LINEARIZATION AND HIGHER ORDER NONLINEAR OSCILLATION THEOREMS USING COMPARISON METHODS

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Abstract. We establish some new oscillation criteria for higher order nonlinear differential equations. The main idea involves comparing our equation with the related linear second order differential equations so that the known oscillation theorems from the literature can be employed directly.

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1. Introduction

In this paper we shall deal with the oscillatory behavior of all solutions of the functional differential equation

$$L_n x(t) + q(t) f(x[g(t)]) = 0, \tag{1.1}$$

where $n \geq 2$, and

$$\begin{cases} L_0 x(t) = x(t), \\ L_k x(t) = \dfrac{1}{a_k(t)} \dfrac{d}{dt} \left(L_{k-1} x(t) \right), \quad k = 1, 2, \ldots, n-1, \\ L_n x(t) = \dfrac{d}{dt} \left([L_{n-1} x(t)]^\alpha \right). \end{cases} \tag{1.2}$$

In what follows we shall assume that
(i). $a_i(t) \in C([t_0, \infty), \mathbb{R}^+ = (0, \infty))$, $t_0 \geq 0$,

$$\int\limits^\infty a_i(s)\,ds = \infty, \quad i = 1, 2, \ldots, n-1, \tag{1.3}$$

(ii). $q(t) \in C([t_0, \infty), \mathbb{R}^+)$,
(iii). $g(t) \in C([t_0, \infty), \mathbb{R} = (-\infty, \infty))$ and $\lim_{t \to \infty} g(t) = \infty$,
(iv). $f \in C(\mathbb{R}, \mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$, and
(v). α is the quotient of positive odd integers.
The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $x : [t_x, \infty) \to \mathbb{R}$, $t_x \geq t_0$, such that $L_j x(t)$, $j = 0, 1, \ldots, n$, exist and are continuous on $[t_x, \infty)$. Our attention is restricted to those solutions $x \in \mathcal{D}(L_n)$ of equation (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq t_x$. We make the standing hypothesis that equation (1.1) does possess such solutions. A solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise,

it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Our interest in this paper is to relate the oscillation of the nonlinear higher order equation (1.1), when f is not required to be a monotone function, to an appropriate linear second order ordinary differential equation. This enables us to employ numerous known results from the literature, see [4], [5].

In addition we establish some new oscillation criteria for higher order nonlinear equations of type (1.1). The obtained results are extended to obtain new oscillation criteria for neutral equations of the form

$$L_n(x(t) + p(t)x[h(t)]) + q(t)f(x[g(t)]) = 0,$$

where L_n and the functions g, q and f are as in equation (1.1), $p(t) \in C([t_0, \infty), \mathbb{R}^+ \cup \{0\})$ and $h(t) \in C([t_0, \infty), \mathbb{R})$ and $\lim_{t\to\infty} h(t) = \infty$.

Finally in Section 4 we consider a special case of equation (1.1), namely, the equation

$$\frac{d}{dt}\left(x^{(n-1)}(t)\right)^\alpha + q(t)f(x[g(t)]) = 0.$$

We note that linearization of nonlinear oscillation theorems has been the subject of intensive study and for some interesting results, we refer to Grace [7]–[8], Kwong and Wong [9] and Philos [11]. We also remark that several other related results are available in [2], [3].

2. Preliminaries

To formulate our results we shall use the following notation: Let $a_i(t) \in C([t_0, \infty), \mathbb{R})$, $i = 1, 2, \ldots$, we define $I_0 = 1$,

$$I_i(t, s; a_i, a_{i-1}, \ldots, a_1) = \int_s^t a_i(u)I_{i-1}(u, s; a_{i-1}, \ldots, a_1)du, \quad i = 1, 2, \ldots.$$

It is easy to verify from the definition of I_i that

$$I_i(t, s; a_1, \ldots, a_i) = (-1)^i I_i(s, t; a_i, \ldots, a_1)$$

and

$$I_i(t, s; a_1, \ldots, a_i) = \int_s^t a_i(u)I_{i-1}(t, u; a_1, \ldots, a_{i-1})du.$$

We shall need the following lemmas.

