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Topics on Hyperbolic Function Theory in Geometric Algebra with a Positive Signature

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Abstract. In this paper we study geometric algebra valued null solutions of the equation

$$D_\ell f - \frac{k}{x_0} Q_0 f = 0$$

on the upper half $\mathbb{R}^{n+1} \cap \{x_0 > 0\}$, where D_ℓ is the Dirac operator and Q_0 is a projection-type mapping. Null solutions are called hypergenic functions. We will also study their local properties and integral representations.

Keywords. Hypergenic function, Cauchy formula, Borel-Pompeiu formula, multivector function.

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1. Preliminaries

Let $\{e_0, \ldots, e_n\}$ be the standard basis in $\mathbb{R}^{n+1}$. The Clifford (geometric) algebra $\mathcal{C}\ell_{n+1}$ is a free associative algebra with a unit generated by the standard basis vectors together with the defining relations

$$e_i e_j + e_j e_i = 2\delta_{ij}$$

for each $i, j = 0, \ldots, n$.

The Clifford algebra $\mathcal{C}\ell_{n+1}$ has dimension 2^{n+1} and the canonical basis is given by $e_A = e_{a_1} \cdots e_{a_k}$ where $A = \{a_1, \ldots, a_k\} \subset N = \{0, \ldots, n\}$ and $a_1 < \ldots < a_k$. Especially $e_\emptyset = 1$ and $e_{\{j\}} = e_j$.

A space of k-vectors is defined by $\mathcal{C}\ell_n^k = \operatorname{span}\{e_A \colon |A| = k\}$. Any $a \in \mathcal{C}\ell_{n+1}$ admits the multivector decomposition:

$$a = [a]_0 + [a]_1 + \cdots + [a]_{n+1}$$

with $[a]_k \in \mathcal{C}\ell_{n+1}^k$. The space of 0-vectors is identified with the set of real numbers $\mathbb{R}$ and the set of 1-vectors is identified with the Euclidean space $\mathbb{R}^{n+1}$.

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The *main involution* is the algebra automorphism

$$' \colon \mathcal{C}\ell_{n+1} \to \mathcal{C}\ell_{n+1}$$

defined by the relations

$$e_j' = -e_j \qquad \text{and} \qquad (ab)' = a'b'$$

for $a, b \in \mathcal{C}\ell_{n+1}$.

Assume that Ω is an open subset of $\mathbb{R}^{n+1}$. In the canonical basis every function $f \colon \Omega \to \mathcal{C}\ell_{n+1}$ admits the representation

$$f = \sum_{A \subset N} e_A f_A$$

where each f_A is a real valued function on Ω. A function f is called differentiable in Ω if each f_A is differentiable in Ω. If f is differentiable we define the left Dirac operator by

$$D_\ell f = \sum_{k=0}^{n} e_k \frac{\partial f}{\partial x_k}$$

and the right Dirac operator by

$$D_r f = \sum_{k=0}^{n} \frac{\partial f}{\partial x_k} e_k.$$

It is easy to see that the square of the Dirac operator is the Laplacian:

$$\Delta := D_\ell D_\ell = D_r D_r = \sum_{j=0}^{n} \frac{\partial^2}{\partial {x_j}^2}.$$

The upper half space is the Riemannian manifold $\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1} \cap \{x_0 > 0\}$ with the metric

$$g_k = \frac{{dx_0}^2 + \cdots + {dx_n}^2}{{x_0}^{2k/(n-1)}}.$$

In the upper half space the coordinate x_0 has a special role. Thus we generate the Clifford algebra $\mathcal{C}\ell_n$ by $\{e_1, \ldots, e_n\}$. Using the Clifford algebra $\mathcal{C}\ell_n$ the Clifford algebra $\mathcal{C}\ell_{n+1}$ admits the following decomposition

$$\mathcal{C}\ell_{n+1} = \mathcal{C}\ell_n \oplus e_0 \mathcal{C}\ell_n.$$

Thus, if $a \in \mathcal{C}\ell_{n+1}$ there exist $b, c \in \mathcal{C}\ell_n$ such that

$$a = b + e_0 c.$$

Let us denote $P_0 a := b$ and $Q_0 a := c$ and use the abbreviations $(P_0 a)' = P_0' a$ and $(Q_0 a)' = Q_0' a$ for each $a \in \mathcal{C}\ell_{n+1}$. Hence if $a \in \mathcal{C}\ell_{n+1}$ we obtain the "projector" decomposition

$$a = P_0 a + e_0 Q_0 a.$$

Using the above decomposition we may define the algebra automorphism

$$\widehat{\ } \colon \mathcal{C}\ell_{n+1} \to \mathcal{C}\ell_{n+1}$$

by

$$\widehat{a} = P_0 a - e_0 Q_0 a.$$

This automorphism is called the *hat involution*. If $a, b \in \mathcal{C}\ell_{n+1}$ it is straightforward to deduce that $\widehat{ab} = \widehat{a}\widehat{b}$. Also $\widehat{e_j} = (-1)^{\delta_{j,0}} e_j$ for $j = 0, \ldots, n$.

