

The Tits Alternative for Tsaranov's Generalized Tetrahedron Groups

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A generalized tetrahedron group is defined to be a group admitting the following presentation: $\langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = W_2^q(y, z) = W_3^r(x, z) = 1 \rangle$, $2 \leq l, m, n, p, q, r$, where each $W_i(a, b)$ is a cyclically reduced word involving both a and b . These groups appear in many contexts, not least as fundamental groups of certain hyperbolic orbifolds or as subgroups of generalized triangle groups. In this paper, we build on previous work to show that the Tits alternative holds for Tsaranov's generalized tetrahedron groups, that is, if G is a Tsaranov generalized tetrahedron group then G contains a non-abelian free subgroup or is solvable-by-finite. The term *Tits alternative* comes from the respective property for finitely generated linear groups over a field (see [10]).

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1. Introduction

If T is a tetrahedron in 3-dimensional Euclidean, hyperbolic or spherical space whose dihedral angles are submultiples of π , then the reflections in the faces of T generate a discrete group of isometries. The index 2 subgroup of orientation-preserving isometries in this group is generated by reflections around the edges of any of the faces of T , and has a presentation of the form

$$\langle x, y, z \mid x^l = y^m = z^n = (xy^{-1})^p = (yz^{-1})^q = (zx^{-1})^r = 1 \rangle, \quad (1)$$

where $2 \leq l, m, n, p, q, r$. These groups are called *ordinary tetrahedron groups*. Coxeter has shown in [1] that an ordinary tetrahedron group of the form (1) is finite if and only if the Coxeter matrix

$$C = \begin{pmatrix} 1 & -\cos(\frac{\pi}{l}) & -\cos(\frac{\pi}{m}) & -\cos(\frac{\pi}{n}) \\ -\cos(\frac{\pi}{l}) & 1 & -\cos(\frac{\pi}{p}) & -\cos(\frac{\pi}{r}) \\ -\cos(\frac{\pi}{m}) & -\cos(\frac{\pi}{p}) & 1 & -\cos(\frac{\pi}{q}) \\ -\cos(\frac{\pi}{n}) & -\cos(\frac{\pi}{r}) & -\cos(\frac{\pi}{q}) & 1 \end{pmatrix} \quad (2)$$

has positive determinant. Following Vinberg we call a group G defined by a presentation

$$G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = W_2^q(y, z) = W_3^r(x, z) = 1 \rangle, \quad (3)$$

$2 \leq l, m, n, p, q, r$, where each $W_i(a, b)$ is a cyclically reduced word involving both a and b , a *generalized tetrahedron group*. For the following we may always assume that each $W_i(a, b)$ also is not a proper power in the free product on a and b (that only would increase the exponents). A generalized tetrahedron group G is called a Tsaranov generalized tetrahedron group if, in addition, $W_1(x, y) = x^\alpha y^\beta$, $W_2(y, z) = y^\gamma z^\delta$, $W_3(x, z) = x^\varepsilon z^\zeta$, with $1 \leq \alpha, \varepsilon \leq l$, $1 \leq \beta, \gamma \leq m$, $1 \leq \delta, \zeta \leq n$.

Certain operations on presentations of this form (3) do not change the groups defined by the presentations. With this in mind, we say that two presentations P_1 and P_2 of the form (3) are equivalent if P_2 can be obtained from P_1 by a sequence of operations of the following type:

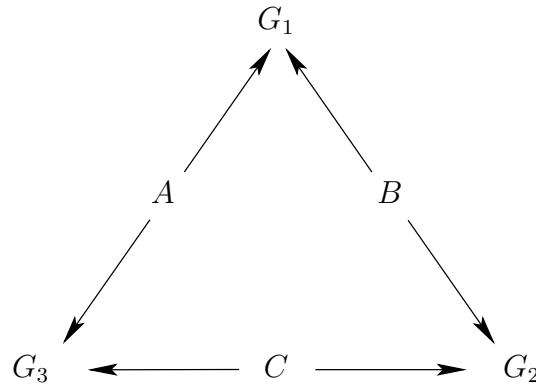
1. Replace a generator a of order k by a new generator $d = a^\alpha$, where α is coprime to k , and then amend the relations accordingly.
2. Apply a permutation to the generators x, y and z .
3. If $V_i(a, b)$ is a cyclically reduced conjugate of $W_i(a, b)$ in the free product on a and b , then replace the relator $W_i^{k_i}(a, b)$ by $V_i^{k_i}(a, b)$, where $k_i \in \{p, q, r\}$, respectively.
4. Replace the relator $W_i^{k_i}(a, b)$ by $V_i^{k_i}(a, b)$, where $V_i(a, b)$ is the inverse of $W_i(a, b)$ and $k_i \in \{p, q, r\}$, respectively.
5. If a is a generator of order 2, if b is a generator of order k , if α and β are coprime to k , and if we have a relator of the form $W = (ab^\alpha)^\beta$, then replace W by $(ab^\beta)^\alpha$.

