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Generalized Lipschitz Functions

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Abstract. In this note the class $\mathrm{Lip}_{\alpha(t)}$ of continuous functions is introduced. The definition is arranged so that for the constant function $\alpha(t) \equiv \alpha$, the class $\mathrm{Lip}_{\alpha(t)}$ is nothing but the classical Lipschitz space Lip_α. Then, to justify that our set of axioms for $\alpha(t)$ are properly chosen, some celebrated theorems of Privalov, Titchmarsh, Hardy and Littlewood about Lip_α functions are shown to be also valid for $\mathrm{Lip}_{\alpha(t)}$ functions.

Keywords. Lipschitz functions, test function, associated test function, Hilbert transform, Hardy classes.

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1. Introduction

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous function. The *modulus of continuity* of f is

$$\omega_f(t) := \sup_{x\in\mathbb{R},\, |y|\le t} |f(x+y) - f(x)|, \qquad t > 0.$$

The function ω_f is used to define different classes of continuous functions. For example, f is called Lip_α, for some α with $0 < \alpha \le 1$, if

$$\omega_f(t) = \mathcal{O}(t^\alpha) \qquad \text{as } t \to 0^+.$$

In this note, our main goal is to replace t^α by $t^{\alpha(t)}$, where $\alpha(t)$ is a function defined for small values of t and *properly* tends to α as $t \to 0^+$. To be more precise, we assume that $\alpha(t)$ is a real continuous function defined in a right neighborhood

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of zero, say $(0, t_0)$, and, as $t \to 0^+$, we have

$$\alpha(t) = \alpha + o(1), \qquad \alpha \in \mathbb{R}, \tag{1.1}$$

$$\int_0^t \tau^{\alpha(\tau)-\beta}\, d\tau = \frac{t^{\alpha(t)-\beta+1}}{\alpha+1-\beta} + o(t^{\alpha(t)-\beta+1}), \qquad \beta < \alpha+1, \tag{1.2}$$

$$\int_t^{t_0} \tau^{\alpha(\tau)-\beta}\, d\tau = \frac{t^{\alpha(t)-\beta+1}}{\beta-\alpha-1} + o(t^{\alpha(t)-\beta+1}), \qquad \beta > \alpha+1. \tag{1.3}$$

Let us call $\alpha(t)$ a *test function*. The space of all continuous functions $f\colon \mathbb{R} \to \mathbb{C}$ satisfying

$$\omega_f(t) = \mathcal{O}(t^{\alpha(t)}) \qquad \text{as } t \to 0^+$$

will be denoted by $\mathrm{Lip}_{\alpha(t)}$. Clearly the classical space Lip_α is a special case corresponding to the test function $\alpha(t) \equiv \alpha$.

Given a test function $\alpha(t)$, let

$$\hat{\alpha}(t) = \alpha + \frac{1}{\log t} \log\left(\int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau \right).$$

In Section 2, we show that $\hat{\alpha}(t)$ is also a test function. We call it the test function *associated* with $\alpha(t)$. The conditions (1.2) and (1.3) deal with cases $\beta < \alpha+1$ and $\beta > \alpha+1$. The associated test function is introduced to deal with the troublesome case $\beta = \alpha + 1$.

In Section 3, we provide a simple sufficient condition for a continuously differentiable function to be a test function. As a consequence we see that

$$\alpha(t) = \alpha - \alpha_1 \frac{\log_2 \frac{1}{t}}{\log \frac{1}{t}} - \alpha_2 \frac{\log_3 \frac{1}{t}}{\log \frac{1}{t}} - \cdots - \alpha_n \frac{\log_{n+1} \frac{1}{t}}{\log \frac{1}{t}}, \tag{1.4}$$

where $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and

$$\log_n := \overbrace{\log \log \cdots \log}^{n \text{ times}},$$

is a test function. For this test function, we have

$$t^{\alpha(t)} = t^\alpha \left(\log \frac{1}{t}\right)^{\alpha_1} \left(\log_2 \frac{1}{t}\right)^{\alpha_2} \cdots \left(\log_n \frac{1}{t}\right)^{\alpha_n}.$$

In Section 4, we generalize three classical theorems: a Theorem of Privalov about the Hilbert transform of bounded Lip_α functions with $0 < \alpha < 1$, a Theorem of Titchmarsh about the Hilbert transform of bounded Lip_1 functions, and a Theorem of Hardy-Littlewood about the boundary behavior of functions in the disc algebra. We show that these theorems remain valid if in the statement of theorem one replaces Lip_α by $\mathrm{Lip}_{\alpha(t)}$. Therefore, we hope that this section will convince our readers that the set of axioms (1.1), (1.2) and (1.3) are properly chosen.

2. The associated test function

The associated test function is defined so that

$$t^{\hat{\alpha}(t)} = t^{\alpha} \int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau.$$

Here we show that $\hat{\alpha}(t)$ is really a test function.

Lemma 2.1. *Let $\alpha(t)$ be a test function and let*

$$\hat{\alpha}(t) = \alpha + \frac{1}{\log t} \log\left(\int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau\right).$$

Then $\hat{\alpha}(t)$ is a test function.

