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SOME INTEGRAL OPERATORS WHICH PRESERVE A SUBCLASS OF UNIFORMLY QUASICONVEX FUNCTIONS

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Abstract. In this paper we define a subclass of uniformly quasiconvex functions and show that this class is preserved under the Alexander and Bernardi integral operators.

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1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, and $S = \{f \in A : f$ is univalent in $U\}$.

Denote by I the Alexander integral operator $I : A \to A$,

$$F(z) = If(z) = \int_0^z \frac{f(t)}{t}\, dt, \tag{1}$$

and by I_c the Bernardi integral operator $I_c : A \to A$,

$$F(z) = I_c f(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)\, dt, \quad c = 1, 2, 3, \dots\ . \tag{2}$$

2. Preliminary Results

We denote by R^n the Ruscheweyh operator (see [11]) defined as

$$R^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \quad z \in U, \quad n \in \mathbb{N},$$

where $*$ is the convolution product.

Remark 2.1. If $h \in A$, $h(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$, then

$$R^n h(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j.$$

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Let D^n be the S'al'agean differential operator (see [9]) defined as

$$D^n : A \to A, \quad n \in \mathbb{N}, \quad \text{and} \quad D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = D\left(D^{n-1} f(z)\right).$$

Definition 2.1 ([4], [5])**.** Let $n \in \mathbb{N}$ and $f \in A$. We say that f is the class $UK_n(\delta)$, $\delta \in [-1, 1)$, if

$$\operatorname{Re}\left(\frac{R^{n+1} f(z)}{R^n f(z)}\right) \geq \left|\frac{R^{n+1} f(z)}{R^n f(z)} - 1\right| + \delta, \quad z \in U.$$

Remark 2.2. Geometric interpretation: $f \in UK_n(\delta)$ if and only if $\frac{R^{n+1} f(z)}{R^n f(z)}$ takes all values in the domain included in right halfplane Ω_δ which is bounded by the parabola $v^2 = 2(1-\delta)u - (1-\delta^2)$. The Carathéodory function is

$$Q_\delta(z) = 1 + \frac{2(1-\delta)}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad z \in U. \tag{3}$$

Thus $f \in UK_n(\delta)$ if and only if $\frac{R^{n+1} f(z)}{R^n f(z)} \prec Q_\delta$, where by $\prec$ we denote the relation of subordination.

The function Q_δ is convex and $\operatorname{Re} Q_\delta > \frac{1+\delta}{2}$.

Remark 2.3. Taking $n=0$ in Definition 2.1, we obtain $UK_0(\delta) = SP\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)$, where $SP(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$ is the class of functions $f \in S$ which satisfy the condition

$$\left|\frac{zf'(z)}{f(z)} - (\alpha + \beta)\right] \leq \operatorname{Re} \frac{zf'(z)}{f(z)} + \alpha - \beta, \quad z \in U.$$

The class $SP(\alpha, \beta)$ was introduced by Rønning in [10].

Remark 2.4. Taking $n = 1$ and $\delta = \frac{1}{2}$ in Definition 2.1 we obtain $UK_1\left(\frac{1}{2}\right) = US^c$, where US^c is the class of uniformly convex functions introduced by Goodman in [3].

Definition 2.2 ([4], [5])**.** Let $f \in A$. We say that f is an n-uniformly starlike function of order δ and type α if

$$\operatorname{Re}\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| + \delta, \quad z \in U,$$

where $\alpha \geq 0$, $\delta \in [-1, 1)$, $\alpha + \delta \geq 0$, $n \in \mathbb{N}$. We denote this class by $US_n(\alpha, \delta)$.

Definition 2.3 ([2])**.** Let $f \in A$. We say that f is an n-uniformly close to convex function of order δ and type α with respect to the n-uniformly starlike function $g(z)$ of order δ and type α, where $\alpha \geq 0$, $\delta \in [-1, 1)$, $\alpha + \delta \geq 0$, if

$$\operatorname{Re}\left(\frac{D^{n+1} f(z)}{D^n g(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1} f(z)}{D^n g(z)} - 1\right| + \delta, \quad z \in U,$$

where $\alpha \geq 0$, $\delta \in [-1, 1)$, $\alpha + \delta \geq 0$, $n \in \mathbb{N}$. We denote this class by $UCC_n(\alpha, \delta)$.

Remark 2.5. We have $UCC_n(\alpha, \delta) \subset CC$, where CC is the class of close to convex functions defined by Kaplan which are univalent.

Remark 2.6. Taking $n = 0$ and $\alpha = 1$ in Definition 2.2, we obtain $US_0(1, \delta) = SP\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)$.

Theorem 2.1 ([4], [5]). *If $f(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1, 1)$ and $c \in \mathbb{C}$ with $\operatorname{Re} c \geq \frac{n(1-\delta)-(1+\delta)}{2}$, then $F(z) = I_c f(z) \in UK_n(\delta)$, where I_c is the Bernardi integral operator defined in* (2).