Let Ω be an open subset of the upper half-space $\mathbb{R}^{n+1}_+$. The *left modified Dirac operator* on Ω is defined by

$$H_k^\ell f = D_\ell f - \frac{k}{x_0} Q_0 f,$$

and similarly the *right modified Dirac operator* is defined by

$$H_k^r f = D_r f - \frac{k}{x_0} Q_0' f$$

(cf. [5, 6, 7]) where k is an arbitrary real number.

Null solutions of the left (resp. right) modified Dirac operator are called *left* (resp. *right*) *k-hypergenic functions.*

We shall also use the abbreviated notations $H^\ell := H^\ell_{(n-1)}$ and $H^r := H^r_{(n-1)}$ for the index $k = n-1$. Null solutions of the previous operators are called *left* (resp. *right*) *hypergenic functions.*

As a technical tool we will need the P_0- and Q_0-parts of the operators H_k^ℓ and H_k^r.

Lemma 1. *Let $f\colon \Omega \to \mathcal{C}\ell_{n+1}$ be a differentiable function. Then*

(a) $P_0(H_k^\ell f) = D_1 P_0 f + \dfrac{\partial Q_0 f}{\partial x_0} - \dfrac{k}{x_0} Q_0 f,$

(b) $Q_0(H_k^\ell f) = \dfrac{\partial P_0 f}{\partial x_0} - D_1 Q_0 f,$

(c) $P_0\left((H_k^\ell)^2 f\right) = \Delta P_0 f - \dfrac{k}{x_0}\dfrac{\partial P_0 f}{\partial x_0},$

(d) $Q_0\left((H_k^\ell)^2 f\right) = \Delta Q_0 f - \dfrac{k}{x_0}\dfrac{\partial Q_0 f}{\partial x_0} + \dfrac{k}{x_0^2} Q_0 f,$

where

$$D_1 = e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

Proof. We will prove only (a) and (b) since the proofs for (c) and (d) are available in the proof of [5, Thm. 1.8]. Assume a function $f = P_0 f + e_0 Q_0 f$ is differentiable. Since

$$D_1 = e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$$

we obtain

$$D_\ell(P_0 f) = e_0 \frac{\partial P_0 f}{\partial x_0} + D_1 P_0 f$$

and

$$D_\ell(e_0 Q_0 f) = \frac{\partial Q_0 f}{\partial x_0} - e_0 D_1 Q_0 f.$$

Previous observations implies that

$$\begin{aligned} H_k^\ell f &= H_k^\ell (P_0 f + e_0 Q_0 f) \\ &= D_\ell (P_0 f) + D_\ell (e_0 Q_0 f) - \frac{k}{x_0} Q_0 f \\ &= \left\{ D_1 P_0 f + \frac{\partial Q_0 f}{\partial x_0} - \frac{k}{x_0} Q_0 f \right\} + e_0 \left\{ \frac{\partial P_0 f}{\partial x_0} - D_1 Q_0 f \right\}. \end{aligned}$$

The proof is complete. ■

2. Local properties of hypergenic functions

In this section we consider local properties of hypergenic functions. If Ω is an open subset in $\mathbb{R}_+^{n+1}$ the operator

$$\Delta_m h = {x_0}^2 \left(\Delta h - \frac{k}{x_0} \frac{\partial h}{\partial x_0} \right)$$

is called *the Modified Laplace-Beltrami operator* for a twice continuously differentiable function $h : \Omega \to \mathcal{C}\ell_{n+1}$. In [5] we obtained the following theorem.

Theorem 2. *Let $\Omega \subset \mathbb{R}_+^{n+1}$ be an open subset and $f : \Omega \to \mathcal{C}\ell_{n+1}$ be a twice continuously differentiable k-hypergenic function. The function $P_0 f$ is a solution of the equation $\Delta_m P_0 f = 0$ and $Q_0 f$ is a solution of the eigenvalue problem $\Delta_m Q_0 f = -k Q_0 f$.*

The Euclidean ball with the radius $r > 0$ and with the center a is denoted by $B_r(a)$.