Proof. Clearly, $\hat{\alpha}(t)$ is a continuous function on $(0, t_0)$. Our first task is to show that $\lim_{t\to 0^+} \hat{\alpha}(t) = \alpha$. For any $\varepsilon > 0$, by (1.3), we have

$$\begin{aligned} t^{\hat{\alpha}(t)-\alpha} &= \int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau = \int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1-\varepsilon}\tau^{\varepsilon}\, d\tau \\ &\geq t^{\varepsilon} \int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1-\varepsilon}\, d\tau = t^{\varepsilon}\left(\frac{t^{\alpha(t)-\alpha-\varepsilon}}{\varepsilon} + o(t^{\alpha(t)-\alpha-\varepsilon})\right). \end{aligned}$$

Hence $\liminf_{t\to 0^+} t^{\hat{\alpha}(t)-\alpha(t)} \geq 1/\varepsilon$. When ε tends to zero we get

$$\lim_{t\to 0^+} t^{\hat{\alpha}(t)-\alpha(t)} = \infty. \tag{2.1}$$

Let $\lambda \in \mathbb{R}$ and let

$$\varphi_\lambda(t) = t^{\hat{\alpha}(t)-\alpha+\lambda}.$$

Then, we have

$$\varphi_\lambda'(t) = t^{\lambda-1-\alpha+\alpha(t)}(\lambda t^{\hat{\alpha}(t)-\alpha(t)} - 1).$$

If $\lambda \leq 0$, then $\varphi_\lambda' < 0$ and thus φ_λ is strictly decreasing. On the other hand, if $\lambda > 0$, by (2.1), $\varphi_\lambda'(t) > 0$ for small values of t and thus φ_λ is strictly increasing. Fix $\varepsilon > 0$. For small enough τ, say $0 < \tau < \tau_0 < \min\{1, t_0\}$, we have

$$\alpha - \varepsilon \leq \alpha(\tau) \leq \alpha + \varepsilon.$$

Hence

$$\tau^{\varepsilon-1} \leq \tau^{\alpha(\tau)-\alpha-1} \leq \tau^{-\varepsilon-1}.$$

For $0 < t < \tau_0$, we thus get

$$\frac{\tau_0^{\varepsilon}}{\varepsilon} - \frac{t^{\varepsilon}}{\varepsilon} = \int_t^{\tau_0} \tau^{\varepsilon-1}\, d\tau \leq \int_t^{\tau_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau \leq \int_t^{\tau_0} \tau^{-\varepsilon-1}\, d\tau = \frac{t^{-\varepsilon}}{\varepsilon} - \frac{\tau_0^{-\varepsilon}}{\varepsilon}.$$

Therefore, for $0 < t < \tau_0$,

$$\frac{1}{\log t} \log\left(\frac{t^{-\varepsilon}}{\varepsilon} - \frac{\tau_0^{-\varepsilon}}{\varepsilon}\right) \leq \frac{1}{\log t} \log\left(\int_t^{\tau_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau\right) \leq \frac{1}{\log t} \log\left(\frac{\tau_0^{\varepsilon}}{\varepsilon} - \frac{t^{\varepsilon}}{\varepsilon}\right),$$

which implies

$$-\varepsilon \le \liminf_{t\to 0^+} \frac{1}{\log t} \log\left(\int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau \right)$$
$$\le \limsup_{t\to 0^+} \frac{1}{\log t} \log\left(\int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau \right) \le 0.$$

Now we, let $\varepsilon \to 0$ to get

$$\lim_{t\to 0^+} \frac{1}{\log t} \log\left(\int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau \right) = 0.$$

Hence,

$$\lim_{t\to 0^+} \hat{\alpha}(t) = \alpha.$$

Let $\beta < \alpha + 1$. Then, for (1.2),

$$\int_0^t \tau^{\hat{\alpha}(\tau)-\beta}\, d\tau = \int_0^t \tau^{\alpha-\beta}\tau^{\hat{\alpha}(\tau)-\alpha}\, d\tau \ge t^{\hat{\alpha}(t)-\alpha} \int_0^t \tau^{\alpha-\beta}\, d\tau = \frac{t^{\hat{\alpha}(t)-\beta+1}}{\alpha-\beta+1}.$$

On the other hand, for $0 < \varepsilon < 1 + \alpha - \beta$, we have

$$\int_0^t \tau^{\hat{\alpha}(\tau)-\beta}\, d\tau = \int_0^t \tau^{\alpha-\beta-\varepsilon}\tau^{\hat{\alpha}(\tau)-\alpha+\varepsilon}\, d\tau \le t^{\hat{\alpha}(t)-\alpha+\varepsilon} \int_0^t \tau^{\alpha-\beta-\varepsilon}\, d\tau$$
$$= \frac{t^{\hat{\alpha}(t)-\beta+1}}{\alpha-\beta+1-\varepsilon}.$$

Hence

$$\int_0^t \tau^{\hat{\alpha}(\tau)-\beta}\, d\tau = \frac{t^{\hat{\alpha}(t)-\beta+1}}{\alpha-\beta+1} + o(t^{\hat{\alpha}(t)-\beta+1}).$$

Finally, let $\beta > \alpha + 1$. Then, for (1.3),

$$\int_t^{t_0} \tau^{\hat{\alpha}(\tau)-\beta}\, d\tau = \int_t^{t_0} \tau^{\alpha-\beta}\tau^{\hat{\alpha}(\tau)-\alpha}\, d\tau \le t^{\hat{\alpha}(t)-\alpha} \int_t^{t_0} \tau^{\alpha-\beta}\, d\tau \le \frac{t^{\hat{\alpha}(t)-\beta+1}}{\beta-\alpha-1}.$$