Theorem 2.2 ([1]). *If $f(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1, 1)$ and $\delta \geq \frac{n-1}{n+1}$ then $F(z) = If(z) \in UK_n(\delta)$, where I is the Alexander integral operator defined in* (1).

The next theorem is a result of the so-called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [6], [7], [8]).

Theorem 2.3. *Let q be convex in U and $j : U \to \mathbb{C}$ with $\operatorname{Re}[j(z)] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ and satisfies $p(z) + j(z) \cdot zp'(z) \prec q(z)$, then $p(z) \prec q(z)$.*

3. Main Results

Definition 3.1. Let $n \in \mathbb{N}$ and $f \in A$. We say that f is in the class $UQ_n(\delta)$, $\delta \in [-1, 1)$, with respect to the function $g(z) \in UK_n(\delta)$, $\delta \in [-1, 1)$ if

$$\operatorname{Re}\left(\frac{R^{n+1}f(z)}{R^n g(z)}\right) \geq \left|\frac{R^{n+1}f(z)}{R^n g(z)} - 1\right| + \delta, \quad z \in U.$$

Remark 3.1. Geometric interpretation: $f \in UQ_n(\delta)$ if and only if $\dfrac{R^{n+1}f(z)}{R^n g(z)}$, where $g(z) \in UK_n(\delta)$, takes all values in the domain Ω_δ which is bounded by the parabola $v^2 = 2(1-\delta)u - (1-\delta^2)$. The Carathéodory function Q_δ defined in (3) is convex and $\operatorname{Re} Q_\delta > \frac{1+\delta}{2}$. Thus $f \in UQ_n(\delta)$ with respect to the function $g \in UK_n(\delta)$ if and only if $\dfrac{R^{n+1}f(z)}{R^n g(z)} \prec Q_\delta(z)$.

Remark 3.2. Taking $n = 0$ in Definition 3.1, we obtain that the subclass $UQ_0(\delta)$, $\delta \in [-1, 1)$, is the class of functions $f \in A$ such that

$$\operatorname{Re}\frac{zf'(z)}{g(z)} \geq \left|\frac{zf'(z)}{g(z)} - 1\right| + \delta, \quad z \in U,$$

where $g \in UK_0(\delta) = US_0(1, \delta) = SP\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)$ (see Remarks 2.3 and 2.6). But this class is the class $UCC_0(1, \delta)$ and thus from Remark 2.5 we have that the functions from $UQ_0(\delta)$ are univalent.

Remark 3.3. Is easy to see that the function $id(z) = z$, $z \in U$, satisfies $id(z) \in UK_n(\delta)$ for all $n \in \mathbb{N}$ and $\delta \in [-1, 1)$. It follows that $id(z) \in UQ_n(\delta)$ with respect to the function $id(z) \in UK_n(\delta)$ for all $n \in \mathbb{N}$ and $\delta \in [-1, 1)$.

Theorem 3.1. *If $f(z) \in UQ_n(\delta)$ with respect to the function $g(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1, 1)$ and $\delta \geq \frac{n-1}{n+1}$, then $F(z) = If(z) \in UQ_n(\delta)$ with respect to the function $G(z) = Ig(z) \in UK_n(\delta)$, where I is the Alexander integral operator defined by* (1).

Proof. By Theorem 2.2 we have $G(z) = Ig(z) \in UK_n(\delta)$ in the conditions from the hypothesis.

If we consider $w(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j, \ \ z \in U$, then

$$W(z) = Iw(z) = z + \sum_{j=2}^{\infty} \frac{1}{j} a_j z^j, \quad z \in U. \tag{4}$$

By Remark 2.1 we have

$$R^n w(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad n \in \mathbb{N}, \quad z \in U. \tag{5}$$

Using (4) and (5) by simple calculations we obtain

$$(n+1) R^{n+1} w(z) - n R^n w(z) = z \left(R^n w(z) \right)', \quad n \in \mathbb{N}, \tag{6}$$

and

$$z \left(R^n W(z) \right)' = R^n w(z), \quad n \in \mathbb{N}, \quad z \in U. \tag{7}$$

From here we have

$$(n+1) R^{n+1} W(z) - n R^n W(z) = R^n w(z), \quad n \in \mathbb{N}, \quad z \in U, \tag{8}$$

or

$$(n+1) \frac{R^{n+1} W(z)}{R^n W(z)} - n = \frac{R^n w(z)}{R^n W(z)}, \quad n \in \mathbb{N}, \quad z \in U. \tag{9}$$