It is clear that, if P_1 and P_2 are equivalent, then P_1 and P_2 define the same group. In the following we often replace a presentation of the form (3) by an equivalent one and work in fact up to equivalence. The purpose of this paper is to prove the following

Theorem 1.1. *Let G be a Tsaranov generalized tetrahedron group. Then G satisfies the Tits alternative, that is, G contains a non-abelian free subgroup or is solvable-by-finite.*

2. Preliminary results

For the benefit of the reader, we will list here some preliminary definitions and results we will need in this paper. Suppose that G is defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = W_2^q(y, z) = W_3^r(x, z) = 1 \rangle$ of the form (3). Let (after conjugation if necessary) $W_1(x, y) = x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_{k_1}} y^{\beta_{k_1}}$ with $k_1 \geq 1$, $1 \leq \alpha_i < l$, $1 \leq \beta_i < m$ for each i . We similarly take k_2 and k_3 to be half the length of the cyclically reduced words $W_2(y, z)$ and $W_3(x, z)$ respectively. G can be realized as a triangle of groups, that is, as the colimit of the diagram of groups and injective homomorphisms shown in the figure



in which

$$\begin{aligned} G_1 &= \langle x, y \mid x^l = y^m = W_1^p(x, y) = 1 \rangle, \\ G_2 &= \langle y, z \mid y^m = z^n = W_2^q(y, z) = 1 \rangle, \\ G_3 &= \langle x, z \mid x^l = z^n = W_3^r(x, z) = 1 \rangle, \end{aligned}$$

$A = \langle x \mid x^l = 1 \rangle$, $B = \langle y \mid y^m = 1 \rangle$ and $C = \langle z \mid z^n = 1 \rangle$. We refer G_1 , G_2 and G_3 as vertex groups and A, B, C as edge groups. Groups with a presentation as G_1 , G_2 and G_3 are called *generalized triangle groups*. Using the improved Spelling Theorem for generalized triangle groups Howie and Kopteva [8] were able to show the following.

Theorem 2.1. *Let G be a generalized tetrahedron group of the form (3) as above.*

- a) *If $1/pk_1 + 1/qk_2 + 1/rk_3 < 1$ then G contains a non-abelian free subgroup.*
- b) *If $1/pk_1 + 1/qk_2 + 1/rk_3 = 1$ then G contains a non-abelian free subgroup except in the case of (up to equivalence)*

$$\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = 1 \rangle$$

with $1/p + 1/q + 1/r = 1$, where G is abelian-by-finite.

Hence, the Tits alternative holds, if the triangle of groups for G is negatively curved or Euclidean. Therefore we are left with the spherical cases (up to equivalence)

$$S1 : \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = (y^\gamma z^\delta)^2 = (x^\epsilon z^\zeta)^2 = 1 \rangle, \quad p \geq 2.$$

- $S2 : \langle x, y, z \mid x^l = y^m = z^n = (x^\alpha y^\beta)^p = (y^\gamma z^\delta)^q = (x^\varepsilon z^\zeta)^r = 1 \rangle$, $p \geq 3$, and $q \geq 3$, $r = 2$ with $1/p + 1/q > 1/2$ or $q = 2$, $r \geq 3$ with $1/p + 1/r > 1/2$.
- $S3 : \langle x, y, z \mid x^l = y^m = z^n = (x^{\alpha_1} y^{\beta_1} x^{\alpha_2} y^{\beta_2})^2 = (y^\gamma z^\delta)^q = (x^\varepsilon z^\zeta)^r = 1 \rangle$, $(q, r) = (2, 3)$ or $(q, r) = (3, 2)$.

