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Metabelian Product of a Free Nilpotent Group with a Free Abelian Group

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Dedicated to Benjamin Fine on the occasion of his 60th birthday.

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If two groups are residually-$\mathcal{P}$, their free product is not necessarily so; however, it is known that the free product of residually torsion-free nilpotent groups is again residually torsion-free nilpotent. In this paper it is shown that the free metabelian product of a free nilpotent group of class two with a free abelian group is residually torsion-free nilpotent.

Keywords: Free product, free metabelian product, residually torsion-free nilpotent, metabelian groups

1. Introduction

1.1. Background

In the variety of all groups, A. I. Mal'cev [5] proved in 1949 that the free product of two residually torsion-free nilpotent groups is again residually torsion-free nilpotent. This paper is motivated by the analogous question in the variety of metabelian groups: can we determine whether free metabelian products of residually torsion-free nilpotent metabelian groups are residually torsion-free nilpotent metabelian.

The first answers to this question came in the 1950's and 60's. In 1957, K. W. Gruenberg [3] showed that every free metabelian group is residually torsion-free nilpotent, so a free metabelian product of free metabelian groups is residually torsion-free nilpotent.

There exist a number of different proofs that a free metabelian product of free or torsion-free abelian groups is residually torsion-free nilpotent. In 1957, R. Ree [7] showed that every free metabelian product of torsion-free abelian groups is residually torsion-free nilpotent. In 1963, G. Baumslag [1] proved that every free metabelian product of free abelian groups is residually torsion-free nilpotent, as

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a by-product of a more general theorem involving product varieties. In 1965, G. Baumslag and F. Levin [2] found a representation for the free metabelian product of two torsion-free abelian groups in a graded ring. Baumslag and Levin's representation provides another proof that every free metabelian product of torsion-free abelian groups is residually torsion-free nilpotent.

In this paper, a further result is obtained:

Theorem 1.1. *The free metabelian product of a free nilpotent group of class 2 with a free abelian group of countable rank is residually torsion-free nilpotent.*

Much of the material herein comes from the author's Ph.D. dissertation.

1.2. Notation and definitions

As usual, if x, y are any elements of G, denote the conjugate $y^{-1}xy$ of x by y as x^y, and the commutator $x^{-1}y^{-1}xy$ of x and y as $[x, y]$. The derived group G' is $[G, G]$, the commutator subgroup of G.

Definition 1.2. Let $\mathcal{P}$ be a property of groups that is isomorphism invariant and inherited by subgroups. A group satisfying $\mathcal{P}$ is referred to as a $\mathcal{P}$-group. A group G is said to be **residually** $\mathcal{P}$ *if given any non-trivial element $g \in G$ there exists a homomorphism of G to a $\mathcal{P}$-group H such that the image of g is non-trivial in H.*

A group G is said to be **fully residually** $\mathcal{P}$ *if given any non-empty finite set X of non-trivial words in G there exists a homomorphism of G to a $\mathcal{P}$-group H such that the words in X are mapped distinctly and nontrivially to words in H.*

Definition 1.3. A non-empty class $\mathcal{V}$ of groups is called a **variety of groups** *if it is closed under subgroups, epimorphic images and unrestricted direct products.*

Throughout this paper, $\mathcal{A}$ will represent the variety of abelian groups; $\mathcal{A}^2$ will represent the variety of metabelian groups. Note that any nontrivial commutator in a metabelian group will be a left-normed commutator.

We will repeatedly take advantage of the following well-known commutator identities, which are due to P. Hall [4]:

For any group G and any $x, y, z \in G$,

$$\begin{aligned}[xy, z] &= [x, z]^y[y, z] \\ [x, y^{-1}] &= [x, y]^{-y^{-1}} \\ [x^{-1}, y] &= [x, y]^{-x^{-1}}\end{aligned}$$

The following lemmas will prove useful.