Theorem 3. *Let $\Omega \subset \mathbb{R}_+^{n+1}$ be open. A differentiable function $f : \Omega \to \mathcal{C}\ell_{n+1}$ is hypergenic if and only if for any $a \in \Omega$ there exists $r > 0$ satisfying $B_r(a) \subset \Omega$ and a map $g : B_r(a) \to \mathcal{C}\ell_{n,0}$ satisfying*

$$f = D_\ell g \qquad \text{and} \qquad \Delta_m g = 0.$$

Proof. Assume first that the radius $r > 0$ and the function g exists. Since $f = D_\ell g$ we get in the neighborhood of a

$$H_k^\ell f = \Delta g - \frac{k}{x_0} Q_0 (D g) = \Delta g - \frac{k}{x_0} \frac{\partial g}{\partial x_0} = \frac{1}{{x_0}^2} \Delta_{LB} g = 0.$$

Let $f : \Omega \to \mathcal{C}\ell_{n+1}$ be a hypergenic function. Let us denote $x = (x_0, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\Delta_1 := D_1 D_1$. Assume that s_A is a solution of the Poisson problem

$$\Delta_1 s_A(\tilde{x}) = (P_0 f)_A (a_0, \tilde{x})$$

in $B_r(a)\cap\mathbb{R}^n$ and let $s=\sum_{A\subset\{1,\dots,n\}} s_A$. The solution of the Poisson problem obviously exists (cf. [1]). Then we define

$$g(x)=\int_{a_0}^{x_0} Q_0 f(t,\tilde{x})s\,dt+D_1 s(\tilde{x})$$

and obtain

$$D_\ell g(x)=e_0 Q_0 f(x)+\int_{a_0}^{x_0} D_1 Q_0 f(t,\tilde{x})\,dt+\Delta_1 s(\tilde{x}).$$

Using Lemma 1 we obtain

$$D_1 Q_0 f=\frac{\partial P_0 f}{\partial x_0}$$

and

$$D_\ell g(x)=e_0 Q_0 f(x)+\int_{a_0}^{x_0}\frac{\partial P_0 f}{\partial x_0}(t,\tilde{x})\,dt+P_0 f(a_0,\tilde{x})=f(x).$$

Similarly we see that

$$0=D_\ell f-\frac{k}{x_0}Qf=\Delta g-\frac{k}{x_0}\frac{\partial g}{\partial x_0},$$

which completes the proof. ∎

3. On the Euler operator in the class of hypergenic functions

An operator

$$E=\sum_{i=0}^{n} x_i\frac{\partial}{\partial x_i}.$$

is called an Euler operator. It is a scalar operator and measures the degree of homogenicity of a function. In this section we shall deduce that the class of hypergenic functions is invariant under the Euler operator. First we recall the following well known result.

Lemma 4. *If D_ℓ is the Dirac operator and E is the Euler operator we have*

$$D_\ell E-ED_\ell=[D_\ell,E]=D_\ell.$$

The similar result holds in the class of hypergenic functions.

Theorem 5. *If H_k^ℓ is the modified Dirac operator and E is the Euler operator we have*

$$H_k^\ell E-EH_k^\ell=[H_k^\ell,E]=H_k^\ell.$$

Proof. Let

$$D_\ell f - \frac{k}{x_0} Q_0 f = 0.$$

Applying E to

$$\frac{k}{x_0} Q_0 f$$

we obtain

$$\frac{k}{x_0} Q_0 E f = \frac{k}{x_0} Q_0 f + E\left(\frac{k}{x_0} Q_0 f\right).$$

Using the previous lemma we obtain

$$\begin{aligned} H_k^\ell E f &= D_\ell E f - \frac{k}{x_0} Q_0 E f \\ &= D_\ell f + E D f - \frac{k}{x_0} Q_0 f - E\left(\frac{k}{x_0} Q_0 f\right) \\ &= D_\ell f - \frac{k}{x_0} Q_0 f + E\left(D f - \frac{k}{x_0} Q_0 f\right) = 0. \end{aligned}$$

The proof is complete. ∎

Corollary 6. *Let $f\colon \Omega \to \mathcal{C}\ell_{n+1}$ be a differentiable function. If the function f is k-hypergenic then Ef is k-hypergenic.*

If f is a hypergenic function then the previous theorem gives us a method how to construct more hypergenic functions.

For every k-hypergenic function $f\colon \Omega \to \mathcal{C}\ell_{n+1,0}$ there exists the sequence of hypergenic functions defined by

$$E^m f := \underbrace{E \cdots E}_{m \text{ copies}} f$$

for each $m \in \mathbb{N}$ and $E^0 f = f$. This sequence $\{E^m f\colon m = 0, 1, \ldots\}$ is called the *homogenized sequence* of f.

