An update on Hurwitz groups

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Abstract

A Hurwitz group is any non-trivial finite quotient of the $(2, 3, 7)$ triangle group, that is, any non-trivial finite group generated by elements x and y satisfying $x^2 = y^3 = (xy)^7 = 1$. Every such group G is the conformal automorphism group of some compact Riemann surface of genus $g > 1$, with the property that $|G| = 84(g - 1)$, which is the maximum possible order for given genus g. This paper provides an update on what is known about Hurwitz groups and related matters, following up the author's brief survey in *Bull. Amer. Math. Soc.* 23 (1990).

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1 Introduction

The $(2,3,7)$ *triangle group* is the abstract group $\Delta = \Delta(2,3,7)$ with presentation

$$\Delta(2,3,7) = \langle\, x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 \,\rangle,$$

and a *Hurwitz group* is any non-trivial quotient of Δ — that is, any non-trivial finite group generated by two elements u and v satisfying $u^2 = v^3 = (uv)^7 = 1$. The significance of such groups comes from an 1893 theorem of Hurwitz, which states that any compact Riemann surface X with genus $g > 1$ admits at most $84(g-1)$ conformal automorphisms — that is, homeomorphisms of the surface onto itself which preserve the local structure — and that this upper bound is attained if and only if the conformal automorphism group of X is isomorphic to the quotient group Δ/K for some normal subgroup K of Δ (in which case K is isomorphic to the fundamental group of X).

A brief survey of known properties and examples of Hurwitz groups was given by the author in [6]. The purpose of this paper is to provide an update on [6], at the request of the chief editors of this new journal.

One of the key properties of every Hurwitz group G is that it is perfect — its abelianisation G/G' is trivial — and hence every Hurwitz group is an extension of some maximal normal subgroup by a non-abelian simple group. Knowledge about the simple groups that can arise in this context was fairly limited at the time of writing of [6], but has progressed a lot since then, and much of it is summarised in Section 2. In particular, much more is now known about groups of Lie type that are Hurwitz (or not Hurwitz, as the case may be), and so Section 2 includes recent results about these also. In contrast, relatively little has been done on the subject of what kinds of group extensions (and covering surfaces) are possible, however a few pertinent observations are made briefly in Section 3.

Recent results that provide relevant information about the associated surfaces (and arithmetic curves) are given in Section 4, and some new information about certain finitely-presented quotients of the $(2,3,7)$ triangle group is presented in Section 5. Finally, in Section 6, some further recent discoveries of interest are described, about actions of groups on Riemann surfaces and related structures.

2 Simple and linear Hurwitz groups

At the time of writing of [6], the only finite simple groups known to be Hurwitz groups were all but 64 of the alternating groups A_n, certain of the projective linear groups $\mathrm{PSL}_2(q)$, the Ree groups ${}^2G_2(3^p)$ for all odd primes p, and 11 of the 26 sporadic finite simple groups. It was also known that $\mathrm{PSL}_3(q)$ is a Hurwitz group only when $q = 2$, and that 13 of the other sporadic finite simple groups are not Hurwitz. Shortly afterwards, related information was provided also by Lino Di Martino and Chiara Tamburini in a nice survey paper about generating sets for finite simple groups [13].

Considerable progress has been made since then.

Following Andrew Woldar's paper [61] dealing with many of the sporadic finite simple groups, Rob Wilson proved in [58] that the Baby Monster B is not a Hurwitz group, and then in [60] that the Monster M is a Hurwitz group. The latter was a remarkable piece of work, made possible by some clever computations. To begin with, the size of the Monster and the degree of its smallest faithful representation make calculations very difficult, but Wilson used carefully coded matrix-vector computations to reduce the time and memory required. Effectively, by random search he found a pair of elements (u, v) of orders 2 and 3 with product uv of order 7, such that the subgroup H generated by u and v contains elements of sufficiently many different orders that H cannot be contained in any proper subgroup of M. As a result, the following is now known, thanks largely to Wilson and Woldar's work (in [61, 58, 60] and earlier papers):

> *Of the* 26 *sporadic finite simple groups, the following* 12 *are Hurwitz groups:* J_1, J_2, J_4, Fi_{22}, $\mathrm{Fi}_{24'}$, Co_3, He, Ru, HN, Ly, Th and M; *while the other* 14 *are not.*

Quite naturally, attention more recently has focussed on the simple groups of Lie type.