Lemma 1.4 (von Dyck). *Let G and H be a pair of groups, and let $G = \langle X; R\rangle$, with a presentation map φ. Let $X' = \{x' | x \in X, x' \in H\}$ be a set of elements* *of H in one to one correspondence with X. Then the map $x\varphi\theta \mapsto x'\,(x \in X)$ can be extended to a homomorphism from G into H, provided that for every $r(x_1, \dots, x_n) \in R$, $r(x'_1, \dots, x'_n) = 1$ in H.*

Von Dyck's lemma is well-known and its proof is straightforward.

Lemma 1.5. *Let N be a normal subgroup of a group G. Let $Q = G/N$, and suppose that conjugation by the elements of Y, a complete set of representatives of N in G, constitutes an action of Q on N. Suppose also that*

$$1 = N_0 < N_1 < \cdots < N_m = N$$

is a series of normal subgroups of G such that $[N, N_i] \leq N_{i-1}$. If Q acts trivially on N_i/N_{i-1} for $i = 1, \dots, m$ and if Q is nilpotent, then so is G.

Proof. Suppose Q is nilpotent of class c. It follows easily that $\gamma_{c+1}G \leq N$.

Now, every element $g \in G$ can be written uniquely as $g = yn$ (for some $y \in Y$, $n \in N$); hence, $[G, N_i] \leq N_{i-1}$, for

$$[g, n_i] = [yn, n_i] = [y, n_i]^n [n, n_i] \leq N_{i-1},$$

by hypothesis. Thus, $\gamma_{c+m+2}G = 1$, and G is nilpotent of class at most $c + m + 1$. $\square$

Lemma 1.6. *If A and B are any groups, and $P = A * B$ is their free product, then $P' = \mathrm{gp}(A', B', [A, B])$.*

Proof. Let $H = \mathrm{gp}(A', B', [A, B])$. Clearly, $P' \geq H$. It suffices now to show that H is normal:

If $[a_1, b] \in [A, B]$, then $[a_1, b]^a = [a_1 a, b][b, a] \in H$. If $[a_1, a_2] \in [A, A]$, then $[a_1, a_2]^b = [a_1, a_2][a_1, a_2, b] \in H$. In the same way, it can be shown that $[a, b_1]^b$, $[b_1, b_2]^a \in H$. $\square$

1.3. Metabelian groups

Although varieties are not generally closed under free products, H. Neumann [6] found a way to define an analogue, the verbal product, under which $\mathcal{V}$ is closed. In the case of $\mathcal{A}^2$, we use the term **free metabelian product**:

If $A_1, A_2 \in \mathcal{A}^2$ then the free metabelian product of A_1 and A_2 is:

$$A_1 *_{\mathcal{A}^2} A_2 = (A_1 * A_2)/(A_1 * A_2)''.$$

A useful commutator identity known as the Jacobi Identity for metabelian groups (derived from a more general identity of the same name for all groups) is:

For any metabelian group G and any $x, y, z \in G$,

$$[x, y, z][y, z, x][z, x, y] = 1. \tag{1}$$

We need the following lemmas about metabelian commutators, which are well-known (e.g., see [6]).

Lemma 1.7. *If M is a metabelian group and $w, x, y, z \in M$, then*

$$[x, y, z, w] = [x, y, w, z].$$

Lemma 1.8. *If M is a metabelian group and $x, y, z_i \in M, (i = 1, \ldots, k)$, then*

$$[y, x, z_1, z_2, \ldots, z_k] = [x, y, z_1, z_2, \ldots, z_k]^{-1}.$$

2. Proof of the theorem

2.1. The free metabelian product of a free 2-nilpotent group with a free abelian group

We begin by proving Theorem 1.1 for the free metabelian product of a free 2-nilpotent group of rank 2 with an infinite cyclic group. Then we show that a free metabelian product of a free nilpotent group of class 2 with a free abelian group of countable rank is residually a free metabelian product of a free 2-nilpotent group of rank 2 with an infinite cyclic group.

Lemma 2.1. *The free metabelian product of a free nilpotent group of class 2 on two generators with an infinite cyclic group is residually torsion-free nilpotent.*

Proof. Let $N = \langle x, y, z\ ;\ z = [y, x], [z, x] = [z, y] = 1\rangle,\ A = \langle a\rangle$.