As an example we consider the monomials

$$X_i = \sum_{j=0}^{n} (-1)^{\delta_{ij}} x_j e_j$$

where $i = 1, \ldots, n$. Since $H^\ell X_i = 0$ the monomials X_i are hypergenic. Moreover, since

$$E X_i = X_i$$

for each $i = 1, \ldots, n$, the homogenized sequence of X_i is just $\{X_i\}$.

4. Further representation results

In this section we will consider the improved version of Cauchy's formula. The "original" Cauchy's formula is represented in [5]. The second aim is to prove the Borel-Pompeiu formula using the language of the modified Dirac operators.

First we recall briefly some preliminaries from integration theory. Let M be a k-dimensional manifold-with-boundary in $\mathbb{R}^{n+1}$, see e.g. [14]. The boundary of M is denoted by ∂M. If moreover

$$\Lambda^* M = \bigoplus_{p=0}^{n+1} \Lambda^p M$$

is the exterior algebra over $\mathbb{R}^{n+1}$ with the basis $\{dx_0, dx_1, \ldots, dx_n\}$ we then construct the bundle $\mathcal{C}\ell_{n+1} \otimes_{\mathbb{R}} \Lambda^k M$. If $\omega(x)$ is a section of the previous bundle over $x \in M$ it is of the form

$$\omega(x) = \sum_{A,B} \omega_{A,B}(x) e_A dx_B,$$

where $B = \{b_1, \ldots, b_k\} \subset N = \{1, \ldots, n\}$ and $dx_B = dx_{b_1} \wedge \cdots \wedge dx_{b_k}$. The meaning of the symbol $e_A dx_B$ is clear. Furthermore let M be an oriented k-dimensional manifold-with-boundary in $\mathbb{R}^{n+1}$, then we define

$$\int_M \omega(x) = \sum_{A,B} e_A \int_M \omega_{A,B}(x)\, dx_B.$$

In this paper M will be $(n+1)$ dimensional or n-dimensional (i.e. the boundary of M). The $(n+1)$-form

$$dV = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n$$

on M is called the (Riemannian) volume element. In surface integrals we shall often use the n-form

$$d\sigma = \sum_{i=0}^{n} (-1)^i e_i d\check{x}_i$$

where

$$d\check{x}_i = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$$

for each $i = 0, 1, \ldots, n$. The exterior derivative d for Clifford algebra valued differential forms is defined componentwise, i.e. if

$$\omega = \sum_{A,B} \omega_A e_B$$

is a k-form then

$$d\omega = \sum_{A,B} d\omega_A e_B.$$

Applying the classical Stokes Theorem (see [14]) it is easy to prove the following theorem.

Theorem 7 (Stokes)**.** *Let ω be a $\mathcal{C}\ell_{n+1}$-valued k-form on an oriented k-dimensional manifold-with-boundary M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

We recall the following version[1] of Cauchy's formula given in [5].

Theorem 8. *Let Ω be an open subset of $\mathbb{R}^{n+1}_+$ and $f\colon \Omega \to \mathcal{C}\ell_{n+1}$ be hypergenic. Let $M \subset \Omega$ be an oriented $(n+1)$-dimensional manifold-with-boundary. Then*

$$f(y) = \frac{2^{n-1}{y_0}^{n-1}}{\omega_{n+1}} \int_{\partial M} \frac{(x-y)^{-1} d\sigma(x) f(x) - (\widehat{x}-y)^{-1} \widehat{d\sigma}(x) \widehat{f}(x)}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}}$$

where y is an interior point of the manifold M.

If we decompose

$$d\sigma(x) f(x) = P_0(d\sigma(x) f(x)) + e_0 Q_0(d\sigma(x) f(x))$$

we obtain

$$\begin{aligned} (1) \qquad & (x-y)^{-1} d\sigma(x) f(x) - (\widehat{x}-y)^{-1} \widehat{d\sigma}(x) \widehat{f}(x) \\ &= \left((x-y)^{-1} - (\widehat{x}-y)^{-1}\right) P_0(d\sigma(x) f(x)) \\ &\quad + \left((x-y)^{-1} + (\widehat{x}-y)^{-1}\right) e_0 Q_0(d\sigma(x) f(x)) . \end{aligned}$$

If we define the kernel by

$$\Lambda(x,y) := \frac{(x-y)^{-1} - (\widehat{x}-y)^{-1}}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}},$$

we obtain the following lemma.