In [32], Gunter Malle used character-theoretic arguments and knowledge about maximal subgroups to prove that the Chevalley group $G_2(q)$ is a Hurwitz group for every prime power $q \geq 5$, and that the Ree group ${}^2G_2(3^{2m+1})$ is a Hurwitz group for every $m \geq 1$. (Moreover, he showed also that the remaining groups $G_2(2)$, $G_2(3)$, $G_2(4)$ and ${}^2G_2(3)$ are not Hurwitz groups, but are still factor groups of the modular group $\langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$.) The result about the Ree groups was proved independently by Gareth Jones in [23], giving more detailed information, and explicit generators were subsequently provided also by Kerope Tchakerian in [49]. Malle went further in [33] to prove that the exceptional simple group ${}^3D_4(p^n)$ is a Hurwitz group if and only if $p \neq 3$ and $p^n \neq 4$, and that ${}^2F_4(2^{2n+1})'$ is a Hurwitz group if and only if $n \equiv 1 \bmod 3$.

Taking a quite different approach for other families of linear groups (that are not necessarily simple), Chiara Tamburini and Salvatore Vassallo considered the groups $\mathrm{SL}_4(q)$ in [45], where they used a classification of irreducible groups generated by transvections to prove that $\mathrm{SL}_4(q)$ is never a Hurwitz group.

A significant step forward was then taken by Andrea Lucchini, Chiara Tamburini and John Wilson in [28], proving that most finite simple classical groups of sufficiently large dimension are Hurwitz groups. Specifically, they proved that if R is a finitely generated ring with identity, and $E_n(R)$ is the group of invertible $n \times n$ matrices generated by the set $\{I_n + re_{ij} \colon r \in R, 1 \leq i \neq j \leq n\}$ of elementary transvections, then $E_n(R)$ can be $(2, 3, 7)$-generated for all but finitely many n. Their proof takes the permutation matrices X and Y coming from particular $(2, 3, 7)$-generators of the alternating group A_n (as given by this author in [5]), and replaces X by an adjusted matrix $\tilde{X}$ with the property that $\tilde{X}^2 = Y^3 = (\tilde{X}Y)^7 = I_n$ and $\langle \tilde{X}, Y \rangle = E_n(R)$. As corollaries of their main theorem, it follows that the following are Hurwitz groups: the special linear group $\mathrm{SL}_n(q)$ and the projective special linear group $\mathrm{PSL}_n(q)$ for every $n \geq 287$ and every prime-power q; the special linear group $SL_n(\mathbb{Z})$ for every $n \geq 287$; and the derived group of $Aut(F_n)$ for every $n \geq 329$. Further, in [27] Lucchini and Tamburini showed that the classical groups $Sp_{2n}(q), \Omega^+_{2n}(q)$ and $SU_{2n}(q)$ are Hurwitz for every $n \geq 371$ and every prime-power q, and that $\Omega_{2n+7}(q)$ and $SU_{2n+7}(q)$ are Hurwitz for every $n \geq 371$ and every odd prime-power q. It follows that simple (projective) quotients of all of these groups are Hurwitz as well.

A nice summary of many of these results was presented by John Wilson in [57], including an amusing observation that the direct product of 5 million copies of $SL_{1000}(7)$ is a Hurwitz group. Wilson extended this work to quotients of $(2, 3, k)$ triangle groups for $k \geq 7$ in [56].

At about the same time, in [14] Lino Di Martino, Chiara Tamburini and Alexandre Zalesskii proved that in contrast, most quasi-simple classical groups of small rank (such as $SL_n(q)$ and $SU_n(q^2)$ for $n \leq 19$ and various q) are not Hurwitz. This was achieved using representation-theoretic arguments and the application of a necessary condition (due to Leonard Scott) for 2-element generation of a matrix group in terms of dimensions of the subspaces fixed pointwise by the generators and their product.