Denote the free metabelian product of N and A by

$$P = (N * A)/(N * A)'' = \langle x, y, z, a\ ;\ z = [y, x], [z, x] = [z, y] = 1\rangle_{\mathcal{A}^2}\ .$$

Let $M = \mathrm{gp}(x, a) = \langle x, a\rangle_{\mathcal{A}^2}$, which is a free metabelian group of rank 2. Let $K = \mathrm{gp}_P(y)$. Then $K = \mathrm{gp}(y, [y, a]^{a^i x^j}, z^{a^l} (i, j, l \in \mathbb{Z}))$. We have the following short exact sequence:

$$1 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 1.$$

Now, M acts on K by conjugation, and the sequence splits. Thus,

$$P = K \rtimes M.$$

It is easy to show that $[x^i, z^{a^n}] = 1, [y^i, z^{a^n}] = 1\ (i, j, n \in \mathbb{Z})$. To learn more about K, let

$$R = (\langle \tilde{y}, z_l\ (l \in \mathbb{Z}), w_{i,j}(i, j \in \mathbb{Z}); [z_i, z_j], [w_{i,j}, z_k], [w_{i,j}, w_{k,l}], [z_i, \tilde{y}]\rangle \rtimes \langle \tilde{x}, \tilde{a}\rangle)_{\mathcal{A}^2}$$

where the actions of $\tilde{x}$ and $\tilde{a}$ are defined by:

$$\begin{aligned}
\tilde{y}^{\tilde{x}} &= \tilde{y}z_0 \\
z_l^{\tilde{x}} &= z_l \\
w_{ij}^{\tilde{x}} &= w_{i.j+1} \\
\tilde{y}^{\tilde{a}} &= \tilde{y}w_{00} \\
z_l^{\tilde{a}} &= z_{l+1} \\
w_{ij}^{\tilde{a}} &= w_{i+1,j}\, .
\end{aligned}$$

By von Dyck's Lemma (1.4), the mapping φ defined on the generators of P by

$$\begin{aligned}
\varphi : x &\mapsto \tilde{x} \\
\varphi : y &\mapsto \tilde{y} \\
\varphi : z &\mapsto z_0 \\
\varphi : a &\mapsto \tilde{a}
\end{aligned}$$

extends to a homomorphism of P onto R. Note that R is generated by $\{\tilde{x}, \tilde{y}, z_0, \tilde{a}\}$; hence φ is an epimorphism.

Now, the image of the subgroup K under φ is the subgroup of R generated by the images of the generators of K:

$$y \mapsto \tilde{y} \qquad [y,a]^{a^i x^j} \mapsto w_{ij} \qquad z^{a^l} \mapsto z_l \qquad (i,j,l \in \mathbb{Z}).$$

The set $\{z_l \,|\, l \in \mathbb{Z}\} \subset R$ freely generates a free abelian group of infinite rank; hence, so does its pre-image $\{z^{a^l} \,|\, l \in \mathbb{Z}\} \subset K$. Denote z^{a^l} by z_l. Note that $\mathrm{gp}(z_l \,|\, l \in \mathbb{Z}) \le \zeta(K)$.

Likewise, the set $\{w_{i,j} \,|\, i,j \in \mathbb{Z}\} \subset R$ freely generates a free abelian group of infinite rank; hence, so does its pre-image $\{[y,a]^{a^i x^j} \,|\, i,j \in \mathbb{Z}\} \subset K$. Denote $[y,a]^{a^i x^j}$ by $w_{i,j}$. Let

$$\begin{aligned}
A_0 &= \mathrm{gp}(w_{i,j}\ (i,j \in \mathbb{Z})) \le K, \\
A_k &= \mathrm{gp}(w_{i,j}^{y^k}\ (i,j \in \mathbb{Z})) \le K, \\
A &= \prod_{k \in \mathbb{Z}} A_k .
\end{aligned}$$

Then $\mathrm{gp}(y, A_0)$ is a wreath product, with:

$$A_k^y = A_{k+1}$$

and we have

$$P = K \rtimes M = ((A_0 \wr \langle y \rangle) \times \mathrm{gp}(z_l\ (l \in \mathbb{Z}))) \rtimes M.$$

Suppose $1 \neq w \in P$. The goal is to find a torsion-free nilpotent group L, and a homomorphism from P to L such that w has a nontrivial image in L.