On the other hand, for any $\varepsilon > 0$, we have

$$\int_t^{\tau_0} \tau^{\hat{\alpha}(\tau)-\beta}\, d\tau = \int_t^{\tau_0} \tau^{\alpha-\beta-\varepsilon}\tau^{\hat{\alpha}(\tau)-\alpha+\varepsilon}\, d\tau \ge t^{\hat{\alpha}(t)-\alpha+\varepsilon} \int_t^{\tau_0} \tau^{\alpha-\beta-\varepsilon}\, d\tau$$
$$= \frac{t^{\hat{\alpha}(t)-\beta+1}}{\beta-\alpha-1+\varepsilon} \left(1 - \left(\frac{t}{\tau_0}\right)^{\beta+\varepsilon-\alpha-1} \right).$$

Hence

$$\int_t^{t_0} \tau^{\hat{\alpha}(\tau)-\beta}\, d\tau = \frac{t^{\hat{\alpha}(t)-\beta+1}}{\beta-\alpha-1} + o(t^{\hat{\alpha}(t)-\beta+1}).$$

■

Using similar calculations, we can show that

$$\alpha(t) + \frac{1}{\log t} \log\left(\int_t^{t_0} \tau^{\alpha(\tau)-\alpha-1}\, d\tau \right)$$

is also a test function. But, we will not use this result in what follows.

The following simple result will be used several times in what follows. Let us mention that $f(t) \asymp g(t)$, as $t \to 0^+$, means that there are constants $c, C > 0$ such that

$$cg(t) \leq f(t) \leq Cg(t)$$

in a right neighborhood of zero.

Corollary 2.2. *Let $\alpha(t)$ be a test function with $\lim_{t\to 0^+} \alpha(t) = \alpha$. Let β, γ, C be real constants such that $C > 0$, $\gamma > 0$ and $\beta - 1 < \alpha$. Then,*

$$\int_0^{t_0} \frac{\tau^{\alpha(\tau)-\beta}}{\tau^\gamma + Ct^\gamma}\, d\tau \asymp \begin{cases} t^{\alpha(t)+1-\beta-\gamma} & \text{if } \alpha < \gamma+\beta-1, \\ t^{\hat{\alpha}(t)-\alpha} & \text{if } \alpha = \gamma+\beta-1, \\ 1 & \text{if } \alpha > \gamma+\beta-1, \end{cases} \qquad \text{as } t \to 0^+.$$

Proof. We decompose the integral over two intervals $(0, t)$ and (t, t_0) and then we use (1.2) and (1.3). Hence

$$\begin{aligned} \int_0^{t_0} \frac{\tau^{\alpha(\tau)-\beta}}{\tau^\gamma + Ct^\gamma}\, d\tau &= \left(\int_0^t + \int_t^{t_0} \right) \frac{\tau^{\alpha(\tau)-\beta}}{\tau^\gamma + Ct^\gamma}\, d\tau \\ &\asymp \int_0^t \frac{\tau^{\alpha(\tau)-\beta}}{t^\gamma}\, d\tau + \int_t^{t_0} \frac{\tau^{\alpha(\tau)-\beta}}{\tau^\gamma}\, d\tau \\ &\asymp \frac{t^{\alpha(\tau)-\beta+1}}{t^\gamma} + t^{\alpha(\tau)-\beta-\gamma+1}. \end{aligned}$$

The first estimate is true since $\alpha > \beta - 1$, and the second one holds since $\alpha < \beta + \gamma - 1$. Similarly, by (1.2) and (2.1), we have

$$\begin{aligned} \int_0^{t_0} \frac{\tau^{\alpha(\tau)-\alpha-1+\gamma}}{\tau^\gamma + Ct^\gamma}\, d\tau &= \left(\int_0^t + \int_t^{t_0} \right) \frac{\tau^{\alpha(\tau)-\alpha-1+\gamma}}{\tau^\gamma + Ct^\gamma}\, d\tau \\ &\asymp \int_0^t \frac{\tau^{\alpha(\tau)-\alpha-1+\gamma}}{t^\gamma}\, d\tau + \int_t^{t_0} \frac{\tau^{\alpha(\tau)-\alpha-1+\gamma}}{\tau^\gamma}\, d\tau \\ &\asymp t^{\alpha(t)-\alpha} + t^{\hat{\alpha}(t)-\alpha} \asymp t^{\hat{\alpha}(t)-\alpha}. \end{aligned}$$

The last case is proved similarly. ∎

3. A sufficient condition

In this section we give a simple and sufficient criterion verifying that the function $\alpha(t)$ defined in (1.4) is a test function.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ and let $\alpha(t)$ be a real continuously differentiable function defined on $(0, t_0)$ with $\lim_{t\to 0^+} \alpha(t) = \alpha$. Suppose that*

$$\lim_{t\to 0^+} \alpha'(t)\, t \log t = 0.$$

Then $\alpha(t)$ is a test function.

Proof. Since

$$\frac{d(t^{\alpha(t)-\gamma})}{dt} = (\alpha(t) - \gamma + \alpha'(t)\, t \log t) t^{\alpha(t)-\gamma-1},$$

then

$$\alpha(t) - \gamma + \alpha'(t)\, t \log t \to \alpha - \gamma \qquad \text{as } t \to 0^+.$$

Hence, for $\beta < \alpha + 1$, we have

$$\begin{aligned}\int_0^t \tau^{\alpha(\tau)-\beta}\, d\tau &= \int_0^t \tau^{\alpha(\tau)-\alpha} \tau^{\alpha-\beta}\, d\tau \\ &= \frac{\tau^{\alpha(\tau)-\beta+1}}{\alpha-\beta+1}\bigg|_{\tau=0}^{\tau=t} \\ &\quad - \frac{1}{\alpha-\beta+1} \int_0^t (\alpha(\tau) - \alpha + \alpha'(\tau)\, \tau \log \tau) \tau^{\alpha(\tau)-\beta}\, d\tau \\ &= \frac{t^{\alpha(t)-\beta+1}}{\alpha-\beta+1} + o(1) \int_0^t \tau^{\alpha(\tau)-\beta}\, d\tau.\end{aligned}$$