We also need several preliminary results about linear representations. Let G be a generalized tetrahedron group given by a presentation of the form (3). If L is a linear group and $G \rightarrow L$ is a representation of G in L , we say that ρ is *essential* if the elements $\rho(x)$, $\rho(y)$, $\rho(z)$, $\rho(W_1(x, y))$, $\rho(W_2(y, z))$ and $\rho(W_3(x, z))$ have orders l, m, n, p, q, r respectively in $\rho(G)$. In this case, we also have essential representations of the three generalized triangle groups G_1 , G_2 and G_3 , defined above as the vertex groups of the triangle of groups for G . In the following we use the notation G_1 , G_2 and G_3 for these vertex groups for G .

Theorem 2.2 ([6]). *Every generalized tetrahedron group admits an essential representation in $\text{PSL}(2, \mathbb{C})$.*

The next Theorem (see [7] and [3]) is very useful to prove Theorem 1.1.

Theorem 2.3 (Fortsetzungssatz). *Let G be the generalized tetrahedron group defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = W_2^q(y, z) = W_3^r(x, z) = 1 \rangle$ and let G_1 be the generalized triangle group defined by the presentation $G_1 = \langle x, y \mid x^l = y^m = W_1^p(x, y) = 1 \rangle$. Suppose that ρ_1 is an essential representation of G_1 into $\text{PSL}(2, \mathbb{C})$ with $X = \rho_1(x)$ and $Y = \rho_1(y)$ and that one of the following two possibilities occurs:*

- (1) $\text{tr}([X, Y]) \neq 2$;
- (2) $(n, q, r) \neq (2, 2, 2)$ and $\langle X, Y \rangle$ is an infinite metabelian subgroup of $\text{PSL}(2, \mathbb{C})$.

Then there is an essential representation $\rho : G \rightarrow \text{PSL}(2, \mathbb{C})$ such that $X = \rho(x)$ and $Y = \rho(y)$.

Moreover, if in case (1) the group $\langle X, Y \rangle$ is non-elementary, then G contains a non-abelian free subgroup in both cases.

Remark 2.4. A subgroup of $\text{PSL}(2, \mathbb{C})$ is said to be non-elementary if it is not solvable-by-finite; such a subgroup must contain a free subgroup of rank 2.

If G_1 is as in Theorem 2.3 and $\langle X, Y \rangle$ is an abelian group other than the elementary abelian group of order 4, then there always exists an essential representation $\sigma : G_1 \rightarrow \text{PSL}(2, \mathbb{C})$ such that $\langle \sigma(x), \sigma(y) \rangle$ is an infinite metabelian group.

From this we get the following extensions for which we have to look at certain special cases for G_1 . A proof is given in [7] together with a little correction in [2].

Theorem 2.5. *Let G be the generalized tetrahedron group defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = W_2^q(y, z) = W_3^r(x, z) = 1 \rangle$ and suppose $l \leq m$ and that $W_1(x, y) = x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_{k_1}} y^{\beta_{k_1}}$, $1 \leq \alpha_i < l$, $1 \leq \beta_i < m$ for all i , where $k_1 \geq 2$ and $W_1(x, y)$ is not a proper power in the free product on x and y . Assume further that one of the following holds*

- 1) $m \geq 4$ and $p \geq 3$;
- 2) $p \geq 4$;
- 3) $l \geq 3$ and $p \geq 3$.

Then G contains a non-abelian free subgroup.

Theorem 2.6. *Let G be the generalized tetrahedron group defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = W_2^q(y, z) = W_3^r(x, z) = 1 \rangle$ and suppose $l \leq m$ and that $W_1(x, y) = x^{\alpha_1}y^{\beta_1} \dots x^{\alpha_{k_1}}y^{\beta_{k_1}}$, $k_1 \geq 1$ and $1 \leq \alpha_i < l$, $1 \leq \beta_i < m$ for all i . Suppose that $1/l + 1/m + 1/p < 1$ and $(n, q, r) \neq (2, 2, 2)$. Then G contains a non-abelian free subgroup.*

Theorems 2.5 and 2.6 hold in a symmetric manner for the vertex groups G_2 and G_3 also.

Using the above preliminary results we were able to prove the following.