Lemma 2.1. *Suppose condition (1.3) holds. If* $x \in \mathcal{D}(\overline{L}_n)$ *where* $\overline{L}_n$ *is* L_n *defined by (1.2) with* $\alpha = 1$, *is eventually of one sign, then there exist a* $t_x \geq t_0 \geq 0$ *and an integer* ℓ, $0 \leq \ell \leq n$ *with* $n + \ell$ *even for* $x(t)\overline{L}_n x(t)$ *nonnegative eventually, or* $n + \ell$ *odd for* $x(t)\overline{L}_n x(t)$ *nonpositive eventually and such that for every* $t \geq t_x$,

$$\begin{cases} \ell > 0 \text{ implies } x(t)\overline{L}_k x(t) > 0, \ k = 0, 1, \ldots, \ell, \\ \ell \leq n - 1 \text{ implies } (-1)^{\ell-k}x(t)\overline{L}_k x(t) > 0, \ k = \ell, \ell+1, \ldots, n. \end{cases} \tag{2.1}$$

This lemma generalizes a well-known lemma of Kiguradze and can be proved similarly.

It will be convenient to make use of the following notation in the remainder of this paper. For any $T \geq t_0$ and all $t \geq T$, we let

$$w_\ell[t, T] = a_1(t) \int_T^t I_{\ell-2}(t, s; a_2, \ldots, a_{\ell-1}) a_\ell(s) I_{n-\ell-1}(t, s; a_{n-1}, \ldots, a_{\ell+1}) ds$$

$$\text{for } 2 \leq \ell \leq n - 1,$$

$$w[t, T] = \min_{2 \leq \ell \leq n-1} w_\ell[t, T],$$

$$\overline{w}_\ell[t, T] = \int_T^t I_{\ell-1}(t, s; a_1, \ldots, a_{\ell-1}) a_\ell(s) I_{n-\ell-1}(t, s; a_{n-1}, \ldots, a_{\ell+1}) ds,$$

$$1 \leq \ell \leq n - 1,$$

$$\overline{w}[t, T] = \min_{1 \leq \ell \leq n-1} \overline{w}_\ell[t, T].$$

For $t \geq s \geq T$, we let

$$w_1[t, s] = a_1(s) I_{n-2}(t, s; a_{n-1}, \ldots, a_2),$$
$$w_\lambda[t, T] = \min \{w_1[t, \lambda t], \ w_\ell[\lambda t, T], \ 2 \leq \ell \leq n - 1 \text{ and } 0 < \lambda < 1\}$$
$$\text{for } t \geq T/\lambda,$$
$$w_0[t, s] = I_{n-1}(t, s; a_{n-1}, \ldots, a_1), \ w_0^*[t, T] = \max_{1 \leq i \leq n-1} I_i(t, T; a_1, \ldots, a_i)$$

and

$$\overline{w}_0[t, s] = a_1(s) I_{n-2}(t, s; a_{n-1}, \ldots, a_2).$$

The following two lemmas can be found in [1,6].

Lemma 2.2. *Let $x \in \mathcal{D}(L_n)$ be eventually positive with condition (1.3) holding. Then*

(i_1). For $1 \leq \ell \leq n - 1$ and all $t \geq T \geq t_0$,

$$x(t) \geq \overline{w}[t, T] L_{n-1} x(t). \tag{2.2}$$

(i_2). For $2 \leq \ell \leq n - 1$ and all $t \geq T \geq t_0$,

$$x'(t) \geq w[t, T] L_{n-1} x(t). \tag{2.3}$$

(i_3). For $1 \leq \ell \leq n - 1$, and λ a constant with $0 < \lambda < 1$, there exists a $T^* \geq T/\lambda$ such that

$$x'[\lambda t] \geq w_\lambda[t, T] L_{n-1} x(t) \text{ for } t \geq T^*. \tag{2.4}$$

