the following observation. Let for $m \in \mathbb{N}$,

$$\begin{aligned} U_{\mathbf{k}(m)} &= [\xi^+_{\mathbf{k}(m)-1}, \xi^-_{\mathbf{k}(m)}], \\ V_{\mathbf{k}(m)} &= (\xi_{\mathbf{k}(m)-1}, \xi_{\mathbf{k}(m)}), \\ \Gamma &= \Gamma(X^+_{\mathbf{k}(m)-1}, X^-_{\mathbf{k}(m)}; z_{\mathbf{k}(m)-1}, z_{\mathbf{k}(m)}; D_0), \\ \Gamma' &= \phi_0(\Gamma) = \Gamma(\xi^+_{\mathbf{k}(m)-1}, \xi^-_{\mathbf{k}(m)}; \xi_{\mathbf{k}(m)-1}, \xi_{\mathbf{k}(m)}; \mathbb{H}). \end{aligned}$$

By (4.10) and Remark 3 we have

$$\frac{|J_{\mathbf{k}(m)}|}{|I_{\mathbf{k}(m)}|} - 2 \le m(\Gamma) = m(\Gamma') \le \frac{1}{\pi} \log \frac{|V_{\mathbf{k}(m)}|}{|U_{\mathbf{k}(m)}|} + c_1,$$

that is,

$$\begin{aligned} \frac{|U_{\mathbf{k}(m)}|}{|V_{\mathbf{k}(m)}|} &\le c_2 \exp\left(-\pi \frac{|J_{\mathbf{k}(m)}|}{|I_{\mathbf{k}(m)}|}\right) = c_2 \exp\left(-\frac{1}{d_{\mathbf{k}(m)}} + \log \frac{1}{d_{\mathbf{k}(m)}}\right) \\ &= c_2 \frac{|J_{\mathbf{k}(m)}|}{d^2_{\mathbf{k}(m)}} \le c_3 |J_{\mathbf{k}(m)}| \left(\log \frac{1}{|J_{\mathbf{k}(m)}|}\right)^2. \end{aligned} \tag{5.17}$$

Note that

$$|J_{\mathbf{k}(m)}| \le c_4 \, |U_{\mathbf{k}(m)}|^{1/2}. \tag{5.18}$$

Indeed, consider the families of curves

$$\begin{aligned} \Gamma &= \Gamma(X^+_{\mathbf{k}(m)-1}, X^-_{\mathbf{k}(m)}; z_0, \infty; D_0), \\ \Gamma' &= \phi_0(\Gamma) = \Gamma(\xi^+_{\mathbf{k}(m)-1}, \xi^-_{\mathbf{k}(m)}; w_0, \infty; \mathbb{H}). \end{aligned}$$

By (4.3) and Lemma 5

$$\begin{aligned} \frac{1}{\pi} \log \frac{1}{|U_{\mathbf{k}(m)}|} - c_5 \le \frac{1}{\pi} \log \frac{|\xi^+_{\mathbf{k}(m)-1} - w_0|}{|\xi^+_{\mathbf{k}(m)-1} - \xi^-_{\mathbf{k}(m)}|} &\le m(\Gamma') \\ = m(\Gamma) &\le \frac{2}{\pi} \log \frac{1}{|J_{\mathbf{k}(m)}|} + c_6, \end{aligned}$$

from which (5.18) follows immediately.

According to (5.17) and (5.18) we obtain

$$|U_{\mathbf{k}(m)}|^{1/2} \left(\log \frac{1}{|U_{\mathbf{k}(m)}|}\right)^{-2} \le c_7 \, |V_{\mathbf{k}(m)}|. \tag{5.19}$$

Let $m \in \mathbb{N}$ be fixed. We consider the covering of E_0 defined by intervals and points

$$\{U_{\mathbf{k}(m)}\}_{\mathbf{k}(m) \in \mathbb{Z}^m}, \qquad \{\pm 1\}, \qquad \{\xi^\pm_{\mathbf{k}(j)}\}_{\substack{1 \le j \le m-1 \\ \mathbf{k}(j) \in \mathbb{Z}^j}}.$$

By (4.11)

$$\sup_{\mathbf{k}(m)\in\mathbb{Z}^m} |U_{\mathbf{k}(m)}| \to 0 \qquad \text{as } m \to \infty.$$

Note that $\{V_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbb{Z}^m}$ are disjoint subsets of $(-1,1)$. By (5.19) for any ε with $0<\varepsilon<1/2$,

$$\sum_{\mathbf{k}(m)\in\mathbb{Z}^m} |U_{\mathbf{k}(m)}|^{1/2+\varepsilon} \le c(\varepsilon) \sum_{\mathbf{k}(m)\in\mathbb{Z}^m} |V_{\mathbf{k}(m)}| \le 2c(\varepsilon),$$

which shows that $\dim E_0 \le 1/2$.