Theorem 2.7 ([4]). *Let G be a generalized tetrahedron group given by a presentation of the form (3). If $(p, q, r) \neq (2, 2, 2)$ then G satisfies the Tits alternative, that is, G contains a non-abelian free subgroup or is solvable-by-finite.*

From Theorem 2.7 and the result of Howie and Kopteva now the Tits alternative holds unless G has a presentation (up to equivalence)

$$G = \langle x, y, z \mid x^l = y^m = z^n = W_1^2(x, y) = (y^\gamma z^\delta)^2 = (x^\varepsilon z^\zeta)^2 = 1 \rangle, \quad (4)$$

with $W_1(x, y) = x^{\alpha_1}y^{\beta_1} \dots x^{\alpha_k}y^{\beta_k}$, $k \geq 1$, $l, m, n \geq 2$; $1 \leq \alpha_1, \dots, \alpha_k, \varepsilon < l$, $1 \leq \beta_1, \dots, \beta_k, \gamma < m$ and $1 \leq \delta, \zeta < n$. We may assume $2 \leq l \leq m$.

For this situation we have a preliminary result.

Theorem 2.8 ([7]). *Let G be a generalized tetrahedron group given by a presentation of the form 3 with $(p, q, r) = (2, 2, 2)$. Suppose $l \leq m$ and $1/l + 1/m < 1/2$. Then G has a free subgroup of rank two with the possible exceptions $n = 2$ and $(l, m) = (3, 8), (3, 10), (4, 5), (4, 6), (4, 8)$ or $(5, 6)$.*

Theorem 2.8 holds in a symmetric manner for the vertex groups G_2 and G_3 also.

We now would like to prove that the Tits alternative holds in general for $(p, q, r) = (2, 2, 2)$. Unfortunately, all the methods used in [4] do not work analogously in this case. The situation is much more difficult. Hence, to get an impression, we consider in the following Tsaranov's generalized tetrahedron groups.

3. Proof of the Main Theorem (1.1)

In the following let G be a Tsaranov generalized tetrahedron group given by a presentation

$$G = \langle x, y, z \mid x^l = y^m = z^n = (x^\alpha y^\beta)^p = (y^\gamma z^\delta)^q = (x^\varepsilon z^\zeta)^r = 1 \rangle, \quad (5)$$

with $2 \leq l, m, n, p, q, r$ and $1 \leq \alpha, \varepsilon < l$, $1 \leq \beta, \gamma < m$ and $1 \leq \delta, \zeta < n$. We may assume $l \leq m$.

In what follows we always use the known fact that the only finite subgroups of $\text{PSL}(2, \mathbb{C})$ are, up to isomorphism, the cyclic groups \mathbb{Z}_n for all $n \in \mathbb{N}$, the dihedral groups D_n for all $n \in \mathbb{N}$, the alternating groups A_k for $k = 4, 5$ and the symmetric group S_4 .

In many cases a generalized tetrahedron group G has in an obvious manner an ordinary tetrahedron group as a homomorphic image. If this image has a free subgroup of rank 2, then G also has one. For ordinary tetrahedron groups we have a criterion which only depends on the orders of the relators, that is, on the Coxeter Matrix C .

Theorem 3.1 ([4]). *Let G be an ordinary tetrahedron group, given by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = (xy^{-1})^p = (yz^{-1})^q = (zx^{-1})^r = 1 \rangle$, where $2 \leq l, m, n, p, q, r$. Let C be the Coxeter matrix (2).*

- (i) *If $\det C < 0$ then G has a non-abelian free subgroup.*
- (ii) *If $\det C = 0$ then G has a non-abelian free subgroup or is abelian-by-finite. In the latter case, G is isomorphic to the Euclidean group $\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^{f_1} = (yz)^{f_2} = (xz)^{f_3} = 1 \rangle$, where $f_1, f_2, f_3 \geq 2$ and $1/f_1 + 1/f_2 + 1/f_3 = 1$.*

Remark 3.2. We already know that G is finite if and only if $\det C > 0$.

Also, quite often we get in an obvious manner as homomorphic images groups of certain SN -type, that is, groups with a presentation $H = \langle a, b, c \mid a^{e_1} = b^{e_2} = c^{e_3} = R_1^{f_1}(a, b) = R_2^{f_2}(a, c) = 1 \rangle$, $2 \leq e_1, e_2, e_3, f_1, f_2$, where $R_i(x, y)$ is a cyclically reduced word in x, y involving both x and y for $i = 1, 2$ (see [7]).