Lemma 2.3. *Let $x \in \mathcal{D}(L_n)$ be eventually positive with condition (1.3) holding and let $x(t)$ satisfy (2.1) with $\ell = 0$. Then for $t \geq s \geq T \geq t_0$,*

$$x(s) \geq w_0[t, s] L_{n-1} x(t) \tag{2.5}$$

and

$$-x'(s) \geq \overline{w}_0[t, s] L_{n-1} x(t). \tag{2.6}$$

Next, we shall need the following notation and a lemma due to Mahfoud [10]. Let

$$\mathbb{R}_{t_0} = \begin{cases} (-\infty, -t_0] \cup [t_0, \infty) & \text{if } t_0 > 0 \\ (-\infty, 0) \cup (0, \infty) & \text{if } t_0 = 0 \end{cases}$$

and

$$C_B(\mathbb{R}_{t_0}) = \{f \in C(\mathbb{R}, \mathbb{R}) : f \text{ is of bounded variation}$$
$$\text{on any interval } [a, b] \subset \mathbb{R}_{t_0}\}.$$

Lemma 2.4. *Suppose $t_0 > 0$ and $f \in C(\mathbb{R}, \mathbb{R})$. Then, $f \in C_B(\mathbb{R}_{t_0})$ if and only if $f(x) = H(x)G(x)$ for all $x \in \mathbb{R}$, where $G : \mathbb{R}_{t_0} \to \mathbb{R}^+$ is nondecreasing on $(-\infty, -t_0)$ and nonincreasing on (t_0, ∞) and $H : \mathbb{R}_{t_0} \to \mathbb{R}$ is nondecreasing on $\mathbb{R}_{t_0}$.*

We shall also need the following lemma.

Lemma 2.5 ([4], [11]). *Let $r(t) \in C([T, \infty), \mathbb{R}^+)$, $T \geq t_0$. If there exists a function $w(t) \in C^1([T, \infty), \mathbb{R})$ such that*

$$q(t) \leq -w'(t) - r(t)w^2(t) \text{ for every } t \geq T,$$

then the second order linear ordinary differential equation

$$\left(\frac{x'(t)}{r(t)}\right)' + q(t)x(t) = 0$$

is nonoscillatory.

3. Oscillation of the even order equation (1.1)

Here, we shall present oscillation criteria of comparison type for equation (1.1), when n is even. For this, we shall assume that there exists a function $\sigma(t) \in C^1([t_0, \infty), \mathbb{R})$ such that

$$\sigma(t) \leq \inf\{t, g(t)\}, \ \sigma'(t) > 0 \text{ for } t \geq t_0 \text{ and } \lim_{t \to \infty} \sigma(t) = \infty. \tag{3.1}$$

For all large $T \geq t_0$ with $\sigma(t) > T/\lambda$, $0 < \lambda < 1$, we let

$$w^*(t) = \lambda\sigma'(t)w_\lambda[\sigma(t), T]\overline{w}^{\alpha-1}[\lambda\sigma(t), T].$$

Theorem 3.1. *Let $\alpha \geq 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, and let G and H be a pair of continuous components of f with H being the nondecreasing one (and G as described in Lemma 2.4) satisfying*

$$H(x)sgn \ x \geq |x|^\beta \text{ for } x \neq 0, \tag{3.2}$$

where β is the quotient of two positive odd integers. Moreover, assume that the conditions (i)–(v), (1.3) and (3.1) hold. Equation (1.1) is oscillatory if there exists a function $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ and a constant $\lambda \in (0, 1)$ such that for all $T \geq t_0$ with $\sigma(t) > T/\lambda$, the linear second order ordinary differential equation