Tamburini and Zalesskii used Scott's work (and other aspects of linear representation theory) also to consider $(2, 3, k)$-generation of subgroups of $PSL_5(F)$ for any algebraically closed field F of positive characteristic, and proved in [48] that the following 5-dimensional linear groups are Hurwitz groups, for prime p:

- $PSL_5(p)$ if $p \equiv 1 \bmod 5$ and $p \equiv 1, 2$ or $4 \bmod 7$,
- $PSL_5(p^2)$ if $p \equiv 1 \bmod 5$ and $p \equiv 3, 5$ or $6 \bmod 7$,
- $PSU_5(p^2)$ if $p \equiv -1 \bmod 5$ and $p \equiv 1, 2$ or $4 \bmod 7$,
- $PSL_5(p^4)$ if $p = 7$ or if $p \equiv -1 \bmod 5$ and $p \equiv 3, 5$ or $6 \bmod 7$ or if $p \equiv 2 \bmod 5$ and $p \equiv 1, 2$ or $4 \bmod 7$, and
- $PSL_5(p^8)$ if $p \equiv 2 \bmod 5$ and $p \equiv 3, 5$ or $6 \bmod 7$.

For 'intermediate' ranks, Lucchini, Tamburini and Wilson proved in [28] that $SL_n(q)$ is a Hurwitz group for 93 values of $n < 287$ (and all prime powers q), and Maxim Vsemirnov extended this result to another 60 values of $n \leq 287$ in [54], again using $(2, 3, 7)$-generation of alternating groups. In particular, it is now known that $SL_{49}(q)$ is Hurwitz for all q. Analogous results about $SL_n(q)$ and $SL_n(\mathbb{Z})$ for 50 such small ranks n were obtained by Sun Yongzhong in [44].

The approach taken in [28] is explained very nicely in a survey chapter in the *Handbook of Algebra* on Hurwitz groups and Hurwitz generation [47] by Tamburini and Vsemirnov. It was taken further also by Nikita Semenov in [38], to prove that the commutator subgroup of the Weyl group of type D_n is a Hurwitz group for all sufficiently large n (and that the commutator subgroup of the Weyl group of type B_n is $(2, 3, 14)$-generated for all $n \geq 168$).

A quite different approach was introduced by Wilhelm Plesken and Daniel Robertz in [36], to construct linear representations of $\Delta(2, 3, 7)$ of degree up to 7 in characteristic zero, which can then be used to find Hurwitz groups embeddable as subgroups of $GL_n(R)$ for a suitable ring R and some $n \leq 7$. This approach involves a normalization process for constructing irreducible or indecomposable representations (generalising the construction of the companion matrix of a univariate polynomial), and the application of an ultraproduct construction to obtain a link between representations in characteristic zero and representations in positive characteristic, and then relies heavily on a computational implementation of Janet's algorithm [35] for solving polynomial equations (developed in the context of linear PDEs). Plesken and Robertz showed also in [36] how the same approach can work when additional relations are added to the presentation for $\Delta(2,3,7)$, giving, for example, infinite representations of the groups $(2,3,7;m) = \langle\, x, y \mid x^2 = y^3 = (xy)^7 = [x,y]^m = 1 \,\rangle$ over an algebraic number field, for $m = 10, 11, 12, 13$ and 17.

Finally in this Section, we note two other pieces of work of related interest. In [15], Amir Džambić produced an alternative proof of Macbeath's necessary and sufficient conditions on q for $\mathrm{PSL}_2(q)$ to be Hurwitz, using a faithful representation of $\Delta(2,3,7)$ as a Fuchsian group derived from a quaternion algebra over $\mathbb{Q}(\zeta + \zeta^{-1})$, where $\zeta = e^{2\pi i/7}$, and observations about principal congruence subgroups. Also in [46], Tamburini and Vsemirnov determined isomorphism types of the projective images of triples (X, Y, XY) generating an irreducible subgroup of $\mathrm{SL}_n(F)$, for $n \leq 7$ and F an algebraically closed of characteristic $p \geq 0$, such that X^2, Y^3, $(XY)^7$ are scalars, and determined which of these are rigid (in the sense of Strambach and Völklein [42]). The latter approach produced in [46] further new examples of projective linear Hurwitz groups of small rank, including the following for prime p:

- $\mathrm{PSL}_6(p^m)$ if $p \neq 3$, and m is the order of p mod 9, and m is odd,
- $\mathrm{PSU}_6(p^m)$ if $p \neq 3$, and m is the order of p mod 9, and m is even,
- $\mathrm{PSL}_7(p^m)$ if $p \neq 7$, and m is the order of p mod 49, and m is odd,
- $\mathrm{PSU}_7(p^m)$ if $p \neq 7$, and m is the order of p mod 49, and m is even.