Theorem 3.3. *Let H be as above. If at least one of e_2, e_3, f_1, f_2 is greater than 2 then H has a free subgroup of rank 2.*

From now on let G be a Tsaranov generalized tetrahedron group given by a presentation of the form 5. Let, without loss of generality, $l \leq m$. By Theorem 2.7 we know that G satisfies the Tits alternative if $(p, q, r) \neq (2, 2, 2)$. Hence, from now on let $(p, q, r) = (2, 2, 2)$.

Theorem 3.4. *Let G be a Tsaranov generalized tetrahedron group given by a presentation of the form 5 with $(p, q, r) = (2, 2, 2)$. Suppose $l \leq m$ and $1/l + 1/m < 1/2$. Then G has a free subgroup of rank 2.*

Proof. By Theorem 2.8 we only have to consider the cases $n = 2$ and $(l, m) = (3, 8), (3, 10), (4, 5), (4, 6), (4, 8)$ or $(5, 6)$.

- (i) Let $(l, m) = (3, 8)$.
Then, without loss of generality, $G = \langle x, y, z \mid x^3 = y^8 = z^2 = (xy^\beta)^2 = (y^\gamma z)^2 = (xz)^2 = 1 \rangle$ with $\beta \mid 8$ and $1 \leq \beta \leq 4$. Assume that there exists

an essential representation $\rho_1 : G_1 \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_1(G_1)$ elementary. Let $\rho_1(x) = X$ and $\rho_1(y) = Y$. We may assume that $\langle X, Y \rangle$ is cyclic (after factorisation if necessary). Then $(XY^\beta)^2 = X^2Y^{2\beta} = 1$ which contradicts $\text{gcd}(3, \beta) = 1$. Hence, there exists an essential representation $\rho_1 : G_1 \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_1(G_1)$ non-elementary. Hence, $\rho_1(G_1)$ has a free subgroup of rank 2, and so G has one by Theorem 2.3.

The Cases $(l, m) = (3, 10)$, $(4, 5)$ and $(5, 6)$ are analogous.

(ii) Let $(l, m) = (4, 6)$.

Then, without loss of generality, $G = \langle x, y, z \mid x^4 = y^6 = z^2 = (x^\alpha y^\beta)^2 = (y^\gamma z)^2 = (x^\varepsilon z)^2 = 1 \rangle$ with $\alpha \mid 4$, $1 \leq \alpha \leq 2$ and $\beta \mid 6$, $1 \leq \beta \leq 3$. The argument in case (i) works analogously for $\beta \neq 3$, because then $y^{2\beta}$ has order 3 but $x^{2\alpha}$ has order 1 or 2.

Now, let $\beta = 3$.

If $\alpha = 2$ then we introduce the relation $y^3 = 1$ and get a homomorphic image \bar{G} which contains a free subgroup of rank 2 by Theorem 3.3.

If $\alpha = 1$ then there exists an essential representation $\rho_1 : G_1 \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_1(G_1)$ non-elementary.

(iii) Let $(l, m) = (4, 8)$.

Then, without loss of generality, $G = \langle x, y, z \mid x^4 = y^8 = z^2 = (x^\alpha y^\beta)^2 = (y^\gamma z)^2 = (x^\varepsilon z)^2 = 1 \rangle$ with $\alpha \mid 4$, $1 \leq \alpha \leq 2$ and $\beta \mid 8$, $1 \leq \beta \leq 4$. G has the factor group $\bar{G} = \langle x, y, z \mid x^4 = y^8 = z^2 = (x^\alpha y^\beta)^2 = (yz)^2 = (xz)^2 = 1 \rangle$. Let H be the subgroup of \bar{G} generated by x and y . H has a presentation $H = \langle x, y \mid x^4 = y^8 = (x^\alpha y^\beta)^2 = 1 \rangle$, especially, H is a generalized triangle group. H contains a free subgroup of rank 2 since $1/8 + 1/4 + 1/2 < 1$ (see [7]). \square

Theorem 3.4 holds in a symmetric manner for the vertex groups G_2 and G_3 also. Hence, if we argue symmetrically, we now may always assume in the following $l \leq m$ and $1/l + 1/m \geq 1/2$, $1/l + 1/n \geq 1/2$, $1/m + 1/n \geq 1/2$.