We can write w uniquely as

$$w = mk \quad (m \in M, k \in K).$$

Case 1: $m \neq 1$. Let $\varphi : P \to P/K \cong M$ be the canonical homomorphism, with $\varphi : mk \mapsto \bar{m}$. Since M is itself residually torsion-free nilpotent, there is a homomorphism $\theta : M \to L$ to a torsion-free nilpotent group L which maps $\bar{m}$ nontrivially. The composite homomorphism $\varphi\theta : P \to L$ maps w nontrivially.

Case 2: $m = 1$. Then $w \in K$, so w is a nontrivial product whose factors could include powers of y, a finite number of powers of distinct $w_{i,j}^{y^k}$'s, and a finite number of powers of distinct z_l's. We can assume with no loss of generality that $i, j, k, l \geq 0$, for if a conjugate of w has a nontrivial image under a given homomorphism, so will w have a nontrivial image. Hence, suppose that:

- the i's are all integers from the interval 0 to n;
- the j's are all integers from the interval 0 to p;
- the k's are all integers from the interval 0 to q;
- the l's are all integers from the interval 0 to m.

Our first attempt for L is

$$L_1 = H \rtimes T$$

with

$$\begin{aligned} T &= \langle r, t \rangle_{\mathcal{A}^2} \\ H &= \operatorname{gp}(s, \beta_{ijk}(i = 0, \ldots, n, j = 0, \ldots, p, k = 0, \ldots, q), c_l(l = 0, \ldots, m)); \end{aligned}$$

where the β_{ijk}'s commute with one another in H, and additionally:

$$\begin{aligned} \beta_{ijq} &\in \zeta(H) \ (i = 0, \ldots, n, j = 0, \ldots, p) \\ c_l &\in \zeta(H) \ (l = 0, \ldots, m) \\ \beta_{ijk}^s &= \beta_{ijk}\beta_{ij,k+1} \ (i = 0, \ldots, n, j = 0, \ldots, p, k = 0, \ldots, q-1) \end{aligned}$$

and T acts on H as follows:

$$\begin{aligned} s^t &= s\beta_{000} \\ \beta_{ijk}^t &= \beta_{ijk}\beta_{i+1,jk} \ (i = 0, \ldots, n-1, j = 0, \ldots, p, k = 0, \ldots, q) \\ \beta_{njk}^t &= \beta_{njk} \ (j = 0, \ldots, p, k = 0, \ldots, q) \\ c_l^t &= c_l c_{l+1} \ (l = 0, \ldots, m-1) \\ c_m^t &= c_m \\ s^r &= sc_0 \\ \beta_{ijk}^r &= \beta_{ijk}\beta_{i,j+1,k} \ (i = 0, \ldots, n, j = 0, \ldots, p-1, k = 0, \ldots, q) \\ \beta_{ipk}^r &= \beta_{ipk} \ (i = 0, \ldots, n, k = 0, \ldots, q) \\ c_l^r &= c_l \ (l = 0, \ldots, m). \end{aligned}$$

All of the β_{ijk} can be written as words in $\{r, s, t\}$:

$$\begin{aligned}
\beta_{000} &= [s,t] && (2)\\
\beta_{i+1,jk} &= [\beta_{ijk}, t]\ (i=0,\dots,n-1, j=0,\dots,p, k=0,\dots,q) && (3)\\
\beta_{i,j+1,k} &= [\beta_{ijk}, r]\ (i=0,\dots,n, j=0,\dots,p-1, k=0,\dots,q) && (4)\\
\beta_{ij,k+1} &= [\beta_{ijk}, s]\ (i=0,\dots,n, j=0,\dots,p, k=0,\dots,q-1) && (5)\\
c_0 &= [s,r] && (6)\\
c_{l+1} &= [c_l, t]\ (l=0,\dots,m-1). && (7)
\end{aligned}$$

Hence, $L_1 = \mathrm{gp}(r, s, t)$.

