

PRESENTATIONS OF MATRIX RINGS

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Recently, there has been a significant interest in the combinatorial properties the ring of $n \times n$ matrices. The aim of this note is to describe a short (may be the shortest possible) presentation of the matrix ring $\text{Mat}_n(\mathbb{Z})$. This presentation is significantly shorter than the previously known ones, see [7].

Surprisingly, the number of relations in the presentation does not depend on the size of the matrices and all relations have relatively simple form. In contrast, the similar statement for the groups $\text{GL}_n(\mathbb{Z})$ is significantly more difficult to prove and the presentations have more relations, see [5, 4].

Theorem 1. *The ring¹ of $n \times n$ matrices over $\mathbb{Z}$, for $n \geq 2$, has a presentation with 2 generators and 3 relations*

$$\text{Mat}_n(\mathbb{Z}) = \langle x, y \mid x^n = y^n = 0, xy + y^{n-1}x^{n-1} = 1 \rangle.$$

Proof. Let R denote the ring defined by the above presentation. A homomorphism from the free associative ring to $\text{Mat}_n(\mathbb{Z})$ given by

$$x \rightarrow X = \sum e_{i,i+1} \quad y \rightarrow Y = \sum e_{i,i-1}$$

factors through the ring R .² The first two relations are satisfied because both X and Y are nilpotent matrices and the third follows from a direct computation. Moreover, this homomorphism is surjective because the matrices X and Y generate $\text{Mat}_n(\mathbb{Z})$ as a ring.

Thus, it remains to prove that the ring R is not too big. Our first step is to find other relations, which are satisfied in the ring R .

Lemma 2. *For any non-negative integers $k, l, m \geq 0$, such that $m \geq l$, we have*

$$x^k y^l x^m = \begin{cases} y^{l-k} x^k & \text{if } l \geq k \\ x^{k+m-l} & \text{if } l \leq k \end{cases}$$

and

$$y^m x^l y^k = \begin{cases} y^m x^{l-k} & \text{if } l \geq k \\ y^{k+m-l} & \text{if } l \leq k \end{cases}.$$

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¹All rings in this paper are associative and contain a unit element. Also all presentations are in the category of unital associative rings.

²As usual, $e_{i,j}$ denotes the elementary matrix with 1 at i, j -th place and zeroes everywhere else.

Proof. The proof uses induction on l . The base case $l = 0$ is trivial. If $k = 0$ again there is nothing to prove. Thus, without loss of generality we can assume that $k, l \geq 1$.

$$\begin{aligned} x^k y^l x^m &= x^{k-1} (xy) y^{l-1} x^m = x^{k-1} (1 - y^{n-1} x^{n-1}) y^{l-1} x^m = \\ &= x^{k-1} y^{l-1} x^m - x^{k-1} y^{n-1} (x^{n-1} y^{l-1} x^m). \end{aligned}$$

By the induction assumption the part of the second term in the brackets is equal to $x^{n+m-l} = 0$, since $m \geq l$. Thus, the second term vanishes. Another application of the induction assumption shows that

$$x^{k-1} y^{l-1} x^m = \begin{cases} y^{l-k} x^m & \text{if } l \geq k \\ x^{k+m-l} & \text{if } l \leq k \end{cases}$$

and completes the proof of the first part of the lemma.

The proof of the second part uses similar induction. Alternatively, one can use that the transformation $x \rightarrow y$ and $y \rightarrow x$ extends to an anti-automorphism σ of R and apply σ to the identities from the first part. $\square$

Lemma 3. *The ring R is generated, as an additive group, by the elements $y^i x^j$ for $0 \leq i, j < n$.*

Proof. Let T denote the additive subgroup of R generated by the elements $y^i x^j$. It suffices to show that T is closed under, both left and right, multiplication by x and y , because $1 = x^0 y^0 \in T$. The two relations $x^n = y^n = 0$ imply that T is closed under left multiplication by y and right multiplication by x . Thus, it remains to show that $xT, Ty \subseteq T$. The element $x \cdot y^i x^j$ is clearly in T if $i = 0$. If $i \geq 1$ we have

$$\begin{aligned} x y^i x^j &= (xy) y^{i-1} x^j = (1 - y^{n-1} x^{n-1}) y^{i-1} x^j = \\ &= y^{i-1} x^j - (y^{n-1} x^{n-1} y^{i-1}) x^j = y^{i-1} x^j - y^{n-1} x^{n-i} x^j \in T, \end{aligned}$$

where the last equality uses Lemma 2.