3 Covers and extensions

If the Hurwitz group G acts faithfully as a group of $84(g-1)$ conformal automorphisms of a compact Riemann surface X of genus g, and G has a non-trivial proper normal subgroup N, then X is a smooth cover of some Riemann surface Y of smaller genus g' (given by $g-1 = |N|(g'-1)$), with the factor group $H = G/N$ as its conformal automorphism group.

In certain situations, the group G (and associated surface X) can be constructed from a given Hurwitz group H and suitable group N. For example, G can be a semi-direct product of N by H, as in various families of extensions of abelian groups by Hurwitz groups $PSL_2(q)$ discussed in [6, Section 4]. More general conditions for a semi-direct product $G = NH$ to be a quotient of a given triangle group (when H is known to be such a quotient, and N is a p-group) were given by Rick Thomas in [50].

It is rare, however, for the normal subgroup N to be cyclic, and for good reason, as follows.

Conjugation by elements of G gives a homomorphism from G to $\mathrm{Aut}(N)$ with kernel $C_G(N)$. Hence if N is cyclic, then since $\mathrm{Aut}(N)$ is abelian while the Hurwitz group G is perfect, it follows that $G/C_G(N)$ is trivial, and therefore N is central in G. In particular, the index in G of its centre $Z(G)$ divides $|G : N|$. Now by Schur's theorem on centre-by-finite groups (see [37] for example), the order of every element of $G' = [G, G]$ divides $|G : N|$. But again, G is perfect, so $G = G'$, and in particular, G' contains N. Thus $|N|$ *divides* $|G/N|$, *whenever the normal subgroup* N *of the Hurwitz group* G *is cyclic.*

(A similar argument was used by the author and Ravi Kulkarni (with thanks to Peter Neumann for pointing out the usefulness of Schur's theorem) in [9] to prove that whenever p and q are coprime positive integers, there are only finitely many finite groups that can be generated by two elements u and v of orders p and q respectively such that uv generates a subgroup of given index.)

On the other hand, there is no upper bound on the order of abelian groups that can occur as the centre of a Hurwitz group G. This follows easily from the known fact that $\mathrm{SL}_n(q)$ is a Hurwitz group for any given prime power q and sufficiently large n, and was used by the author to prove in [8] that the centre of a Hurwitz group can be any given finite abelian group — thereby giving a complete answer to a 1965 question by John Leech. The proof (which is relatively short) involves simply taking a direct product of suitably chosen special linear groups, and then factoring out however much of the centre of this product is surplus to requirements.

4 Information about Hurwitz curves and surfaces

Murray Macbeath presented a very nice account of his experiences with Hurwitz groups and the curves and surfaces on which they act, in a chapter of the MSRI publication *The eightfold way: the beauty of Klein's quartic curve* (1999); see [30].

In [43], Manfred Streit investigated the complex algebraic curves associated with the Hurwitz groups $\mathrm{PSL}_2(q)$, showing that the a minimal field of definition of these Hurwitz curves is the rational field $\mathbb{Q}$ when $q = 7$ or $q = p^3$ for some prime $p \equiv \pm 2$ or ± 3 modulo 7, and $Q(\zeta + \zeta^{-1})$ where $\zeta = e^{2\pi i/7}$ when $q = p$ for some prime $p \equiv \pm 1$ modulo 7. He also showed that in the latter case, the three curves are mutually conjugate under the action of the absolute Galois group $\mathrm{Gal}(\overline{Q}/Q)$.

The action of automorphisms on the set of Weierstrass points of a Hurwitz curve was considered by Kay Magaard and Helmut Völklein in [31], where they showed that the corresponding Hurwitz group does not act transitively on the Weierstrass points of the curve whenever its genus g is greater than 14. It does for $g = 3$ (as follows immediately from the determination of the 24 Weierstrass points of Klein's quartic by David Singerman and Paul Watson in [40], and shown also later by Noam Elkies [17]), and for $g = 7$ (as shown in [31]), but the question seems to be open for $g = 14$.

In his 2004 University of Helsinki dissertation [52], Roger Vogeler developed a method to encode and classify the conjugacy classes of hyperbolic transformations in the extended $(2, 3, 7)$ triangle group (viewed as the group generated by the reflections about the sides of a hyperbolic triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$), and provided a method for computing the lengths of closed geodesics on a Hurwitz surface in order to produce data about their length spectra. This work was further expanded in [53], where he described a means of exactly determining a large initial portion of the spectrum of $\Delta(2,3,7)$ and hence that of Hurwitz surfaces (having the maximum possible number of conformal automorphisms).