Lemma 2.2. *H is $(q+1)$-nilpotent.*

Proof. H can be written as a semi-direct product of an abelian group by an infinite cyclic group:

$$H = \langle \beta_{ijk}(i=0,\dots,n, j=0,\dots,p, k=0,\dots,q), c_l(l=0,\dots,m)\rangle_{\mathcal{A}} \rtimes \langle s\rangle$$

where s acts trivially on the c_l's but not on the β_{ijk}'s, unless $k=q$. Thus, by (5)

$$\begin{aligned}
\gamma_2(H) =&\ \mathrm{gp}(\beta_{ijk} | i=0,\dots,n,\ j=0,\dots,p,\ k=1,\dots,q)\\
\gamma_3(H) =&\ \mathrm{gp}(\beta_{ijk} | i=0,\dots,n,\ j=0,\dots,p,\ k=2,\dots,q)\\
&\vdots\\
\gamma_q(H) =&\ \mathrm{gp}(\beta_{ij,q-1}, \beta_{ijq} | i=0,\dots,n,\ j=0,\dots,p)\\
\gamma_{q+1}(H) =&\ \mathrm{gp}(\beta_{ijq} | i=0,\dots,n,\ j=0,\dots,p) \le \zeta(H)\\
\gamma_{q+2}(H) =&\ 1.
\end{aligned}$$

□

Lemma 2.3. *w has a nontrivial image under the homomorphism $\theta : P \to L_1$ which maps the generators as follows:*

$$\begin{aligned}
\theta : a &\mapsto t\\
\theta : x &\mapsto r\\
\theta : y &\mapsto s
\end{aligned}$$

Proof. Lemma 1.6 along with von Dyck's Lemma and straightforward applications of the commutator identities confirm that θ is a homomorphism.

We start by examining the images of the factors of w.

$$\begin{aligned}
y &\mapsto s\\
w_{i,j}^{y^k} = [y,a]^{a^i x^k y^k} &\mapsto [s,t]^{t^i r^j s^k} = \beta_{000}^{t^i r^j s^k} \quad (2)\\
z_l = [y,x]^{a^l} &\mapsto [s,r]^{t^l} = c_0^{t^l} \quad (6)
\end{aligned}$$

If w contains a power of y as a factor, then $w\theta$ contains the same power of s, and the image is nontrivial.

If w contains no power of y as a factor, then w is a finite number of powers of distinct $w_{i,j}^{y^k}$'s, and a finite number of powers of distinct z_l's. If there is at least one $w_{ij}^{y^k}$ factor in w, let

$$\begin{aligned} K = & \max\left\{k \mid \beta_{000}^{t^i r^j s^k} \text{ appears in } w\theta\right\} \\ J = & \max\left\{j \mid \beta_{000}^{t^i r^j s^K} \text{ appears in } w\theta\right\} \\ I = & \max\left\{i \mid \beta_{000}^{t^i r^J s^K} \text{ appears in } w\theta\right\}. \end{aligned}$$

Denote the power of $\beta_{000}^{r^I t^J s^K}$ by α, a nonzero integer. Although there is likely to be some cancelling of β_{ijk}'s after all the actions have been consummated, calculations show that the actions on the factor $(\beta_{000}^{r^I t^J s^K})^\alpha$ generate a β_{IJK}^α, which can appear nowhere else in the image; hence, will not cancel, and the image is nontrivial.

If there are only z_l's in w, the image can be seen to be nontrivial by considering

$$L = \max\left\{l \mid c_0^{t^l} \text{ appears in } w\theta\right\}.$$

$c_0^{t^L}$ generates a c_L which can appear nowhere else in the image; hence will not cancel, and the image is nontrivial. ☐

L_1 is promising, but it has a serious problem: L_1 contains T as a subgroup, a free metabelian group of rank 2; hence, it is not nilpotent. We need to modify T in such a way that it still acts in the same way on H, that the image of w under θ is still nontrivial, but that our target group is nilpotent.