By the first part (i) of this theorem and (1.2) $\dim E_0 \ge 1/2$. Hence, $\dim E_0 = 1/2$. This completes the proof of part (ii). ■

6. Proof of Theorem 2

The proof of Theorem 2 is similar in spirit to the proof of Theorem 1. We briefly sketch the major features of this case and leave the details to the reader.

Proof of Theorem 2 (i)**.** It is known (cf. [4, p. 142]) that (1.3) is the same as

$$g_{\Omega_1}(-1-r) \le c_1\sqrt{r}, \qquad 0<r<1,$$

which, by (3.3), is equivalent to the estimate

$$y \le c_2|\xi+1|^{1/2} \tag{6.1}$$

for any $0<y<1$ and $\xi=\phi_1(-1+iy)$.

In order to prove (6.1) we consider the families of crosscuts

$$\Gamma = \Gamma(-1,z;z_0,\infty;D_1) \qquad \text{and} \qquad \Gamma' = \phi_1(\Gamma) = \Gamma(-1,\xi;w_0,\infty;\mathbb{H}),$$

where $z=-1+iy$, $z_0=2i$, $w_0=\phi_1(z_0)$, $\xi=\phi_1(z)$. For the module of Γ we have

$$m(\Gamma) \le \frac{2}{\pi}\log\frac{1}{y} + c_3 \tag{6.2}$$

(cf. Lemma 2).

Hence, by (4.3) and (6.2), we get

$$\frac{2}{\pi}\log\frac{1}{y} + c_3 \ge m(\Gamma) = m(\Gamma') \ge \frac{1}{\pi}\log\frac{|w_0+1|}{|\xi+1|} \ge \frac{1}{\pi}\log\frac{1}{|\xi+1|} - c_4,$$

from which (6.1) follows. ■

Proof of Theorem 2 (ii)**.** This part of the proof needs some preparatory work. Let $X^{\pm}_{\mathbf{k}(j)} = \psi_1(\xi^{\pm}_{\mathbf{k}(j)})$, $j\in\mathbb{N}$, that is $|X^{\pm}_{\mathbf{k}(j)}| = x_{\mathbf{k}(j)}$. For convenience, let

$$\xi^{+}_{\mathbf{k}(j-1),K(j)} = \xi^{+}_{\mathbf{k}(j-1)}, \quad \xi^{-}_{\mathbf{k}(j-1),0} = \xi^{-}_{\mathbf{k}(j-1)-1}, \quad \xi^{-}_{0} = 1,$$
$$\xi_{\mathbf{k}(j-1),K(j)} = \xi_{\mathbf{k}(j-1)}, \quad \xi_{\mathbf{k}(j-1),0} = \xi_{\mathbf{k}(j-1)-1}, \quad \xi_{0} = 2,$$

where $j > 1$. Let

$$U_{\mathbf{k}(j)} = [\xi^+_{\mathbf{k}(j)}, \xi^-_{\mathbf{k}(j)-1}] \qquad \text{and} \qquad V_{\mathbf{k}(j)} = (\xi_{\mathbf{k}(j)}, \xi_{\mathbf{k}(j)-1}),$$

and let, for convenience, $X^-_{\mathbf{k}(0)-1} = 1, X^+_{\mathbf{k}(0)} = -1$. From two prime ends $X^-_{\mathbf{k}(j-1)-1}$ and $X^+_{\mathbf{k}(j-1)}$ we choose the closest to the interval $I_{\mathbf{k}(j)}$ (in the sense of distance between $I_{\mathbf{k}(j)}$ and impressions of these two prime ends) and denote it by $\mathcal{Z}'_{\mathbf{k}(j)}$. Denote the other prime end by $\mathcal{Z}''_{\mathbf{k}(j)}$. Let

$$d_{\mathbf{k}(j)} = |x_{\mathbf{k}(j)} - |\mathcal{Z}'_{\mathbf{k}(j)}|| + |I_{\mathbf{k}(j)}|.$$

The analysis similar to that in the the proof of Theorem 1 shows that for $j \in \mathbb{N}$ the following three inequalities hold:

$$m(\Gamma(X^+_{\mathbf{k}(j)}, z_{\mathbf{k}(j),k(j+1)}; X^-_{\mathbf{k}(j)-1}, \infty; D_1)) \le \frac{2}{\pi} \log \frac{|I_{\mathbf{k}(j)}|}{d_{\mathbf{k}(j)}} + c_5 \tag{6.3}$$

(cf. Remark 1);

$$m(\Gamma(z_{\mathbf{k}(j)-1}, z_{\mathbf{k}(j)}; \mathcal{Z}'_{\mathbf{k}(j)}, \infty; D_1)) \le \frac{1}{\pi} \log \frac{d_{\mathbf{k}(j)}}{|I_{\mathbf{k}(j)}|} + c_6 \tag{6.4}$$

(cf. Remark 2);

$$m(\Gamma(X^+_{\mathbf{k}(j)}, X^-_{\mathbf{k}(j)-1}; z_{\mathbf{k}(j)}, \infty; D_1)) \le \frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|} + c_7 \tag{6.5}$$

(cf. Lemma 4).