5 Related results about finitely-presented quotients of Δ(2,3,7)

Families (and individual instances) of quotients of the $(2,3,7)$ triangle group can be obtained by adding additional relators to its presentation.

For example, the family of quotients $(2,3,7;m) = \langle\, x, y \mid x^2 = y^3 = (xy)^7 = [x,y]^m = 1 \,\rangle$ has already been mentioned (in observations about linear representations of $\Delta(2,3,7)$) in Section 2. These groups are now known to be infinite for all $m \geq 9$. For $m \neq 11$ this was proved by Derek Holt and Wilhelm Plesken using coset diagrams in [20], and (independently) by Jim Howie and Rick Thomas using notions of asphericity in [22]. The difficult case $m = 11$ was resolved by Martin Edjvet using a curvature argument in [16], and (independently) by Holt, Plesken, and Bernd Souvignier by exhibiting homomorphism to a Lie group of type G_2 over an algebraic number field of degree 10 over $\mathbb{Q}$, in [21].

When $m = 84$, the group $(2,3,7;m)$ is not just infinite, but like $\Delta(2,3,7)$ itself, has all but finitely many alternating groups A_n among its quotients; this was proved (in answer to a question by Graham Higman) in [7], again using coset diagrams.

As noted in Section 2, the approach taken by Plesken and Robertz in [36] can be used to produce infinite representations of $(2,3,7;m)$ for various values of m. More recently, Maxim Vsemirnov proved in [55] that for every prime $p \geq 5$, the Chevalley group $G_2(p)$ is a quotient of the group $(2,3,7;2p)$.

Other quotients of the form $\langle\, x, y \mid x^2 = y^3 = (xy)^7 = w(x,y) = 1 \,\rangle$ where w is a word of small length in the generators were investigated by the author of this paper in a piece of joint work with Colin Campbell and Edmund Robertson [4], dedicated to the memory of John Leech. As with Leech's own work, this involved a combination of proofs by hand and the use of coset-enumeration by computer to resolve a number of questions (including some posed by Leech himself about cases in which $w(x,y)$ has the form $([x,y]^r(xy)^{4s})^k$).

Finally, we note that a further consequence of the results of Lucchini, Tamburini and Wilson [28] is that there exist $2^{\aleph_0}$ quotients of $\Delta(2,3,7)$ that are *infinite simple*, and further, that John Wilson showed in [56] that every countable group can be embedded in a simple image of $\Delta(2,3,7)$ — indeed in a simple image of $\Delta(2,3,k)$ for any $k \geq 7$. The latter result generalises work by Gareth Jones and Mary Jones in [25] on infinite quotients of Fuchsian groups.

6 Other matters

We complete this paper by outlining some related matters about actions of groups on Riemann surfaces and other structures.

The *strong symmetric genus* $\sigma^o(G)$ of a finite group G is the smallest genus g of all compact orientable surfaces on which G acts faithfully as a group of orientation-preserving automorphisms. This was defined by Tom Tucker in [51], although the concept dates back to Burnside (or earlier). If G does not act faithfully on the sphere (genus 0) or the torus (genus 1), then by Hurwitz's theorem $|G| \leq 84(\sigma^o(G) - 1)$, so $\sigma^o(G) \geq \frac{|G|}{84} + 1$. This lower bound is achieved precisely when G is a Hurwitz group, and if it is not, then more refined bounds can be obtained using the Riemann-Hurwitz formula (see [51]).

The strong symmetric genus of a large number of groups and families of groups is known, including all of the alternating and symmetric groups, the groups $\mathrm{PSL}(2, q)$, and all of the sporadic simple groups. For instance, Rob Wilson found the strong symmetric genus of the Baby Monster B to be $\frac{|\mathrm{B}|}{48} + 1$ and that of the Fischer group Fi_{23} to be $\frac{|\mathrm{Fi}_{23}|}{48} + 1$, since each is $(2, 3, 8)$-generated [58, 59]. A remarkable recent development is that Coy May and Jay Zimmerman have proved (by determining the strong symmetric genus of the direct products $C_k \times D_n$) that for every non-negative integer g, there exists at least one group G with $\sigma^o(G) = g$; see [34].