Lemma 9.

$$\frac{(x-y)^{-1} + (\widehat{x}-y)^{-1}}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}} e_0 x_0 = \Lambda(x,y)(y - Px).$$

Using the identity (1) and the above lemma we obtain

$$\begin{aligned} &\frac{(x-y)^{-1} d\sigma(x) f(x) - (\widehat{x}-y)^{-1} \widehat{d\sigma}(x) \widehat{f}(x)}{|x-y|^{n-1} |x-\widehat{y}|^{n-1}} \\ &\quad = \Lambda(x,y) P_0(d\sigma(x) f(x)) + \Lambda(x,y) \frac{y - Px}{x_0} Q_0(d\sigma(x) f(x)) . \end{aligned}$$

Hence we obtain the improved version of Cauchy's formula:

[1] In this paper we use the Cauchy's formula with the "manifold assumptions". It is easy to see that the proof for it in [5] is exactly the same since, e.g. the outer unit vector field on the boundary of the oriented manifold-with-boundary always exists (cf. [10]).

Theorem 10. *Assume that* Ω *is an open subset of* $\mathbb{R}^{n+1}_+$ *and* $f\colon \Omega \to \mathcal{C}\ell_{n+1}$ *be hypergenic. Let* $M \subset \Omega$ *be an oriented* $(n+1)$*-dimensional manifold-with-boundary. Then*

$$f(y) = \frac{2^{n-1}{y_0}^{n-1}}{\omega_{n+1}} \int_{\partial M} \Lambda(x,y)\left(P_0(d\sigma(x)f(x)) + \frac{y-Px}{x_0}Q_0(d\sigma(x)f(x))\right),$$

where y *is an interior point of the manifold* M.

Theorem 11 (Borel-Pompeiu)**.** *Let* $\Omega \subset \mathbb{R}^{n+1}_+$ *be open and* $f\colon \Omega \to \mathcal{C}\ell_{n+1}$ *be a differentiable function. Let* $M \subset \Omega$ *be an oriented* $(n+1)$*-dimensional manifold-with-boundary. Then*

$$\begin{aligned} f(y) = {} & \frac{2^{n-1}{y_0}^{n-1}}{\omega_{n+1}} \int_{\partial M} \left\{\frac{1}{{x_0}^{n-1}}P_0(p(x,y)d\sigma(x)f(x)) + e_0Q_0(q(x,y)d\sigma(x)f(x))\right\} \\ & + \frac{2^{n-1}{y_0}^{n-1}}{\omega_{n+1}} \int_M \frac{1}{{x_0}^{n-1}}\left\{P_0\big(p(x,y)H^\ell f(x)\big) + e_0Q_0\big(q(x,y)H^\ell f(x)\big)\right\} dV \end{aligned}$$

where y *is an interior point of the manifold* M. *The kernels*

$$\begin{aligned} p(x,y) &:= {x_0}^{n-1}\frac{(x-y)^{-1} - (x-\widehat{y})^{-1}}{|x-y|^{n-1}|x-\widehat{y}|^{n-1}}, \\ q(x,y) &:= \frac{(x-y)^{-1} + (x-\widehat{y})^{-1}}{|x-y|^{n-1}|x-\widehat{y}|^{n-1}} \end{aligned}$$

are hypergenic with respect to x *on* $\mathbb{R}^{n+1} \setminus \{y, \widehat{y}\}$ *for each* $y \in \mathbb{R}^{n+1}$.

Remark 1. In the Borel-Pompeiu's formula hypergenicity has no role. Assuming that f is hypergenic we obtain the original version of Cauchy's formula.

In order to prove the Borel-Pompeiu's theorem, let us first state a few basic lemmata. For the proofs we refer to [5].

Lemma 12. *The function* $x \mapsto p(x,y)$ *is right* $(n-1)$*-hypergenic and the function* $x \mapsto q(x,y)$ *is right* $-(n-1)$*-hypergenic on* $\mathbb{R}^{n+1} \setminus \{y, \widehat{y}\}$ *for each* $y \in \mathbb{R}^{n+1}$.

Lemma 13. *Let* Ω *be an open subset of* $\mathbb{R}^{n+1}_+$ *and* $f, g\colon \Omega \to \mathcal{C}\ell_{n+1}$ *be differentiable functions. Let* $M \subset \Omega$ *be an oriented* $(n+1)$*-dimensional manifold-with-boundary. If* g *is right* $-(n-1)$*-hypergenic then*

$$\int_{\partial M} \frac{1}{{x_0}^{n-1}}P_0(gd\sigma f) = \int_M \frac{1}{{x_0}^{n-1}}P_0\big(gH^\ell f\big)\ dV$$