Next, for $m \in \mathbb{N}$ and $2 \le k(j) \le K(j) - 1$, $1 \le j \le m$, consider the families of crosscuts

$$\begin{aligned} \Gamma &= \Gamma(z_{\mathbf{k}(m)-1}, z_{\mathbf{k}(m)}; z_0, \infty; D_1), \\ \Gamma' &= \phi_1(\Gamma) = \Gamma(\xi_{\mathbf{k}(m)-1}, \xi_{\mathbf{k}(m)}; w_0, \infty; \mathbb{H}). \end{aligned}$$

By (4.3),

$$m(\Gamma') \ge \frac{1}{\pi} \log \frac{|\xi_{\mathbf{k}(m)} - w_0|}{|\xi_{\mathbf{k}(m)} - \xi_{\mathbf{k}(m)-1}|} \ge \frac{1}{\pi} \log \frac{1}{|V_{\mathbf{k}(m)}|} - c_8. \tag{6.6}$$

Our next objective is to estimate $m(\Gamma)$ from above. Note that

$$\begin{aligned} \Gamma \subset \bigcup_{j=1}^{m} \Big(&\Gamma(z_{\mathbf{k}(j)-1}, z_{\mathbf{k}(j)}; \mathcal{Z}'_{\mathbf{k}(j)}, \infty; D_1) \cup \Gamma(z_{\mathbf{k}(j)}, \mathcal{Z}'_{\mathbf{k}(j)}; \mathcal{Z}''_{\mathbf{k}(j)}, \infty; D_1) \\ &\cup \Gamma(X^+_{\mathbf{k}(j-1)}, X^-_{\mathbf{k}(j-1)-1}; z_{\mathbf{k}(j-1)}, \infty; D_1) \Big) \cup \Gamma_1, \end{aligned}$$

where $z_{\mathbf{k}(0)} = z_0$,

$$m(\Gamma_1) = 0 \qquad \text{and} \qquad m(\Gamma(-1, 1; z_0, \infty; D_1)) \le c_9.$$

Therefore, by (6.3)–(6.5) we have

$$m(\Gamma) \le \sum_{j=1}^{m} \left(\frac{1}{\pi} \log \frac{d_{\mathbf{k}(j)}}{|I_{\mathbf{k}(j)}|} + \frac{2}{\pi} \log \frac{|I_{\mathbf{k}(j-1)}|}{d_{\mathbf{k}(j)}} + \frac{|J_{\mathbf{k}(j-1)}|}{|I_{\mathbf{k}(j-1)}|} + c_{10} \right), \tag{6.7}$$

where, for convenience, we let $|I_{\mathbf{k}(0)}| = |J_{\mathbf{k}(0)}| = 2$. Comparing (6.6) and (6.7) we obtain

$$|I_{\mathbf{k}(m)}|^2 \le c_{11} |I_{\mathbf{k}(m)}| d_{\mathbf{k}(m)} \le c_{12} (A_{\mathbf{k}(m-1)})^2 |V_{\mathbf{k}(m)}|.$$

Furthermore, for the families of crosscuts

$$\Gamma = \Gamma(X^+_{\mathbf{k}(m)}, X^-_{\mathbf{k}(m)-1}; z_{\mathbf{k}(m)}, z_{\mathbf{k}(m)-1}; D_1),$$
$$\Gamma' = \phi_1(\Gamma) = \Gamma(\xi^+_{\mathbf{k}(m)}, \xi^-_{\mathbf{k}(m)-1}; \xi_{\mathbf{k}(m)}, \xi_{\mathbf{k}(m)-1}; \mathbb{H})$$

we have

$$m(\Gamma) \ge \frac{1}{2} \frac{|J_{\mathbf{k}(m)}|}{|I_{\mathbf{k}(m)}|}$$

(cf. Remark 3) and

$$m(\Gamma') \le \frac{1}{\pi} \log \frac{|V_{\mathbf{k}(m)}|}{|U_{\mathbf{k}(m)}|} + c_{13},$$

the last inequality being a consequence of (4.10).