A contiguous stream of research has been carried out on *regular maps* on surfaces. An orientably-regular map M is a 2-cell embedding of a graph (or multigraph) into a closed surface with the property that its group G^o of orientation-preserving automorphisms (which must preserve incidence among vertices, edges and faces) acts transitively on the ordered edges of M. Such maps have uniform structure, with all faces of the same size p and all vertices of same valency q, in which case M is said to have type $\{p, q\}$, and the group G^o is a quotient of the $(2, p, q)$ triangle group. Thus, for example, every Hurwitz group G is the group of all orientation-preserving automorphisms of some orientably-regular map of genus $\frac{|G|}{84} + 1$. There are numerous papers in the literature on regular maps and their groups. One such paper by Jozef Širáň [41] deals with representations of the triangle groups in special linear groups, with applications to maps of arbitrarily large planar width, Hurwitz groups, vertex-transitive non-Cayley graphs, and arc-transitive graphs of given valency and girth.

Actions of groups on other kinds of surfaces have also been considered.

For example, Mischa Belolipetsky and Gareth Jones have investigated automorphism groups of compact *arithmetic* Riemann surfaces (viz. those which are uniformized by an arithmetic Fuchsian group). It is known, for example, that surfaces for which the Hurwitz bound is attained are arithmetic. For a non-arithmetic surface of genus g, Belolipetsky proved in [1] that the largest possible order for the group G of all conformal automorphisms is $\frac{156}{7}(g - 1)$, attainable if and only if G is a quotient of the $(2, 3, 13)$ triangle group. On the other hand, by earlier work of Accola and Maclachlan in the late 1960s, for every positive integer g there exists some compact Riemann surface of genus g having at least $8(g+1)$ conformal automorphisms, but it is known that all surfaces for which the Accola and Maclachlan bound is attained are non-arithmetic. In [2], Belolipetsky and Jones showed that for every $g \geq 2$ there exists a compact arithmetic Riemann surface of genus g with at least $4(g-1)$ conformal automorphisms, and that this bound is attained for infinitely many g (starting with 24).

For *non-orientable* surfaces, the Hurwitz bound becomes $|G| \leq 84(p-2)$ for every group G of automorphisms of a compact non-orientable surface of genus p, as proved by David Singerman [39]. Groups meeting this bound are smooth quotients of the extended $(2,3,7)$ triangle group $\langle\, x, y \mid x^2 = y^3 = (xy)^7 = t^2 = (xt)^2 = (yt)^2 = 1 \,\rangle$ in which the image of the reflection t lies in the image of the subgroup generated by x and y — or equivalently, Hurwitz groups possessing an inner automorphism that inverts the images of the standard generators x and y of the ordinary $(2,3,7)$ triangle group. Examples include all but finitely many alternating groups A_n, and infinitely many $\mathrm{PSL}_2(q)$, by results of this author's earlier work. The analogue of the Accola-Maclachlan bound for non-orientable surfaces is more complicated. Colin Maclachlan, Sanja Todorovic Vasiljevic, Steve Wilson and the author proved in [10] that if $\nu(p)$ is the largest number of automorphisms of a non-orientable surface of genus p, then $\nu(p) \geq 4p$ if p is odd, and $\nu(p) \geq 8(p-2)$ if p is even. For various congruence classes mod 12, these bounds may be improved; for example, if $p \equiv 9 \bmod 12$, then $\nu(p) \geq 6(p+1)$, and this is sharp for infinitely many such p. Sharp bounds are given in [10] for all congruence classes of p mod 12 except in the case $p \equiv 3 \bmod 12$, for which it is suspected (but not yet proved) that the bound $\nu(p) \geq 4p$ is sharp.

Group actions on *Klein surfaces* (which include Riemann surfaces as a special case) are described in an informative monograph by Emilio Bujalance, Javier Etayo, José Manuel Gamboa and Grzegorz Gromadzki [3]. Compact Klein surfaces are determined topologically by three invariants: the topological genus g, the number of boundary components k, and the orientability. The *algebraic genus* of any such surface X is defined as $p = 2g+k-1$ if X is orientable, or $p = g+k-1$ if X is non-orientable. Automorphism groups of Klein surfaces can be investigated via the study of non-Euclidean crystallographic (NEC) groups, which act on the upper half-plane, preserving or reversing orientation. The maximum possible order for a group of automorphisms of a Klein surface of algebraic genus p is $12(p-1)$. Any group for which this bound is attained is called an M^*-group, and is a quotient of the extended modular group $\langle\, x, y \mid x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 \,\rangle \cong \mathrm{PGL}_2(\mathbb{Z})$. Accordingly, M^*-groups are to Klein surfaces what Hurwitz groups are to Riemann surfaces, but it is clear that Hurwitz groups are more rare.