$$\left(\frac{1}{C(t)}z'(t)\right)' + p(t)z(t) = 0 \tag{3.3}$$

is oscillatory, where

$$p(t) = \rho(t)q(t)G(cw_0^*[g(t), T]) + \frac{1}{2}\left(\frac{\rho'(t)}{\rho(t)C(t)}\right)' - \frac{1}{4C(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2,$$

c   is any positive constant, and

$$\frac{\rho(t)}{\beta w^*(t)}C(t) = \begin{cases} c_1, \ c_1 \text{ is any positive constant, when } \beta > \alpha, \\ 1, \ \text{when } \beta = \alpha, \\ c_2(w_0^*[\lambda\sigma(t), T])^{\beta-\alpha}, \ c_2 \text{ is any positive constant, when } \beta < \alpha. \end{cases}$$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. It is easy to see that the sign of L_n and $\overline{L}_n$ are the same, where $\overline{L}_n$ is L_n defined by (1.2) with $\alpha = 1$. Since n is even, there exist $t_1 \geq t_0$ and an integer $\ell \in \{1, 3, \ldots, n-1\}$ such that (2.1) holds. There exist a $t_2 \geq t_1$ and a constant $b_1 > 0$ such that $L_{n-1}x(t) \leq b_1$ for $t \geq t_2$. Integrating the above inequality $n-1$ times, there exist a $t_3 \geq t_2$ and a constant $b > 0$ such that

$$x[g(t)] \leq bw_0^*[g(t), t_2] \text{ for } t \geq t_3. \tag{3.4}$$

From equation (1.1), it follows that

$$-L_n x(t) = q(t)f(x[g(t)]) = q(t)G(x[g(t)])H(x[g(t)])$$
$$\geq q(t)G(bw_0^*[g(t), t_2])x^\beta[\sigma(t)] \quad \text{for } t \geq t_3. \tag{3.5}$$

Define

$$w(t) = \rho(t)\frac{L_{n-1}^\alpha x(t)}{x^\beta[\lambda\sigma(t)]} \quad \text{for } t \geq t_3, \quad \lambda \in (0, 1).$$

Then, for $t \geq t_3$ we have

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$
$$- \lambda\beta\frac{\sigma'(t)}{\rho(t)}w^2(t)\frac{x'[\lambda\sigma(t)]}{L_{n-1}^\alpha x(t)}x^{\beta-1}[\lambda\sigma(t)]. \tag{3.6}$$

By Lemma 2.2 there exists a $t_4 \geq t_3$ such that for all $t \geq t_4/\lambda$,

$$x[\lambda\sigma(t)] \geq \overline{w}[\lambda\sigma(t), t_3]L_{n-1}x(t) \tag{3.7}$$

and

$$x'[\lambda\sigma(t)] \geq w_\lambda[\sigma(t), t_3]L_{n-1}x(t). \tag{3.8}$$

Using (3.7) and (3.8) in (3.6), we have

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$
$$- \frac{\beta}{\rho(t)}w^*(t)x^{\beta-\alpha}[\lambda\sigma(t)]w^2(t) \quad \text{for } t \geq t_4/\lambda. \tag{3.9}$$

Now, we consider the following three cases:

Case 1. $\beta > \alpha$. Since $x'(t) > 0$ eventually, there exist a constant $k_1 > 0$ and a $t_5 \geq t_4/\lambda$ such that

$$x[\lambda\sigma(t)] \geq k_1 \quad \text{for} \quad t \geq t_5. \tag{3.10}$$

Thus, inequality (3.9) becomes

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$

$$- \frac{\beta k_1^{\beta-\alpha}}{\rho(t)} w^*(t)w^2(t) \quad \text{for} \quad t \geq t_5. \tag{3.11}$$

Case 2. $\beta = \alpha$. In this case inequality (3.9) becomes

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$

$$- \frac{\beta}{\rho(t)}w^*(t)w^2(t) \quad \text{for} \quad t \geq t_5. \tag{3.12}$$