Theorem 3.5. *Let G be a Tsaranov generalized tetrahedron group given by a presentation of the form 5 with $(p, q, r) = (2, 2, 2)$. Suppose $l \leq m$ and $1/l + 1/m = 1/2$. Then G has a free subgroup of rank 2 or is infinite solvable.*

Proof. Since $l \leq m$ we have $l = 3$, $m = 6$ and $2 \leq n \leq 3$ or $l = m = 4$ and $2 \leq n \leq 4$ because $1/m + 1/n \geq 1/2$.

(i) $l = 3$, $m = 6$ and $n = 3$. Then, without loss of generality and up to equivalence, we may assume that $G = \langle x, y, z \mid x^3 = y^6 = z^3 = (xy^\beta)^2 = (y^\gamma z)^2 = (xz)^2 = 1 \rangle$ with $1 \leq \beta \leq 3$ and $1 \leq \gamma \leq 5$. If $\beta = 2$ or 3 then there exists an essential representation $\rho_1 : G_1 \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_1(G_1)$ non-elementary, and G has a free subgroup of rank 2 by Theorem 2.3. Now, let $\beta = 1$. Then $G_1 = \langle x, y \mid x^3 = y^6 = (xy)^2 = 1 \rangle$, and G_1 has an essential representation $\rho_1 : G_1 \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_1(G_1)$ infinite metabelian. Hence, by Theorem 2.3, G has a free subgroup of rank 2 because $n = 3$.

(ii) $l = 3$, $m = 6$ and $n = 2$. Then, without loss of generality, $G = \langle x, y, z \mid$

$x^3 = y^6 = z^2 = (xy^\beta)^2 = (y^\gamma z)^2 = (xz)^2 = 1$ with $1 \leq \beta \leq 3$ and $1 \leq \gamma \leq 4$. If $\beta = 2$ or 3 then, as in case (i), there exists an essential representation $\rho_1 : G_1 \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_1(G_1)$ non-elementary, and G has a free subgroup of rank 2. Now, let $\beta = 1$. Up to equivalence, we may assume that $1 \leq \gamma \leq 3$. If $\gamma = 1$ then G is infinite solvable by Theorem 3.1 and $z = z^{-1}$. If $\gamma = 3$ then we introduce the relation $y^3 = 1$ and get a factor group which contains a free subgroup of rank 2. Now, let $\gamma = 2$. Then $G = \langle x, y, z \mid x^3 = y^6 = z^2 = (xy)^2 = (y^2z)^2 = (xz)^2 = 1 \rangle$. Let H be the subgroup of G generated by $a = x, b = y, c = zxz$ and $d = zyz$. H has index 2 in G and a presentation $H = \langle a, b, d \mid a^3 = b^6 = d^6 = b^2d^2 = (ab)^2 = (ad^{-1})^2 = 1 \rangle$. Let K be the subgroup of H generated by $u = a, v = bab^{-1}, w = d, r = bdb^{-1}$ and $s = b^2$. K has index 2 in H and a presentation $K = \langle u, v, w, r, s \mid u^3 = v^3 = w^6 = r^6 = s^3 = uvs = sw^2 = sr^2 = (uw^{-1})^2 = (vr^{-1})^2 = 1 \rangle$. K is a free product with amalgamation, $K = K_1 *_A K_2$ with $A = \langle u, v, s, w^2, r^2 \rangle, K_1 = \langle u, v, s, w \rangle$ and $K_2 = \langle u, v, s, r \rangle$. We have $wuw^{-1} = w^2u^{-1} \in A, wvw^{-1} = u \in A$ and $ws w^{-1} = w^{-2} = s = r^{-2} \in A$. Analogously, $wuw^{-1} \in A, wvw^{-1} \in A$ and $ws w^{-1} \in A$. Hence, A is a normal subgroup of K . Further, $A = \langle u, v, s, w^2, r^2 \rangle = \langle u, v \mid u^3 = v^3 = (uv)^3 = 1 \rangle$ is infinite solvable and $K/A = \langle w \mid w^2 = 1 \rangle * \langle r \mid r^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$ which is infinite solvable, too. Altogether, G is infinite solvable.