Therefore,

$$(1 + o(1)) \int_0^t \tau^{\alpha(\tau)-\beta}\, d\tau = \frac{t^{\alpha(t)-\beta+1}}{\alpha-\beta+1}.$$

Thus (1.2) holds. For $\beta > \alpha + 1$, we have

$$\begin{aligned}\int_t^{t_0} \tau^{\alpha(\tau)-\beta}\, d\tau &= \int_t^{t_0} \tau^{\alpha(\tau)-\alpha} \tau^{\alpha-\beta}\, d\tau \\ &= \frac{\tau^{\alpha(\tau)-\beta+1}}{\alpha-\beta+1}\bigg|_{\tau=t}^{\tau=t_0} \\ &\quad - \int_t^{t_0} (\alpha(\tau) - \alpha + \alpha'(\tau)\, \tau \log \tau) \tau^{\alpha(\tau)-\beta}\, d\tau \\ &= \mathcal{O}(1) - \frac{t^{\alpha(t)-\beta+1}}{\alpha-\beta+1} + o(1) \int_t^{t_0} \tau^{\alpha(\tau)-\beta}\, d\tau.\end{aligned}$$

Hence

$$(1 + o(1)) \int_t^{t_0} \tau^{\alpha(\tau)-\beta}\, d\tau = \frac{t^{\alpha(t)-\beta+1}}{\beta-\alpha-1}.$$

Thus (1.3) holds. ∎

Corollary 3.2. *Let $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n$ be real constants, and let*

$$\alpha(t) = \alpha - \alpha_1 \frac{\log_2 \frac{1}{t}}{\log \frac{1}{t}} - \alpha_2 \frac{\log_3 \frac{1}{t}}{\log \frac{1}{t}} - \cdots - \alpha_n \frac{\log_{n+1} \frac{1}{t}}{\log \frac{1}{t}}.$$

Then $\alpha(t)$ is a test function. Moreover,

$$t^{\hat{\alpha}(t)} \asymp \begin{cases} t^{\alpha} & \text{if } \alpha_1 < -1, \\ t^{\alpha} \left(\log \frac{1}{t}\right)^{1+\alpha_1} \left(\log_2 \frac{1}{t}\right)^{\alpha_2} \cdots \left(\log_n \frac{1}{t}\right)^{\alpha_n} & \text{if } \alpha_1 > -1. \end{cases}$$

If $\alpha_1 = \alpha_2 = \ldots = \alpha_{k-1} = -1$, then

$$t^{\hat{\alpha}(t)} \asymp \begin{cases} t^{\alpha} & \text{if } \alpha_k < -1, \\ t^{\alpha} \left(\log_k \frac{1}{t}\right)^{1+\alpha_k} \left(\log_{k+1} \frac{1}{t}\right)^{\alpha_{k+1}} \cdots \left(\log_n \frac{1}{t}\right)^{\alpha_n} & \text{if } \alpha_k > -1. \end{cases}$$

Finally, if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = -1$, then

$$t^{\hat{\alpha}(t)} \asymp t^{\alpha} \log_{n+1} \tfrac{1}{t}.$$

4. Applications

In this section, we generalize a theorem of Privalov and a theorem of Titchmarsh about the Hilbert transform of bounded Lip_α functions, and a theorem of Hardy-Littlewood about the boundary behavior of functions in the disc algebra.

4.1. Hilbert transform on $\mathbb{R}$. Let $u\colon \mathbb{R} \to \mathbb{R}$ be in $L^1(dt/(1+t^2))$. Then the integral

$$\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2} \right) u(t)\, dt$$

converges uniformly on compact subsets of the upper half plane and thus it defines an analytic function there. Hence its real and imaginary parts

$$U(z) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} z}{|z-t|^2} u(t)\, dt,$$

$$\widetilde{U}(z) := \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\operatorname{Re} z - t}{|z-t|^2} + \frac{t}{1+t^2} \right) u(t)\, dt \tag{4.1}$$

are a pair of harmonic conjugates in the upper half plane. According to the well known *approximate identity* property of the Poisson kernel $\operatorname{Im} z/\pi|z-t|^2$, the non-tangential limit of U exists at almost every $x \in \mathbb{R}$ and is equal to $u(x)$. Moreover, the non-tangential limit of $\widetilde{U}$ also exists almost everywhere on $\mathbb{R}$. This fact is a deep and fundamental result in the theory of functions. This limit, wherever it exists, is denoted by $\tilde{u}$ and is called the *Hilbert transform* of u; $\tilde{u}$ can also be found by the singular integral

$$\tilde{u}(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) u(t)\, dt. \tag{4.2}$$

This subject has been comprehensively studied in [1] and [5]. It is also briefly discussed in [4] and [6]. There are two celebrated results about the Hilbert transform of bounded Lipschitz functions.

Theorem (Privalov [7]). *If u is a bounded Lip_α, $0<\alpha<1$, function, then $\tilde{u}(x)$ exists for all $x \in \mathbb{R}$, and besides $\tilde{u}$ is also Lip_α.*

Theorem (Titchmarsh [8]). *If u is a bounded Lip_1 function, then $\tilde{u}(x)$ exists for all $x \in \mathbb{R}$, and besides*

$$|\tilde{u}(x+t)-\tilde{u}(x)| \le \mathrm{const}\, t \log \tfrac{1}{t}$$

for all $x \in \mathbb{R}$ and for small values of $t>0$.