and if g *is right hypergenic then*

$$\int_{\partial M} Q_0(gd\sigma f) = \int_M Q_0\big(gH^\ell f\big)\ dV.$$

Proof. This is a corollary of Lemma 2.6 and Lemma 2.9 in [5]. ∎

Lemma 14. *If $f\colon \Omega \to \mathcal{C}\ell_{n+1,0}$ is continuous and $B_r(y) \subset \Omega \subset \mathbb{R}^{n+1}_+$ then*

$$\frac{2^{n-1}{y_0}^{n-1}}{\omega_{n+1}} \int_{\partial B_r(y)} \frac{1}{{x_0}^{n-1}} P_0(p(x,y)\, d\sigma(x) f(x)) \to P_0 f(y),$$

$$\frac{2^{n-1}{y_0}^{n-1}}{\omega_{n+1}} \int_{\partial B_r(y)} Q_0(q(x,y) d\sigma(x) f(x)) \to Q_0 f(y)$$

as $r \to 0$.

Proof. See the proofs of Theorem 2.4 and 2.7 in [5]. ■

Proof of Theorem 11. For each interior point $y \in M$ there exists $r > 0$ such that $\overline{B_r(y)} \subset M$. We denote $M_r(y) := M \setminus B_r(y)$. Then

$$\begin{aligned} &\int_{\partial M} \frac{1}{{x_0}^{n-1}} P_0(p(x,y) d\sigma(x) f(x)) \\ &\quad + e_0 \int_{\partial M} Q_0(q(x,y) d\sigma(x) f(x)) \\ &= \int_{\partial M_r(y)} \frac{1}{{x_0}^{n-1}} P_0 \qquad (p(x,y) d\sigma(x) f(x)) \\ &\quad + e_0 \int_{\partial M_r(y)} Q_0 \qquad (q(x,y) d\sigma(x) f(x)) \\ &\quad + \int_{\partial B_r(y)} \frac{1}{{x_0}^{n-1}} P_0(p(x,y) d\sigma(x) f(x)) \\ &\quad + e_0 \int_{\partial B_r(y)} Q_0(q(x,y) d\sigma(x) f(x)) . \end{aligned} \tag{2}$$

Using Lemma 14 we obtain

$$\begin{aligned} &\int_{\partial B_r(y)} \frac{1}{{x_0}^{n-1}} P_0(p(x,y) d\sigma(x) f(x)) \\ &\quad + e_0 \int_{\partial B_r(y)} Q_0(q(x,y) d\sigma(x) f(x)) \to \frac{\omega_{n+1}}{2^{n-1} y_0^{n-1}} f(y) \end{aligned}$$

as $r \to 0$. Using Lemma 12 and 13 we obtain

$$\int_{\partial M_r(y)} \frac{1}{{x_0}^{n-1}} P_0(p(x,y) d\sigma(x) f(x)) = \int_{M_r(y)} \frac{1}{{x_0}^{n-1}} P_0\big(p(x,y) H^\ell f(x)\big)\, dx,$$

$$\int_{\partial M_r(y)} Q_0(q(x,y) d\sigma(x) f(x)) = \int_{M_r(y)} Q_0\big(q(x,y) H^\ell f(x)\big)\, dx.$$

Since $\int_{M_r(y)} \to \int_M$ as $r \to 0$, we obtain the result just taking $r \to 0$ in (2). ■

5. Multivector calculus

In [5] we also studied some connections between Q_0-operator, H_k^ℓ-operators and the left- and right contraction -operators. In this section we will continue our study. First we recall some notions and properties.

Let $x \in \mathcal{C}\ell_{n+1}^1$ be a vector and $u \in \mathcal{C}\ell_{n+1}$ be an arbitrary Clifford number. The left contraction is defined by

$$x \lrcorner u = \frac{1}{2}(xu - u'x), \tag{3}$$

and the right contraction by

$$u \llcorner x = \frac{1}{2}(ux - xu').$$

Marcel Riesz has introduced in 1958 the exterior product in Clifford algebras by

$$x \wedge u = \frac{1}{2}(xu + u'x). \tag{4}$$

Adding (3) and (4) together we obtain the Clifford product

$$xu = x \lrcorner u + x \wedge u.$$

Moreover, if u is a k-vector we have $u' = (-1)^k u$. In this case we obtain

$$x \lrcorner u = (-1)^{k-1} u \llcorner x$$

and

$$x \wedge u = (-1)^k u \wedge x$$

for each vector x. Historical remarks and more comprehensive introduction to the contraction products and the exterior product, see [13].

Remark 2. If $x, y \in \mathcal{C}\ell_{n+1}^1$, then

$$x \lrcorner y = y \llcorner x = (x, y)$$

where $(\cdot, \cdot)$ is the usual Euclidean inner product.