Similarly, $y^i x^j y$ is in T if $j = 0$ and if $j \geq 1$ we have

$$\begin{aligned} y^i x^j y &= y^i x^{j-1} (xy) = y^i x^{j-1} (1 - y^{n-1} x^{n-1}) = \\ &= y^i x^{j-1} - y^i (x^{j-1} y^{n-1} x^{n-1}) = y^i x^{j-1} - y^i y^{n-j} x^{n-1} \in T. \end{aligned}$$

$\square$

Lemma 3 together with the surjection for R to $\text{Mat}_n(\mathbb{Z})$ is sufficient to show that R is isomorphic to the matrix ring, but one can build the isomorphism directly:

Definition 4. Let $a_{i,j}$, for $0 \leq i, j < n$, denote the elements

$$a_{i,j} = y^i x^j - y^{i+1} x^{j+1}.$$

Lemma 5. *We have that*

$$a_{i,j} x = a_{i,j+1} \quad a_{i,j} y = a_{i,j-1} \quad x a_{i,j} = a_{i-1,j} \quad y a_{i,j} = a_{i+1,j}.$$

Here, we assume that $a_{i,j} = 0$ if either i or j is outside the interval $[0, n-1]$.

Proof. The first and the fourth identities follow from the associativity of the ring and the definition of the elements $a_{i,j}$. Thus, we need to prove only that $a_{i,j}y = a_{i,j-1}$ and $xa_{i,j} = a_{i-1,j}$. By the definition of the element $a_{i,j}$ we have

$$a_{i,j}y = y^i x^j y - y^{i+1} x^{j+1} y$$

If $j = 0$ the right side is equal to $y^{i+1} - y^{i+1}xy = y^{i+1}y^{n-1}x^{n-1} = 0$. Otherwise, we can use the proof of Lemma 3:

$$\begin{aligned} &= (y^i x^{j-1} - y^{n+i-j} x^{n-1}) - (y^{i+1} x^j - y^{n+i-j} x^{n-1}) = \\ &= y^i x^{j-1} - y^{i+1} x^j = a_{i,j-1}. \end{aligned}$$

The proof of the second relation $xa_{i,j} = a_{i-1,j}$ is similar, i.e., as in Lemma 3 (assuming that $j > 0$) we have:

$$\begin{aligned} xa_{i,j} &= xy^i x^j - xy^{i+1} x^{j+1} = (1 - y^n x^n) (y^{i-1} x^j - y^i x^{j+1}) = \\ &= y^{i-1} x^j - y^i x^{j+1} = a_{i-1,j}. \end{aligned}$$

□

Lemma 6. *The product $a_{i,j}a_{p,q}$ is equal to 0 if $j \neq p$ and is equal to $a_{i,q}$ if $j = p$. Moreover we have*

$$1 = \sum a_{i,i} \quad x = \sum a_{i,i+1} \quad y = \sum a_{i+1,i}.$$

Proof. These equalities follows directly from Lemma 5 and the definition of the elements a_{ij} . □

Thus, the map $a_{i,j} \rightarrow e_{i,j}$ extends to an isomorphism between the ring R and $\text{Mat}_n(\mathbb{Z})$, which completes the proof of Theorem 1. □

Remark 7. From the isomorphism between R and $\text{Mat}_n(\mathbb{Z})$ it follows that the elements x and y also satisfy the relation

$$yx + x^{n-1}y^{n-1} = 1,$$

i.e., the map $x \rightarrow y$ and $y \rightarrow x$ can be extended to an automorphism of R .