If $\alpha(t) \equiv 1$, then, by Corollary 3.2, $t^{\hat{\alpha}(t)} \asymp t \log(1/t)$. Therefore, Titchmarsh's Theorem can be rephrased in the following way: *If u is a bounded Lip_1 function, then $\tilde{u}(x)$ exists for all $x \in \mathbb{R}$, and besides it is $\mathrm{Lip}_{\hat{1}}$.* Now, we may generalize these two theorems in the following way.

Theorem 4.1. *Let $\alpha(t)$ be a test function with $\lim_{t\to 0^+}\alpha(t) \le 1$. Let u be a bounded $\mathrm{Lip}_{\alpha(t)}$ function on $\mathbb{R}$. Then $\tilde{u}(x)$ exists for all $x \in \mathbb{R}$, and furthermore*

$$\tilde{u} \in \begin{cases} \mathrm{Lip}_{\alpha(t)} & \text{if } \lim_{t\to 0^+}\alpha(t) < 1, \\ \mathrm{Lip}_{\hat{\alpha}(t)} & \text{if } \lim_{t\to 0^+}\alpha(t) = 1. \end{cases}$$

Proof. To estimate $|\tilde{u}(x+t)-\tilde{u}(x)|$, instead of taking the real line as our straight path to go from x to $x+t$, we go from x up to $x+it$, then to $x+t+it$ and finally down to $x+t$:

$$\begin{aligned} |\tilde{u}(x)-\tilde{u}(x+t)| \le\ & |\tilde{u}(x)-\widetilde{U}(x+it)| \\ & +|\widetilde{U}(x+it)-\widetilde{U}(x+t+it)| \\ & +|\widetilde{U}(x+t+it)-\tilde{u}(x+t)|. \end{aligned} \tag{4.3}$$

Hence we proceed to study each term of the right side.

By (4.1) and (4.2), we have

$$\begin{aligned} |\tilde{u}(x)-\widetilde{U}(x+it)| &= \Bigg| \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-\tau|>\varepsilon}\left(\frac{1}{x-\tau}+\frac{\tau}{1+\tau^2}\right)u(\tau)\,d\tau \\ &\quad -\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{x-\tau}{(x-\tau)^2+t^2}+\frac{\tau}{1+\tau^2}\right)u(\tau)\,d\tau\Bigg| \\ &= \left|\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-\tau|>\varepsilon}\left(\frac{1}{x-\tau}-\frac{x-\tau}{(x-\tau)^2+t^2}\right)u(\tau)\,d\tau\right| \\ &= \left|\lim_{\varepsilon\to 0}\frac{t^2}{\pi}\int_{\varepsilon}^{\infty}\frac{u(x-\tau)-u(x+\tau)}{\tau(\tau^2+t^2)}\,d\tau\right| \\ &\le \frac{t^2}{\pi}\int_0^{\infty}\frac{|u(x+\tau)-u(x-\tau)|}{\tau(\tau^2+t^2)}\,d\tau. \end{aligned}$$

Since u is *bounded* and $\mathrm{Lip}_{\alpha(t)}$, there is a constant C such that

$$|u(x+\tau)-u(x-\tau)| \le C(2\tau)^{\alpha(2\tau)}$$

for *all* values of $x \in \mathbb{R}$, small values of τ, say $0<\tau<1$. Hence, by Corollary 2.2, we have

$$\begin{aligned}|\tilde{u}(x)-\widetilde{U}(x+it)| &\le \frac{t^2}{\pi}\int_0^1 \frac{C(2\tau)^{\alpha(2\tau)}}{\tau(\tau^2+t^2)}\,d\tau + \frac{t^2}{\pi}\int_1^\infty \frac{2\|u\|_\infty}{\tau^3}\,d\tau\\ &\le \frac{4Ct^2}{\pi}\int_0^2 \frac{\tau^{\alpha(\tau)-1}}{\tau^2+4t^2}\,d\tau + \frac{\|u\|_\infty}{\pi}t^2\\ &\le C't^{\alpha(t)}+C''t^2.\end{aligned}$$

Therefore, for any $x\in\mathbb{R}$ and for any test function $\alpha(t)$ with $\lim_{t\to 0^+}\alpha(t)\le 1$, we have

$$|\tilde{u}(x)-\widetilde{U}(x+it)| = \mathcal{O}(t^{\alpha(t)}). \tag{4.4}$$

Incidentally, the preceding calculation also shows that $\tilde{u}(x)$ exists for all $x\in\mathbb{R}$.

By the Mean Value Theorem,

$$|\widetilde{U}(x+it)-\widetilde{U}(x+t+it)| \le t\sup_{s\in\mathbb{R}}\left|\frac{\partial\widetilde{U}}{\partial x}(s+it)\right|,$$

where

$$\begin{aligned}\frac{\partial\widetilde{U}}{\partial x}(s+it) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\frac{x-\tau}{(x-\tau)^2+t^2}+\frac{\tau}{1+\tau^2}\right)\Bigg|_{x=s}u(\tau)\,d\tau\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t^2-(s-\tau)^2}{(t^2+(s-\tau)^2)^2}u(\tau)\,d\tau\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t^2-\tau^2}{(t^2+\tau^2)^2}u(s+\tau)\,d\tau\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t^2-\tau^2}{(t^2+\tau^2)^2}(u(s+\tau)-u(s))\,d\tau.\end{aligned}$$