Extension of actions of Hurwitz groups on surfaces (2-manifolds) to 3-manifolds was considered by Monique Gradolato and Bruno Zimmermann in [19], with special focus on the Hurwitz actions of the groups $\mathrm{PSL}_2(q)$.

Finally, there is now an analogue of Hurwitz's theorem for 3-manifolds.

An orientable n-dimensional hyperbolic manifold is the quotient space $M = \mathbb{H}^n/\Lambda$, where Λ is some torsion-free discrete subgroup of group $\mathrm{Isom}^+(\mathbb{H}^n)$ of all orientation-preserving isometries of n-dimensional hyperbolic space $\mathbb{H}^n$. The group $\mathrm{Isom}^+(M)$ of all orientation-preserving isometries of the manifold M is then isomorphic to Γ/Λ where Γ is the normalizer of Λ in $\mathrm{Isom}^+(\mathbb{H}^n)$. Letting O be the orientable n-dimensional orbifold $\mathbb{H}^n/\Gamma$, we have $\mathrm{vol}(O) = \mathrm{vol}(M)/|\mathrm{Isom}^+(M)|$ and $O = \mathbb{H}^n/\Gamma \cong (\mathbb{H}^n/\Lambda)/(\Gamma/\Lambda) \cong M/\mathrm{Isom}^+(M)$, and so the ratio $\frac{|\mathrm{Isom}^+(M)|}{\mathrm{vol}(M)}$ is largest precisely when $O = \mathbb{H}^n/\Gamma$ is of minimum possible volume.

When $n = 2$, the minimum value of $\mathrm{vol}(\mathbb{H}^2/\Gamma)$ is attained precisely when Γ is isomorphic to the $(2, 3, 7)$ triangle group — this is Hurwitz's theorem.

The discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ of uniquely smallest co-volume has recently been shown by Fred Gehring, Gaven Martin and Tim Marshall in a series of papers culminating in [18] to be the normaliser in $\mathrm{Isom}(\mathbb{H}^3)$ of a subgroup isomorphic to the $[3, 5, 3]$-Coxeter group

$$[3,5,3] = \langle\, a,b,c,d \mid a^2 = b^2 = c^2 = d^2 = (ab)^3 = (bc)^5 = (cd)^3 = (ac)^2 = (ad)^2 = (bd)^2 = 1 \,\rangle,$$

obtainable by adjoining the involutory graph automorphism t that interchanges a with d and b with c. Its orientation-preserving subgroup is isomorphic to the index 2 subgroup Φ generated by ab, bc, cd and t, which is a split extension of $\langle ab, bc, cd\rangle = [3, 5, 3]^+$ by $\langle t\rangle \cong C_2$. It follows that the ratio $\frac{|\mathrm{Isom}^+(M)|}{\mathrm{vol}(M)}$ is maximized for an orientable hyperbolic 3-manifold M precisely when $\mathrm{Isom}^+(M)$ is a smooth quotient of the latter group Φ.

Gareth Jones and Sasha Mednykh have shown that the smallest such quotient of Φ is $\mathrm{PGL}_2(9)$, of order 720, and investigated the 3-manifolds associated with these and a number of other such quotients of small order in [26]. In joint work with Gaven Martin and Anna Torstensson [11], the author of this paper has proved the following:

(A) *For every prime p there is some $q = p^k$ (with $k \le 8$) such that either* $\mathrm{PSL}_2(q)$ *or* $\mathrm{PGL}_2(q)$ *is a quotient of Φ by some torsion-free normal subgroup.*

(B) *For all but finitely many n, both the alternating group A_n and the symmetric group S_n are quotients of both Φ and its normalizer in* $\mathrm{Isom}^+(\mathbb{H}^3)$*, by torsion-free normal subgroups.*

These results give two infinite families of hyperbolic 3-manifolds with the maximum possible number of orientation-preserving automorphisms with respect to volume.

The situation for hyperbolic 4-manifolds is still open ...

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