The following variation of Theorem 1 gives presentation of the ring of matrices over $\mathbb{Z}/N\mathbb{Z}$:

Theorem 8. *For any integer N the matrix ring $\text{Mat}_n(\mathbb{Z}/N\mathbb{Z})$ has presentation*

$$\langle x, y \mid x^n = y^n = 0, xy + (N+1)y^{n-1}x^{n-1} = 1 \rangle.$$

Proof. The argument is a slight modification of the proof of Theorem 1. The map $x \rightarrow X$ and $y \rightarrow Y$ is a homomorphism onto $\text{Mat}_n(\mathbb{Z}/N\mathbb{Z})$. The proof of injectivity follows the same outline — Lemmas 2, 3, 5 and 6 still hold, because their proofs only use that $(1 - xy)x = y(1 - yx) = 0$ and these relations follow from the defining relations.

Finally, one observes that $a_{n-1,n-1} = y^{n-1}x^{n-1}$ and $xy = \sum_{i=0}^{n-2} a_{i,i}$. Thus the last relation in the presentation is equivalent to

$$\begin{aligned} \sum_{i=0}^{n-1} a_{i,i} &= 1 = xy + (N+1)y^{n-1}x^{n-1} = \\ &= \sum_{i=0}^{n-2} a_{i,i} + (N+1)a_{n-1,n-1} \end{aligned}$$

therefore $Na_{n-1,n-1} = 0$. Multiplying this relation on the left by $a_{i,n-1}$ and on the right by $a_{n-1,j}$ and using Lemma 6, we obtain that $Na_{i,j} = 0$ for all i, j , i.e., the additive group of R has exponent N . $\square$

Remark 9. In the case $N = p$ is a prime number, one can use that the matrix algebra $\text{Mat}_n(\mathbb{F}_p)$ is a cyclic algebra, thus it is possible to obtain a presentation (as an algebra over $\mathbb{F}_p$) with 2 generators and 3 relations. This leads to a presentation of the ring $\text{Mat}_n(\mathbb{F}_p)$ with 2 generators and 4 relations. The presentation obtained using this approach uses a presentation of the finite field $\mathbb{F}_{p^n}$, which involves an irreducible polynomial of degree n over $\mathbb{F}_p$. Thus, the relations in this presentation will be more “complicated” than the ones Theorem 8. In some cases, one can modify the presentation of the cyclic algebra and save one relation:³

$$\text{Mat}_p(\mathbb{F}_p) = \langle x, y \mid y^p = 1, x^p = x, xy = y(x+1) \rangle.$$

Remark 10. In some sense the presentation in Theorem 1 is a variant of the presentation of cyclic algebra, where one uses the nilpotent ring $\mathbb{Z}[x]/(x^n)$ instead of the maximal subfield, however it is not completely clear what is the analog of the “field” automorphism in this picture.

One would like to say that the presentations in Theorems 1 and 8 are the simplest possible. Unfortunately, we were not able to prove that presentations of the matrix rings with 2 generators and only 2 relations do not exist. The following result shows that there isn’t any presentation of the matrix ring $\text{Mat}_n(\mathbb{Z})$ with a single relation:

Theorem 11. *The number of relations in any presentation of the matrix ring $\text{Mat}_n(\mathbb{Z})$ is at least equal to the number of generators.*

Proof. The main idea of the proof is to “translate” the notion of the “relation module” from groups rings and use it to obtain a lower bound for the number of relations in a presentation of ring, see [6].

Let $0 \rightarrow I \rightarrow \mathbb{Z}\langle S \rangle \rightarrow R \rightarrow 0$ be a presentation of the ring R . The quotient I/I^2 is called *relation module associated to this presentation* and is naturally a R bi-module. It is clear that the projection of any generating set of the ideal I to I/I^2 is a generating set of this bi-module. Thus, the minimal number of generators of relation module gives a lower bound for

³This presentation was found by Robert Guralnick [3].

the number of generators of the ideal I and the number of relations in a presentation of the ring R .