Hence

$$\begin{aligned}|\widetilde{U}(x+it)-\widetilde{U}(x+t+it)| &\le \sup_{s\in\mathbb{R}}\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{|u(s+\tau)-u(s)|}{t^2+\tau^2}\,d\tau\\ &\le Ct\int_0^1\frac{\tau^{\alpha(\tau)}}{t^2+\tau^2}\,d\tau + \frac{t}{\pi}\int_1^\infty\frac{2\|u\|_\infty}{\tau^2}\,d\tau.\end{aligned}$$

The asymptotic behavior of

$$\int_0^1 \frac{\tau^{\alpha(\tau)}}{t^2+\tau^2}\,d\tau$$

depends on $\lim_{t\to 0^+}\alpha(t)$. According to Corollary 2.2, we have

$$(4.5)\qquad |\widetilde{U}(x+it)-\widetilde{U}(x+t+it)| = \begin{cases} \mathcal{O}(t^{\alpha(t)}) & \text{if } \lim_{t\to 0^+}\alpha(t)<1, \\ \mathcal{O}(t^{\hat{\alpha}(t)}) & \text{if } \lim_{t\to 0^+}\alpha(t)=1. \end{cases}$$

Finally, (4.3), (4.4) and (4.5) give the required result. ■

In the light of Corollary 3.2 and Theorem 4.1, we get the following special result.

Corollary 4.2. *Let $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{R}$. Let u be a bounded function on $\mathbb{R}$ such that*

$$|u(x+t)-u(x)| \le Ct\left(\log\tfrac{1}{t}\right)^{\alpha_1}\left(\log_2\tfrac{1}{t}\right)^{\alpha_2}\cdots\left(\log_n\tfrac{1}{t}\right)^{\alpha_n}$$

for all $x\in\mathbb{R}$ and for all $t\in(0,t_0)$, where t_0 is a small enough positive constant. Then, $\tilde{u}(x)$ exists for all values of $x\in\mathbb{R}$ and besides, for all $x\in\mathbb{R}$ and for all $t\in(0,t_0)$, we have

$$|\tilde{u}(x+t)-\tilde{u}(x)| \le \begin{cases} Ct & \textit{if } \alpha_1<-1, \\ Ct\left(\log\tfrac{1}{t}\right)^{1+\alpha_1}\left(\log_2\tfrac{1}{t}\right)^{\alpha_2}\cdots\left(\log_n\tfrac{1}{t}\right)^{\alpha_n} & \textit{if } \alpha_1>-1; \end{cases}$$

if $\alpha_1=\alpha_2=\ldots=\alpha_{k-1}=-1$, then

$$|\tilde{u}(x+t)-\tilde{u}(x)| \le \begin{cases} Ct & \textit{if } \alpha_k<-1, \\ Ct\left(\log_k\tfrac{1}{t}\right)^{1+\alpha_k}\left(\log_{k+1}\tfrac{1}{t}\right)^{\alpha_{k+1}}\cdots\left(\log_n\tfrac{1}{t}\right)^{\alpha_n} & \textit{if } \alpha_k>-1; \end{cases}$$

finally, if $\alpha_1=\alpha_2=\ldots=\alpha_n=-1$, then

$$|\tilde{u}(x+t)-\tilde{u}(x)| \le Ct\log_{n+1}\tfrac{1}{t}.$$

Now, we give an example to show that Corollary 4.2 gives a *sharp* result. Let

$$u(t) = \begin{cases} |t|\log\frac{1}{|t|} & \text{if } |t|\le\frac{1}{e}, \\ \frac{1}{e} & \text{if } |t|\ge\frac{1}{e}, \end{cases}$$

and let

$$\alpha(t) = 1-\frac{\log_2\frac{1}{t}}{\log\frac{1}{t}}, \qquad 0<t<\frac{1}{e}.$$

Hence,

$$t^{\alpha(t)} = t\log\frac{1}{t},$$

and u is a bounded $\mathrm{Lip}_{\alpha(t)}$ function on $\mathbb{R}$. We show that $\tilde{u}(t)-\tilde{u}(0)$ behaves asymptotically like $t^{\hat{\alpha}(t)}$ as $t\to 0^+$. Indeed, if we write

$$\tilde{u}(t)-\tilde{u}(0) = \left(\tilde{u}(t)-\widetilde{U}(t+it)\right)+\left(\widetilde{U}(t+it)-\widetilde{U}(it)\right)+\left(\widetilde{U}(it)-\tilde{u}(0)\right),$$

then, by (4.4), we have

$$|\tilde{u}(t)-\widetilde{U}(t+it)| = \mathcal{O}(t^{\alpha(t)}), \qquad |\widetilde{U}(it)-\tilde{u}(0)| = \mathcal{O}(t^{\alpha(t)}) \qquad \text{as } t\to 0^+.$$

On the other hand, by the Mean Value Theorem,

$$\widetilde{U}(t+it)-\widetilde{U}(it)=\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{t^2-\tau^2}{(t^2+\tau^2)^2}u(\eta+\tau)\,d\tau$$

for some η in $(0,t)$. First of all, we have

$$\left|\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{t^2-\tau^2}{(t^2+\tau^2)^2}u(\eta+\tau)\,d\tau-\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{t^2-\tau^2}{(t^2+\tau^2)^2}u(\tau)\,d\tau\right|$$
$$\leq\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{\omega_u(\eta)}{t^2+\tau^2}\,d\tau\leq\omega_u(t)\leq Ct^{\alpha(t)}.$$