Next we recall a few handy formulae.

Lemma 15. *If x and y are vectors and u is a k-vector. Then*

(a) $x \lrcorner (y \wedge u) = (x, y)u - y \wedge (x \lrcorner u)$,
(b) $x \lrcorner (y \lrcorner u) = -y \lrcorner (x \lrcorner u)$.

Proof. See [7, 13]. ■

In the case of k-vectors it is convenient to represent P_0- and Q_0-operators using the Riesz's products as follows.

Proposition 16. *If $u \in \mathcal{C}\ell_{n+1}$ then*

(a) $P_0 u = e_0 \lrcorner (e_0 \wedge u)$ *and* $Q_0 u = e_0 \lrcorner u$,
(b) $P_0' u = (e_0 \wedge u) \llcorner e_0$ *and* $Q_0' u = u \llcorner e_0$.

Proof. (a) See the proof of $Qu = e_0 \lrcorner u$ in [5]. Using

$$e_0 u + u' e_0 = e_0 P u + Q u + P' u e_0 - e_0 Q' u e_0 = 2 e_0 P u$$

we obtain

$$P_0 u = e_0 \frac{1}{2}(e_0 u + u' e_0) = e_0 (e_0 \wedge u) = e_0 \lrcorner (e_0 \wedge u),$$

since $e_0 \wedge e_0 \wedge u = 0$.

(b) Recall $P_0' u = (P_0 u)'$ and $Q_0' u = (Q_0 u)'$. Since

$$(e_0 \lrcorner v)' = \frac{1}{2}(e_0 v - v' e_0)' = \frac{1}{2}(v e_0 - e_0 v') = v \llcorner e_0$$

for each $v \in \mathcal{C}\ell_{n+1}$ we obtain

$$P_0' u = (e_0 \lrcorner (e_0 \wedge u))' = (e_0 \wedge u) \llcorner e_0$$

and $Q_0' u = u \llcorner e_0$. ■

Let $\mathcal{E}^k(\Omega)$ be the set of k-vector valued functions on Ω. If $f \in \mathcal{E}^k(\Omega)$ is differentiable then the modified Dirac operators are represented in the form

$$\begin{aligned} H^\ell f &= D_\ell f - \frac{n-1}{x_0} e_0 \lrcorner f, \\ H^r f &= D_r f - \frac{n-1}{x_0} f \llcorner e_0, \end{aligned}$$

on Ω.

We interpret the Dirac operator $D := D_\ell$ as a vector derivative. As in [9] we also interpret Df as a product of a vector and a Clifford number, i.e.,

$$\begin{aligned} D_\ell f &= D \lrcorner f + D \wedge f, \\ D_r f &= f \llcorner D + f \wedge D. \end{aligned}$$

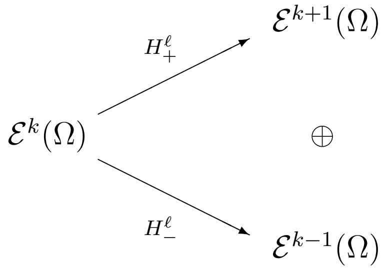
Thus if $f \in \mathcal{E}^k(\Omega)$ we define

$$\begin{aligned} H^\ell_+ f &= [H^\ell f]_{k+1} = D \wedge f, \\ H^\ell_- f &= [H^\ell f]_{k-1} = D \lrcorner f - \frac{n-1}{x_0} e_0 \lrcorner f, \\ H^r_+ f &= [H^r f]_{k+1} = f \wedge D, \\ H^r_- f &= [H^r f]_{k-1} = f \llcorner D - \frac{n-1}{x_0} f \llcorner e_0. \end{aligned}$$

Hence

$$H^\ell = H^\ell_+ + H^\ell_- \quad \text{and} \quad H^r = H^r_+ + H^r_-.$$

Mapping properties of the operator H^ℓ are illustrated in the following picture:



Theorem 17. *Let $f \in \mathcal{E}^k(\Omega)$ be a differentiable function on open $\Omega \subset \mathbb{R}_+^{n+1}$. Then*

$$H_-^\ell f = 0,$$
$$H_+^\ell f = 0$$

if and only if

$$H^\ell f = 0.$$

Proof. Assume

$$H^\ell f = \left(D - \frac{n-1}{x_0} e_0\right) \lrcorner f + D \wedge f = 0.$$

Since

$$[H^\ell f]_{k-1} = \left(D - \frac{n-1}{x_0} e_0\right) \lrcorner f = 0$$

and

$$[H^\ell f]_{k+1} = D \wedge f = 0$$

we obtain the result. ∎

Proposition 18. *Let $f \in \mathcal{E}^k(\Omega)$ be a differentiable function. Then*