With notation $\dfrac{R^{n+1} F(z)}{R^n G(z)} = p(z)$ and $\dfrac{R^{n+1} G(z)}{R^n G(z)} = h(z)$ we have

$$z \cdot' (z) = \frac{z \cdot \left(R^{n+1} F(z) \right)'}{R^n G(z)} - \frac{R^{n+1} F(z)}{R^n G(z)} \cdot \frac{z \cdot \left(R^n G(z) \right)'}{R^n G(z)},$$

and from (7) with $w(z) = f(z), \ W(z) = F(z)$ and $w(z) = g(z), \ W(z) = G(z)$ we have

$$\begin{aligned} z \cdot p'(z) &= \frac{R^{n+1} f(z)}{R^n G(z)} - p(z) \cdot \frac{R^n g(z)}{R^n G(z)} \\ &= \frac{R^{n+1} f(z)}{R^n g(z)} \cdot \frac{R^n g(z)}{R^n G(z)} - p(z) \cdot \frac{R^n g(z)}{R^n G(z)}. \end{aligned} \tag{10}$$

From (9) with $w(z) = g(z)$ and $W(z) = G(z)$ we have

$$\frac{R^n g(z)}{R^n G(z)} = (n+1) \cdot h(z) - n. \tag{11}$$

Now, (10) and (11) imply

$$z \cdot p'(z) = [(n+1)h(z) - n] \cdot \left[\frac{R^{n+1}f(z)}{R^n g(z)} - p(z) \right]$$

or

$$\frac{R^{n+1}f(z)}{R^n g(z)} = p(z) + \frac{1}{(n+1)h(z) - n} \cdot zp'(z). \tag{12}$$

By Remarks 3.1 and 2.2 we have $\frac{R^{n+1}f(z)}{R^n g(z)} \prec Q_\delta(z)$ and $h(z) \prec Q_\delta(z)$, where $Q_\delta(z)$ is given by (3) and is convex with Re $Q_\delta(z) > \frac{1+\delta}{2}$. Using this results and the hypothesis we obtain $\mathrm{Re}\,[(n+1)h(z) - n] > 0, \quad z \in U$ and $p(z) + \frac{1}{(n+1)h(z) - n} \cdot zp'(z) \prec Q_\delta(z)$, with $Q_\delta(z)$ convex in U.

In the conditions of Theorem 2.3 we have $\frac{R^{n+1}F(z)}{R^n G(z)} = p(z) \prec Q_\delta(z)$. Thus, by Remark 3.1, we conclude that $F(z) = If(z) \in UQ_n(\delta)$ with respect to the function $G(z) = Ig(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1, 1)$ and $\delta \geq \frac{n-1}{n+1}$. ☐

Theorem 3.2. *If $f(z) \in UQ_n(\delta)$ with respect to the function $g(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1, 1)$ and $c \in \mathbb{C}$ with* Re $c \geq \frac{n(1-\delta)-(1+\delta)}{2}$*, then $F(z) = I_c f(z) \in UQ_n(\delta)$ with respect to the function $G(z) = I_c g(z) \in UK_n(\delta)$, where I_c is the Bernardi integral operator defined by* (2)*.*

By Theorem 2.1 we have $G(z) = I_c g(z) \in UK_n(\delta)$ in the conditions of the hypothesis.

If we consider $w(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$, then we have

$$W(z) = I_c w(z) = z + \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j, \quad z \in U. \tag{13}$$

In a similar way, with the proof of the Theorem 3.1 (see relations (5), (6)), we obtain

$$(c+1)R^n w(z) - cR^n W(z) = z \left(R^n W(z) \right)', \quad n \in \mathbb{N}, \tag{14}$$

and

$$(c+1)R^n w(z) = (n+1)R^{n+1}W(z) + (c-n)R^n W(z), \quad n \in \mathbb{N}. \tag{15}$$

From (15) we have

$$(c+1)\frac{R^n w(z)}{R^n W(z)} = (n+1)\frac{R^{n+1}W(z)}{R^n W(z)} + (c-n), \quad n \in \mathbb{N}, \quad z \in U. \tag{16}$$

With the notations $\frac{R^{n+1}F(z)}{R^n G(z)} = p(z)$ and $\frac{R^{n+1}G(z)}{R^n G(z)} = h(z)$, in a similar way as in the proof of the above theorem we have

$$\frac{R^{n+1}f(z)}{R^n g(z)} = p(z) + \frac{1}{(n+1)h(z) + (c-n)} \cdot zp'(z). \tag{17}$$

By Remarks 3.1 and 2.2 we have $\dfrac{R^{n+1}f(z)}{R^n g(z)} \prec Q_\delta(z)$ and $h(z) \prec Q_\delta(z)$, where $Q_\delta(z)$ is given by (3) and is convex with Re $Q_\delta(z) > \frac{1+\delta}{2}$. Using this results and the hypothesis, we obtain $\mathrm{Re}\,[(n+1)h(z)+c-n] > 0,\ \ z \in U$, and

$$p(z) + \frac{1}{(n+1)h(z)+(c-n)} zp'(z) \prec Q_\delta(z),$$

with $Q_\delta(z)$ convex in U.

In the conditions of Theorem 2.3 we have $\dfrac{R^{n+1}F(z)}{R^n G(z)} = p(z) \prec Q_\delta(z)$, and thus the proof is complete.

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