- (iii) $l = m = 4$ and $n = 4$. Then, $G = \langle x, y, z \mid x^4 = y^4 = z^4 = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^2 = (x^\varepsilon z^\zeta)^2 = 1 \rangle$. If $\alpha = 2$ then we introduce the relation $x^2 = 1$ and get as a factor group of SN -type with at least one exponent ≥ 3 or a free product $N_1 * N_2$ with $|N_2| \geq 3$. In each case we get a free subgroup of rank 2 (see Theorem 3.3). Analogously we may argue if $\beta = 2$. Hence, without loss of generality, let $\alpha = \beta = 1$. Then we use Theorem 2.3 to get a free subgroup of rank 2 (recall that $n = 4$).
- (iv) $l = m = 4$ and $n = 3$. Then, without loss of generality, $G = \langle x, y, z \mid x^4 = y^4 = z^3 = (x^\alpha y^\beta)^2 = (y^\gamma z)^2 = (x^\varepsilon z)^2 = 1 \rangle$. If $\gamma = 2$ then $G_2 = \langle y, z \mid y^4 = z^3 = (y^2z)^2 = 1 \rangle$ has an essential representation $\rho_2 : G_2 \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_2(G_2)$ non-elementary, and we may use Theorem 2.3 for G_2 instead of G_1 . Analogously we may argue if $\varepsilon = 2$. Hence, let $\gamma \neq 2$ and $\varepsilon \neq 2$. If $\alpha = 2$ or $\beta = 2$ we get, as in (iii), a factor group which has a free subgroup of rank 2. Now, let also $\alpha \neq 2$ and $\beta \neq 2$. Then we may use again Theorem 2.3 to get a free subgroup of rank 2 (recall that $n = 3$).
- (v) $l = m = 4$ and $n = 2$. Then $G = \langle x, y, z \mid x^4 = y^4 = z^2 = (x^\alpha y^\beta)^2 = (y^\gamma z)^2 = (x^\varepsilon z)^2 = 1 \rangle$. Up to equivalence, we may assume that $1 \leq \alpha, \beta, \gamma, \varepsilon \leq 2$. Let first $\alpha = 2$ or $\beta = 2$. Then G has the factor group $\bar{G} = \langle x, y, z \mid x^4 = y^4 = z^2 = (x^\alpha y^\beta)^2 = (yz)^2 = (xz)^2 = 1 \rangle$. Let H be the subgroup of \bar{G} generated by x and y . H has a presentation $H = \langle x, y \mid x^4 = y^4 = (x^\alpha y^\beta)^2 = 1 \rangle$, and H has a free subgroup of rank 2 (see [7]). Now, let $\alpha = \beta = 1$. If $\gamma = 2$ we introduce the relation $y^2 = 1$ and apply Theorem 3.3 to get a free subgroup of rank 2. Analogously we

may argue if $\varepsilon = 2$. Hence, finally, let $\alpha = \beta = \gamma = \varepsilon = 1$. Then we have an ordinary tetrahedron group and apply Theorem 3.1. \square

Theorem 3.5 holds in a symmetric manner for the vertex groups G_2 and G_3 also. Hence, if we argue symmetrically, from now on we may always assume that $l \leq m$ and $1/l + 1/m > 1/2$, $1/l + 1/n > 1/2$, $1/m + 1/n > 1/2$. Up to equivalence, we have to consider the cases $(l, m, n) = (3, 3, 5), (3, 3, 4), (3, 3, 3), (3, 3, 2), (3, 4, 2), (3, 5, 2)$ and $(2, 2, n)$.

Theorem 3.6. *Let G be a Tsaranov generalized tetrahedron group given by a presentation of the form 5 with $(p, q, r) = (2, 2, 2)$. Suppose $l \leq m$ and $1/l + 1/m > 1/2$, $1/l + 1/n > 1/2$, $1/m + 1/n > 1/2$. Then G is finite, infinite solvable or has a free subgroup of rank 2.*

Proof. If G is equivalent to an ordinary tetrahedron group then we may apply Theorem 3.1. Tsaranov (see [11] and [12]) classified those finite G which are not equivalent to a presentation for an ordinary tetrahedron group. Up to equivalence, they are given by the following list:

- (a) $G = \langle x, y, z \mid x^3 = y^3 = z^3 = (xy)^2 = (yz)^2 = (xz)^2 = 1 \rangle$ with $|G| = 60$,
- (b) $G = \langle x, y, z \mid x^3 = y^3 = z^4 = (xy)^2 = (yz)^2 = (xz)^2 = 1 \rangle$ with $|G| = 192$,
- (c) $G = \langle x, y, z \mid x^3 = y^3 = z^5 = (xy)^2 = (yz)^2 = (xz)^2 = 1 \rangle$ with $|G| = 7200$,
- (d) $G = \langle x, y, z \mid x^3 = y^3 = z^5 = (xy)^2 = (yz^2)^2 = (xz)^2 = 1 \rangle$ with $|G| = 7200$,
- (e) $G = \langle x, y, z \mid x^3 = y^3 = z^5 = (xy)^2 = (yz^3)^2 = (xz)^2 = 1 \rangle$ with $|G| = 7200$.