Secondly, if we write

$$\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{t^2-\tau^2}{(t^2+\tau^2)^2}u(\tau)\,d\tau=\frac{4t^3}{\pi}\int_{0}^{\infty}\frac{u(\tau)}{(t^2+\tau^2)^2}\,d\tau-\frac{2t}{\pi}\int_{0}^{\infty}\frac{u(\tau)}{t^2+\tau^2}\,d\tau$$

as in the proof of (4.4), we have

$$\frac{4t^3}{\pi}\int_{0}^{\infty}\frac{u(\tau)}{(t^2+\tau^2)^2}\,d\tau=\mathcal{O}(t^{\alpha(t)}).$$

Hence

$$\tilde{u}(t)-\tilde{u}(0)=\mathcal{O}(t^{\alpha(t)})-\frac{2t}{\pi}\int_{0}^{\infty}\frac{u(\tau)}{t^2+\tau^2}\,d\tau.$$

Moreover, by Corollary 2.2,

$$\frac{2t}{\pi}\int_{0}^{\infty}\frac{u(\tau)}{t^2+\tau^2}\,d\tau=\frac{2t}{\pi}\int_{0}^{t_0}\frac{\tau^{\alpha(\tau)}}{t^2+\tau^2}\,d\tau+\frac{2t}{\pi}\int_{t_0}^{\infty}\frac{t_0^{\alpha(t_0)}}{t^2+\tau^2}\,d\tau\asymp t^{\hat{\alpha}(t)}\asymp t\left(\log\tfrac{1}{t}\right)^2.$$

Therefore, by (2.1),

$$\tilde{u}(0)-\tilde{u}(t)\asymp t\left(\log\tfrac{1}{t}\right)^2.$$

4.2. Boundary values of analytic functions on the unit disc. The disc algebra consists of all analytic functions on the unit disc $\mathbb{D}$ which have a continuous extension on $\bar{\mathbb{D}}$. Hardy and Littlewood characterized the subspace of the disc algebra consisting of all functions whose restriction to the unit circle is a Lip_α function.

Theorem (Hardy-Littlewood [2, 3])**.** *Let f be analytic on $\mathbb{D}$. Then f is continuous on $\bar{\mathbb{D}}$ and is Lip_α on $\mathbb{T}$ if and only if $f'(re^{i\theta})=\mathcal{O}((1-r)^{\alpha-1})$ as $r\to 1^-$.*

There is a bijection between the family of 2π periodic functions on $\mathbb{R}$ and the class of all functions on $\mathbb{T}$. Hence, the notion Lip $\alpha(t)$ is well defined on $\mathbb{T}$ too. Now, we can generalize the Hardy-Littlewood Theorem in the following way.

Theorem 4.3. *Let $\alpha(t)$ be a test function with $\lim_{t\to 0^+}\alpha(t)>0$, and let f be analytic on $\mathbb{D}$. Then f is continuous on $\bar{\mathbb{D}}$ and is $\mathrm{Lip}_\alpha(t)$ on $\mathbb{T}$ if and only if $f'(re^{i\theta})=\mathcal{O}((1-r)^{\alpha(1-r)-1})$ as $r\to 1^-$.*

Proof. Suppose that $f'(re^{i\theta}) = \mathcal{O}((1-r)^{\alpha(1-r)-1})$ as $r \to 1^-$. According to (1.1), this assumption ensures that the limit

$$\lim_{R\to 1^-} \int_\rho^R f'(re^{i\theta})e^{i\theta}\,dr$$

exists and is finite at all points $e^{i\theta} \in \mathbb{T}$. Hence, the radial limit

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

exists at *all* points of $e^{i\theta} \in \mathbb{T}$, and

$$f(e^{i\theta}) - f(\rho e^{i\theta}) = \int_\rho^1 f'(re^{i\theta})e^{i\theta}\,dr.$$

Thus, by (1.2),

$$\begin{aligned} |f(e^{i\theta}) - f(\rho e^{i\theta})| &\le \int_\rho^1 |f'(re^{i\theta})|\,dr \le C\int_\rho^1 (1-r)^{\alpha(1-r)-1}\,dr \\ &= C\int_0^{1-\rho} \tau^{\alpha(\tau)-1}\,d\tau \le C(1-\rho)^{\alpha(1-\rho)}. \end{aligned} \tag{4.6}$$

Moreover, we have (cf. Figure 1)

$$f(e^{i\varphi}) - f(e^{i\theta}) = \int_\Gamma f'(\zeta)\,d\zeta$$

for all θ and φ with $0 < \varphi - \theta < 1$ and $\rho = 1 - (\varphi - \theta)$.

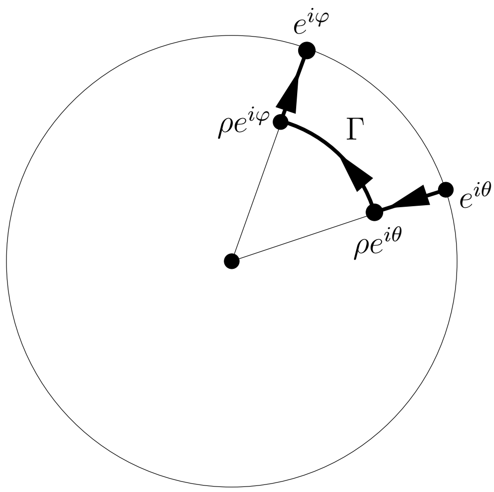

Figure 1.