Thus, by (3.1) and (3.2) we obtain

$$|U_{\mathbf{k}(m)}| \le 2 \frac{|U_{\mathbf{k}(m)}|}{|V_{\mathbf{k}(m)}|} \le c_{14} \exp\left(-\frac{\pi}{2} \frac{|J_{\mathbf{k}(m)}|}{|I_{\mathbf{k}(m)}|} \right) \le c_{14} \exp\left(-\frac{\pi}{4} \frac{|J_{\mathbf{k}(m-1)}|}{\delta_{\mathbf{k}(m-1)}} \right)$$
$$\le c_{14} \left(\frac{\delta_{\mathbf{k}(m-1)}}{A_{\mathbf{k}(m-1)}} \right)^m \le {c_{15}}^m |V_{\mathbf{k}(m)}|^{m/2}.$$

Note that above inequality

$$|U_{\mathbf{k}(m)}| \le {c_{15}}^m |V_{\mathbf{k}(m)}|^{m/2}$$

is proved for the case $2 \le k(j) \le K(j) - 1$, $1 \le j \le m$. Reasoning in the same manner, we can extend this inequality also to the case of all values $\mathbf{k}(m)$ under consideration.

Now, for any fixed $m \in \mathbb{N}$ consider the covering of $E_1 \setminus \{-1\}$ by intervals $\{U_{\mathbf{k}(m)}\}$. By (4.11)

$$\sup_{\mathbf{k}(m)} |U_{\mathbf{k}(m)}| \to 0 \qquad \text{as } m \to \infty.$$

Note that $V_{\mathbf{k}(m)}$ are disjoint subsets of $(-1, 2)$. Let $0 < \varepsilon < 1$ be arbitrary. Taking $m > 2/\varepsilon$ we obtain

$$\sum_{\mathbf{k}(m)} |U_{\mathbf{k}(m)}|^\varepsilon \le c(\varepsilon) \sum_{\mathbf{k}(m)} |V_{\mathbf{k}(m)}| \le 3c(\varepsilon),$$

which shows that $\dim E_1 = 0$. ■

Acknowledgement. This research was supported by the Alexander von Humboldt foundation while the author was visiting the Katholische Universitätt Eichstätt. The author wishes to thank the members of this university for the pleasant mathematical atmosphere they offered him. We also thank M. Nesterenko for helping to improve this manuscript in many ways.

References

1. L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Princeton, NJ, Van Nostrand, 1966.
2. V. V. Andrievskii, The highest smoothness of the Green function implies the highest density of a set, *Arkiv för Matematik* **42** (2004), 217-238.
3. V. V. Andrievskii and H.-P. Blatt, *Discrepancy of Signed Measures and Polynomial Approximation*, Springer-Verlag, Berlin/New York, 2002.
4. A. Baernstein, Integral means, univalent functions and circular symmetrization, *Acta Math.* **133** (1974), 139–169.
5. L. Białas and A. Volberg, Markov's property of the Cantor ternary set, *Studia Math.* **104** (1993), 259–268.
6. L. Carleson and V. Totik, Hölder continuity of Green's functions, *Acta Sci. Math. (Szeged)* **70** (2004), 558–608.
7. J. A. Jenkins and K. Oikawa, On results of Ahlfors and Hayman, *Illinois J. Math.* **15** (1971), 664–671.
8. M. A. Lavrentiev, Sur la continuité des fonctions univalentes, *C. R.: Akad. Sci. USSR* **4** (1936), 215–217.
9. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd ed., Springer-Verlag, Berlin, 1973.
10. J. Lithner, Comparing two versions of Markov's property, *J. Approx. Theory* **77** (1994), 202–211.
11. V. G. Maz'ja, On the modulus of continuity of a harmonic function at a boundary point, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **135** (1984), 87–95.
12. W. Pleśniak, A Cantor regular set which does not have Markov's property, *Ann. Pol. Math.* **LI** (1990), 269–274.
13. Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag, Berlin/New York, 1992.
14. T. Ransford, *Potential Theory in the Complex plane*, Cambridge University Press, Cambridge, 1995.
15. E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag, New York/Berlin, 1997.
16. V. Totik, Markoff constants for Cantor sets, *Acta Sci Math. (Szeged)* **60** (1995), 715–734.
17. ———, On Markoff's inequality, *Constr. Approx.* **18** (2002), 427–441.
18. ———, Metric properties of harmonic measure, manuscript.

Vladimir V. Andrievskii E-MAIL: `andriyev@math.kent.edu`
ADDRESS: *Department of Mathematical Sciences, Kent State University, Kent, OH 44242, U.S.A.*