Up to equivalence, this covers completely the cases $(l, m, n) = (3, 3, 5), (3, 3, 3), (3, 3, 2)$ and $(3, 5, 2)$. If we consider for the cases $(l, m, n) = (3, 4, 2)$ subgroups of index 2, if necessary, then this covers easily also the cases $(l, m, n) = (3, 3, 4)$ and $(3, 4, 2)$ with the exception that, up to equivalence, one of the vertex groups has the presentation $G_\nu = \langle a, b \mid a^3 = b^4 = (ab^2)^2 = 1 \rangle$. But G_ν has an essential representation $\rho_\nu : G_\nu \rightarrow \text{PSL}(2, \mathbb{C})$ with $\rho_\nu(G_\nu)$ non-elementary and we may apply Theorem 2.3.

Hence, finally we are left with the case $(l, m, n) = (2, 2, n)$. Then $G = \langle x, y, z \mid x^2 = y^2 = z^n = (xy)^2 = (yz^\delta)^2 = (xz^\zeta)^2 = 1 \rangle$. We have several cases to consider.

- (i) $\text{gcd}(\delta, n) = \text{gcd}(\zeta, n) = 1$. Then, up to equivalence, $\delta = \zeta = 1$, and we may apply Theorem 3.1.
- (ii) $\text{gcd}(\delta, n) = 1$ or $\text{gcd}(\zeta, n) = 1$. Let, without loss of generality, $\text{gcd}(\zeta, n) = 1$. Then we may assume $\zeta = 1$. If $\text{gcd}(\delta, n) = 1$ then we are in case (i). Let $\text{gcd}(\delta, n) = g \geq 2$. If $g \geq 3$ then we introduce the relation $z^g = 1$, and we get the factor group $\bar{G} = \langle x, y, z \mid x^2 = y^2 = z^g = (xy)^2 = (xz)^2 = 1 \rangle$ which has a free subgroup of rank 2 by Theorem 3.3.

Now, let $g = 2$. Then, without loss of generality $\delta = g = 2$ and $n = 2k$ with $\text{gcd}(\delta, k) = 1$. Then $G = \langle x, y, z \mid x^2 = y^2 = z^{2k} = (xy)^2 = (yz^2)^2 = (xz)^2 = 1 \rangle$. Let H be the subgroup of G generated by $a = yx$ and $b = z$. H has index 2 in G and a presentation $H = \langle a, b \mid a^2 = b^{2k} = ab^{-2}a^{-1}b^2 = 1 \rangle$. Let Z be the center of H , we have $Z = \langle b^2 \mid (b^2)^k = 1 \rangle$ and $H/Z = \langle a \mid a^2 =$

- 1). Hence, G is solvable. If we introduce in G the relation $z^2 = 1$ then we get the infinite factor group $\bar{G} = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^2 = (xz)^2 = 1 \rangle$, see [7]. Altogether, here G is infinite solvable.
- (iii) $\gcd(\delta, n) = g \geq 2$ and $\gcd(\zeta, n) = d \geq 2$. If $\gcd(g, d) = r \geq 2$ then we introduce the relation $z^r = 1$ and get the factor group $\bar{G} = \langle x, y, z \mid x^2 = y^2 = z^r = (xy)^2 = 1 \rangle$ which has a free subgroup of rank 2. Now, let $\gcd(g, d) = 1$. Then $d \geq 3$ or $g \geq 3$, and let $d \geq 3$. If we introduce the relation $z^d = 1$ then we get the factor group $\bar{G} = \langle x, y, z \mid x^2 = y^2 = z^d = (xy)^2 = (yz)^2 = 1 \rangle$ which has a free subgroup of rank 2 by Theorem 3.3. \square

This completes the proof of Theorem 1.1.

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