Therefore, by (1.2),

$$
\begin{aligned}
|f(e^{i\varphi}) - f(e^{i\theta})| &\le \int_\rho^1 |f'(re^{i\theta})|\, dr + \int_\theta^\varphi |f'(\rho e^{it})|\, \rho dt + \int_\rho^1 |f'(re^{i\varphi})|\, dr \\
&\le 2C \int_\rho^1 (1-r)^{\alpha(1-r)-1}\, dr + C|\varphi-\theta|(1-\rho)^{\alpha(1-\rho)-1} \\
&= 2C \int_0^{|\varphi-\theta|} \tau^{\alpha(\tau)-1}\, d\tau + C|\varphi-\theta|^{\alpha(|\varphi-\theta|)} \\
&\le C|\varphi-\theta|^{\alpha(|\varphi-\theta|)}.
\end{aligned}
\tag{4.7}
$$

Hence, by (4.6) and (4.7), f is continuous on $\bar{\mathbb{D}}$ and is $\mathrm{Lip}_{\alpha(t)}$ on $\mathbb{T}$.

Now, suppose that f is continuous on $\bar{\mathbb{D}}$ and is $\mathrm{Lip}_{\alpha(t)}$ on $\mathbb{T}$. Hence, the Poisson representation formula

$$
f(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} f(e^{it})\, \frac{dt}{2\pi}, \qquad re^{i\theta} \in \mathbb{D},
$$

is valid. Taking the derivative of both sides with respect to θ gives

$$
\begin{aligned}
ire^{i\theta} f'(re^{i\theta}) &= \int_{-\pi}^{\pi} \frac{-2r(1-r^2)\sin(\theta-t)}{(1+r^2-2r\cos(\theta-t))^2} f(e^{it})\, \frac{dt}{2\pi} \\
&= \int_{-\pi}^{\pi} \frac{2r(1-r^2)\sin t}{(1+r^2-2r\cos t)^2} f(e^{i(\theta+t)})\, \frac{dt}{2\pi} \\
&= \int_{-\pi}^{\pi} \frac{2r(1-r^2)\sin t}{(1+r^2-2r\cos t)^2} (f(e^{i(\theta+t)}) - f(e^{i\theta}))\, \frac{dt}{2\pi}.
\end{aligned}
$$

Thus, by Corollary 2.2,

$$
\begin{aligned}
|f'(re^{i\theta})| &\le \frac{2(1-r)}{\pi} \int_{-\pi}^{\pi} \frac{|t|\omega_f(|t|)}{((1-r)^2 + 4r(\frac{t}{\pi})^2)^2}\, dt \\
&\le C(1-r) \int_0^\pi \frac{t^{\alpha(t)+1}}{((1-r)^2 + 2\frac{t^2}{\pi^2})^2}\, dt \\
&\le C(1-r)^{\alpha(1-r)-1} \qquad \text{as } r \to 1^-.
\end{aligned}
$$

■

The boundary values of a function in the Hardy space $H^p(\mathbb{D})$ is far away from being continuous. Nevertheless, for $f \in H^p(\mathbb{D})$, the function

$$
\Omega_f(t) = \|f_t - f\|_p,
$$

where $f_t(z) = f(e^{it}z)$, represents a continuous function [5, p. 9]. In the same papers, Hardy and Littlewood proved the following result.

Theorem (Hardy-Littlewood [2, 3])**.** *Let $f \in H^p(\mathbb{D})$. Then, the the function Ω_f is* $\mathrm{Lip}\,\alpha$*, $0 < \alpha \le 1$, if and only if $\|f_r'\|_p = \mathcal{O}((1-r)^{\alpha-1})$ as $r \to 1^-$.*

Similarly, this theorem can also be generalized as follows.

Theorem 4.4. *Let $\alpha(t)$ be a test function and let $f \in H^p(\mathbb{D})$. Then, Ω_f is* $\mathrm{Lip}_{\alpha(t)}$ *if and only if* $\|f_r'\|_p = \mathcal{O}((1-r)^{\alpha(1-r)-1})$ *as* $r \to 1^-$.

5. Further generalization

In the definition of $\alpha(t)$, instead of (1.2) and (1.3), we may assume that

$$\int_0^t \tau^{\alpha(\tau)-\beta}\, d\tau = \mathcal{O}(t^{\alpha(t)-\beta+1}), \qquad \beta < \alpha + 1,$$
$$\int_t^{t_0} \tau^{\alpha(\tau)-\beta}\, d\tau = \mathcal{O}(t^{\alpha(t)-\beta+1}), \qquad \beta > \alpha + 1.$$

Then, instead of Corollary 2.2, we would have

$$\int_0^{t_0} \frac{\tau^{\alpha(\tau)-\beta}}{\tau^\gamma + Ct^\gamma}\, d\tau = \begin{cases} \mathcal{O}(t^{\alpha(t)+1-\beta-\gamma}) & \text{if } \alpha < \gamma+\beta-1, \\ \mathcal{O}(t^{\hat{\alpha}(t)-\alpha}) & \text{if } \alpha = \gamma+\beta-1, \\ \mathcal{O}(1) & \text{if } \alpha > \gamma+\beta-1, \end{cases} \qquad \text{as } t \to 0^+.$$

Nevertheless, theorems like 4.1, 4.3 and 4.4 are still valid. One of the main reasons we did not pick this more generalized definition was that we were interested in the test function

$$t^{\alpha(t)} = t^\alpha \left(\log \tfrac{1}{t}\right)^{\alpha_1} \left(\log_2 \tfrac{1}{t}\right)^{\alpha_2} \cdots \left(\log_n \tfrac{1}{t}\right)^{\alpha_n},$$

for which (1.2) and (1.3) hold.

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