We begin by examining some commutators of L_1.

Let $\nu = \max\{m, n+1\} + 1$.

Lemma 2.4. $[r, \overbrace{t, \dots, t}^{\nu}, s] = 1$.

Proof. The Jacobi Identity for metabelian groups (1) is useful here: in our group L_1,

$$[r, t, s][t, s, r][s, r, t] = 1$$

$$\text{or,} \qquad \begin{aligned} [r, t, s] &= [t, [s, r]\,]\,[r, [t, s]\,] \\ &= [t, c_0]\,[r, \beta_{000}^{-1}] \\ &= [c_0, t]^{-1}[r, \beta_{000}]^{-1} \\ &= c_1^{-1}\beta_{010} \quad (4), (7) \\ &= \beta_{010}c_1^{-1} \end{aligned}$$

$$\begin{aligned}\text{Hence,}\quad [r,t,t,s] &= [r,t,s,t] \quad \text{(Lemma 1.7)}\\ &= [\beta_{010}c_1^{-1},t]\\ &= [\beta_{010},t]\,[c_1,t]^{-1}\\ &= \beta_{110}c_2^{-1} \quad (3),(7).\end{aligned}$$

Inductively, for $i < min\{m, n+1\}$, we get

$$[r,\overbrace{t,\ldots,t}^{i},s] = [\beta_{i-2,10},t][c_{i-1},t]^{-1} = \beta_{i-1,10}c_i^{-1}.$$

Recall that t acts trivially on both c_m and β_{n10}. Suppose, for definiteness, that $m \le n+1$, so $\nu = n+2$.

$$\begin{aligned}[r,\overbrace{t,\ldots,t}^{m+1},s] &= [\beta_{m-1,10},t]\,[c_m,t]^{-1}\\ &= \beta_{m10}\\ &\vdots\\ [r,\overbrace{t,\ldots,t}^{n+1},s] &= \beta_{n10}\\ [r,\overbrace{t,\ldots,t}^{\nu},s] &= 1.\end{aligned}$$

The case where $m > n+1$ follows similarly. □

Lemma 2.5. $[r,t,\overbrace{r,\ldots,r}^{p},s] = 1$.

Proof. Starting again from the Jacobi Identity for metabelian groups, and recalling that s acts trivially on β_{0p0}, we get:

$$\begin{aligned}[r,t,s] &= \beta_{010}c_1^{-1}\\ [r,t,s,r] &= [\beta_{010},r][c_1,r]^{-1}\\ &= \beta_{020} \quad (4)\\ &\vdots\\ [r,t,\overbrace{r,\ldots,r}^{p-1},s] &= \beta_{0p0}\\ [r,t,\overbrace{r,\ldots,r}^{p},s] &= 1.\end{aligned}$$

□

Lemma 2.6. *$\gamma_{\nu+p+1}(T)$ acts trivially on H.*

Proof. By Lemmas 1.7 and 1.8, any nontrivial commutator in T of weight $\nu{+}p{+}1$ can be written either in the form

$$[r, \overbrace{t, \ldots, t}^{d}, \overbrace{r, \ldots, r}^{\nu+p-d}]^{\pm 1} \quad (1 \leq d \leq \nu + p)$$

or

$$[r, t, \overbrace{r, \ldots, r}^{\nu+p-d}, \overbrace{t, \ldots, t}^{d-1}]^{\pm 1}.$$

Now, either $d \geq \nu$ or $d < \nu$; i.e., $d \geq \nu$ or $(\nu{+}p{-}d) > p$. Suppose for definiteness that $d \geq \nu$. Then

$$\begin{aligned} s^{[r,t,\ldots,t,r,\ldots,r]^{\pm 1}} &= s[r, \overbrace{t, \ldots, t}^{d}, \overbrace{r, \ldots, r}^{\nu+p-d}, s]^{\pm 1} \\ &= s[r, \overbrace{t, \ldots, t}^{\nu}, s, \overbrace{t, \ldots, t}^{d-\nu}, \overbrace{r, \ldots, r}^{\nu+p-d}]^{\pm 1} \\ &= s \quad \text{(Lemma 2.4)}. \end{aligned}$$