One way to construct a big quotient of the relation module is the following: Let d be the size of the generating set $\bar{S}$ and let M be the free R bi-module on d generators m_i , i.e., $M \simeq (R \otimes R)^{\oplus d}$. We can define a ring structure on the abelian group $R \oplus M$, where the multiplication between elements of R and M is defined using left and right actions of R on M and the product of any two elements in M is equal to zero.

For any generating set $\bar{S}$ of R with d elements one can define a subring $\tilde{R}$ of $R \oplus M$ generated by “extensions” of the generators in $\bar{S}$ by generators of the module M , i.e., $\tilde{s}_i = \bar{s}_i + m_i$. It is easy to see that the relation module corresponding to this presentation maps surjectively onto the intersection of $\tilde{R}$ with M .

In the case of the presentation in Theorem 1 one can directly compute the image of the relation module under the above map: we have the module isomorphism $M \simeq \text{Mat}_n(L)$, where L is free abelian group on $2n^2$ generators x_{ij} and y_{ij} , where m_x and m_y are matrices with entries of x_{ij} and y_{ij} . The intersection of $\tilde{R}$ with M is generated as a $\text{Mat}_n(\mathbb{Z})$ bi-module by the lifts of the relations in the presentation to $\tilde{R}$, thus in order to compute the “dimension” of this module it is enough to compute the entries of the lifts of the relations and find the subgroup of L generated by them. A straight forward computation gives that the entries of the lift of the relations $x^n = 0$ (similarly for $y^n = 0$) span a free factor in L of rank n , and the entries in the lift of $1 = xy + y^{n-1}x^{n-1}$ span a free factor of rank $n^2 - 1$. However all the entries in these three lift only span a free factor of rank $n^2 + n$. Therefore, the relation module can not be generated by only one element as a $\text{Mat}_n(\mathbb{Z})$ bi-module, which proves that the ring $\text{Mat}_n(\mathbb{Z})$ does not have a presentation with 2 generators (mapped to X and Y respectively) and only 1 relation.

Similar computation can be made for any presentation of the ring $\text{Mat}_n(\mathbb{Z})$. If we have a presentation with d generators the module M is isomorphic to $\text{Mat}_n(L)$, where L is free abelian group of rank dn^2 . The relation module is isomorphic to $\text{Mat}_n(\tilde{L})$, where $\tilde{L} \subset L$ is a subgroup of rank at least $n^2(d-1) + n > n^2(d-1)$, because one can pick preimages of the matrixes X and Y and see that any complementary submodule of the relation module for the presentation on X and Y is also a complementary to the relation module associated with the original presentation. This completes the proof of the Theorem, since the relation module can not be generated by than $d - 1$ elements. $\square$

Remark 12. A more carefully computation shows that the relation module, corresponding to the map $\mathbb{Z}\langle x, y \rangle \rightarrow \text{Mat}_n(\mathbb{Z})$ given by $x \rightarrow X$ and $y \rightarrow Y$, can be generated by only two elements as an bi-module (for example the elements $xy + y^{n-1}x^{n-1} - 1$ and $xy^n + yx^n$ generate the relation module). This suggests that it might be possible to “combine” the two relations $x^n = y^n = 0$ into a single one and obtain a presentation of $\text{Mat}_n(\mathbb{Z})$

with 2 generators and only 2 relations. Unfortunately, the usual trick of combining such relations — replacing them with $x^n = y^n$ does not work, because the presentation

$$\langle x, y \mid x^n = y^n, xy + y^{n-1}x^{n-1} = 1 \rangle$$

defines a ring which surjects onto $\text{Mat}_n(\mathbb{Z}[t]/(t^2))$.

Remark 13. One can view the proof of Theorem 11 as an analog of Gaschütz' result [1, 2], which says that the tensor product of the relation module for a finite group G with $\mathbb{Q}$ is isomorphic as G -module to $\mathbb{Q}[G]^{\oplus d-1} \oplus \mathbb{Q}$, therefore the relation module can not be generated by less than d elements.

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