(a) $(H_+^\ell)^2 = (H_-^\ell)^2 = 0$,

(b) $H_-^\ell H_+^\ell = \Delta f - D \wedge (D \lrcorner f) - \dfrac{n-1}{x_0} \dfrac{\partial f}{\partial x_0} + \dfrac{n-1}{x_0} D \wedge (e_0 \lrcorner f)$,

(c) $H_+^\ell H_-^\ell = D \wedge (D \lrcorner f) + \dfrac{n-1}{x_0^2} e_0 \wedge (e_0 \lrcorner f) - \dfrac{n-1}{x_0} D \wedge (e_0 \lrcorner f)$.

Proof. Let $f \in \mathcal{E}^k(\Omega)$ be a twice continuously differentiable function.

(a) First we compute

$$(H_+^\ell)^2 f = \sum_{i,j=0}^{n} e_i \wedge e_j \wedge \frac{\partial^2 f}{\partial x_j \partial x_j} = \sum_{i<j}^{n} e_i \wedge e_j \wedge \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) = 0.$$

Since $D \lrcorner D \lrcorner f = 0$ and $e_0 \lrcorner e_0 \lrcorner f = 0$ we have

$$(H_-^\ell)^2 f = -D \lrcorner \left(\frac{n-1}{x_0} e_0 \lrcorner f\right) - \frac{n-1}{x_0} e_0 \lrcorner (D \lrcorner f).$$

Using Lemma 15 we obtain

$$D\lrcorner\left(\frac{n-1}{x_0}e_0\lrcorner f\right) = -\frac{n-1}{x_0}e_0\lrcorner(D\lrcorner f).$$

Then $(H_-^\ell)^2 = 0$.

(b) Using Lemma 15 we have

$$D\lrcorner(D\wedge f) = \Delta f - D\wedge(D\lrcorner f)$$

and

$$e_0\lrcorner(D\wedge f) = \frac{\partial f}{\partial x_0} - D\wedge(e_0\lrcorner f).$$

Thus

$$\begin{aligned} H_-^\ell H_+^\ell f &= D\lrcorner(D\wedge f) - \frac{n-1}{x_0}e_0\lrcorner(D\wedge f) \\ &= \Delta f - D\wedge(D\lrcorner f) - \frac{n-1}{x_0}\frac{\partial f}{\partial x_0} + \frac{n-1}{x_0}D\wedge(e_0\lrcorner f). \end{aligned}$$

(c) We compute

$$\begin{aligned} H_+^\ell H_-^\ell f &= D\wedge(D\lrcorner f) - D\wedge\left(\frac{n-1}{x_0}e_0\lrcorner f\right) \\ &= D\wedge(D\lrcorner f) + \frac{n-1}{{x_0}^2}e_0\wedge(e_0\lrcorner f) - \frac{n-1}{x_0}D\wedge(e_0\lrcorner f). \end{aligned}$$

∎

Corollary 19. *Let $f \in \mathcal{E}^k(\Omega)$ be a differentiable function. Then*

$$(H^\ell)^2 = H_+^\ell H_-^\ell + H_-^\ell H_+^\ell$$

and especially

$$(H^\ell)^2 f = \Delta f - \frac{n-1}{x_0}\frac{\partial f}{\partial x_0} + \frac{n-1}{{x_0}^2}e_0\wedge(e_0\lrcorner f).$$

Above results are formulated and proved only for the left hypergenic functions. In the next theorem we shall see that this is not a shortcoming of the theory.

Theorem 20. *Let $f \in \mathcal{E}^k(\Omega)$ be a differentiable function. The function f is left hypergenic if and only if it is right hypergenic.*

Proof. Let f be a differentiable k-vector-valued function. Then

$$\begin{aligned} H_-^\ell f &= \sum_{j=0}^n e_j\lrcorner\frac{\partial f}{\partial x_j} - \frac{n-1}{x_0}e_0\lrcorner f \\ &= (-1)^{k-1}\left(\sum_{j=0}^n \frac{\partial f}{\partial x_j}\llcorner e_j - f\llcorner e_0\frac{n-1}{x_0}\right) \\ &= (-1)^{k-1}H_-^r f. \end{aligned}$$

Similarly

$$H_+^\ell f = \sum_{j=0}^{n} e_j \wedge \frac{\partial f}{\partial x_j} = (-1)^k \sum_{j=0}^{n} \frac{\partial f}{\partial x_j} \wedge e_j = (-1)^k H_+^r f.$$

The proof is complete. ■

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