If $d < \nu$, we switch the order of the last $(d-1)$ t's with the $(\nu + p - d)$ r's, and proceed similarly to the same result, this time using Lemma 2.5. □

Thus, the action of T on H factors through $T/\gamma_{\nu+p+1}(T)$, and we define our target group

$$L = H \rtimes T/\gamma_{\nu+p+1}(T).$$

We abuse notation and again use θ to describe the homomorphism $\theta : P \to L$ which is defined by the same generator maps as before. The image of w is still nontrivial under this mapping. It remains to show that L is torsion-free nilpotent.

Let $\mu = \max\{m, n + p + q\}$. Define a sequence

$$1 = B_0 \leq B_1 \leq \ldots \leq B_{\mu+2} = H,$$

where

$$\begin{aligned} B_1 &= \operatorname{gp}(\beta_{ijk}, c_l \mid l, (i + j + k) \geq \mu) \\ B_2 &= \operatorname{gp}(\beta_{ijk}, c_l \mid l, (i + j + k) \geq \mu - 1) \\ &\vdots \\ B_\mu &= \operatorname{gp}(\beta_{ijk}, c_l \mid l, (i + j + k) \geq 1) \\ B_{\mu+1} &= \operatorname{gp}(\beta_{ijk}, c_l \mid l, (i + j + k) \geq 0) \\ B_{\mu+2} &= H. \end{aligned}$$

From the actions of r, s and t on β_{ijk} and on c_l, we see that $B_1, \ldots, B_{\mu+1}$ are all normal in L; also H is normal in L.

Further,

$$\begin{aligned}[H,B_\iota] &= \mathrm{gp}([h,b]\,|\,h\in H, b\in B_\iota)\\ &= \mathrm{gp}([s,\beta_{ijk}],[s^{-1},\beta_{ijk}]\,|\,i+j+k\geq\mu-\iota+1)\\ &= \mathrm{gp}(\beta^{-1}_{ij,k+1},\beta_{ij,k+1}\cdots\beta^{\pm1}_{ijq}\,|\,i+j+k\geq\mu-\iota+1)\\ &\leq B_{\iota-1}\,.\end{aligned}$$

Recalling the action of T on H, it is easy to see that T, and hence $T/\gamma_{\nu+p+1}(T)$, acts trivially on each B_i modulo B_{i-1}. Hence, by Lemma 1.5, L is nilpotent. □

Proof of Theorem 1.1. Suppose now that P is a free metabelian product of a free nilpotent group of class 2 with a free abelian group of countable rank. We will show that P is residually a free metabelian product of a free nilpotent group of class 2 and rank 2 with an infinite cyclic group. It follows from Lemma 2.1, then, that P is residually torsion-free nilpotent.

A nontrivial word w in P is a product made up from a finite set S of non-trivial words in N, a free nilpotent group of class 2, and a finite set R of non-trivial words in A, a free abelian group of countable rank; with the words from S strictly alternating with the words in R. At least one of R and S must be nonempty.

It is known that a free abelian group is fully residually an infinite cyclic group and that a free 2-nilpotent group is fully residually a free 2-nilpotent group of rank 2 (see, for example, [6]).

Now, $(\tilde{N}*\tilde{A})''\cong(N*A)''\ker\theta$; thus, care must be taken to avoid "losing" w to the metabelian relator. We expand both R and S to include every distinct *segment* of any word in the original R and S respectively. Observe that both sets are still finite.

Hence, there is a map (say $\theta_1 : A\longrightarrow\tilde{A}$) taking the elements of R distinctly and nontrivially to words in $\tilde{A}$, an infinite cyclic group, and a map (say $\theta_2 : N\longrightarrow\tilde{N}$) taking the elements of S distinctly and nontrivially to words in $\tilde{N}$.

Since every segment of every word in the original sets R and S is included, then we can map w nontrivially to the free metabelian product $\tilde{N}*_{\mathcal{A}^2}\tilde{A}$ via the homomorphism θ that combines θ_1 and θ_2. □

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