Case 3. $\beta < \alpha$. As in the above proof, there exist a constant $k_2 > 0$ and a $t_6 \geq t_4/\lambda$ such that

$$x[\lambda\sigma(t)] \leq k_2 w_0^*[\lambda\sigma(t), t_4] \quad \text{for} \quad t \geq t_6. \tag{3.13}$$

Using (3.13) in (3.9), we have

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$

$$- \frac{\beta k_2^{\beta-\alpha}}{\rho(t)}(w_0^*[\lambda\sigma(t), t_4])^{\beta-\alpha}w^*(t)w^2(t) \quad \text{for} \quad t \geq t_6. \tag{3.14}$$

Choose $T_1 \geq \max\{t_5, t_6\}$, from inequalities (3.11), (3.12) and (3.14), we get

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t) - C(t)w^2(t), \quad t \geq T_1, \tag{3.15}$$

where $C(t)$ here is given by

$$\frac{\rho(t)}{\beta w^*(t)}C(t) = \begin{cases} k_1^{\beta-\alpha} \text{ when } \beta > \alpha, \\ 1 \text{ when } \beta = \alpha, \\ k_2^{\beta-\alpha}(w_0^*[\lambda\sigma(t), t_4])^{\beta-\alpha} \text{ when } \beta < \alpha. \end{cases}$$

Clearly, $C(t) > 0$ for $t \geq T_1$. Now,

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) - C(t)\left[w^2(t) - \frac{\rho'(t)}{\rho(t)C(t)}w(t)\right]$$

$$= -\rho(t)q(t)G(bw_0^*[g(t), t_2]) - C(t)\left[w(t) - \frac{\rho'(t)}{2\rho(t)C(t)}\right]^2$$

$$+ \frac{1}{4C(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2 \text{ for } t \geq T_1,$$

or

$$\left(w(t) - \frac{\rho'(t)}{2\rho(t)C(t)}\right)' \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) - \left(\frac{\rho'(t)}{2\rho(t)C(t)}\right)'$$
$$+ \frac{1}{4C(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2 - C(t)\left[w(t) - \frac{\rho'(t)}{2\rho(t)C(t)}\right]^2, \quad t \geq T_1.$$

Set

$$y(t) = w(t) - \frac{\rho'(t)}{2\rho(t)C(t)}, \quad t \geq T_1$$

so we have

$$y'(t) \leq -\left[\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \left(\frac{\rho'(t)}{2\rho(t)C(t)}\right)'\right.$$
$$\left. - \frac{1}{4C(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2\right] - C(t)y^2(t) \quad \text{for} \quad t \geq T_1,$$

or

$$y'(t) \leq -p(t) - C(t)y^2(t) \text{ for } t \geq T_1, \tag{3.16}$$

where $p(t)$ is given by

$$p(t) = \rho(t)q(t)G(bw_0^*[g(t), t_2]) + \left(\frac{\rho'(t)}{2\rho(t)C(t)}\right)' - \frac{1}{4C(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2.$$

Applying Lemma 2.5 to inequality (3.16), we conclude that the linear equation (3.3) is nonoscillatory, which is a contradiction. This completes the proof. □

Next, we shall consider equation (1.1) with $0 < \alpha \leq 1$ and prove the following result.

Theorem 3.2. *Let $0 < \alpha \leq 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$ and let G and H be a pair of continuous components of f with H being the nondecreasing one (and G as described in Lemma 2.4). Moreover, assume that the conditions (i)–(v), (1.3), (3.1) and (3.2) hold. Equation (1.1) is oscillatory if there exist a function $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ and a constant $\lambda, 0 < \lambda < 1$ such that for all $T \geq t_0$ with $\sigma(t) > T/\lambda$, the linear second order ordinary differential equation*

$$\left(\frac{1}{Q^*(t)}y'(t)\right)' + P(t)y(t) = 0 \tag{3.17}$$

is oscillatory, where

$$Q^*(t) = \lambda\beta\sigma'(t)\rho^{-1/\alpha}(t)w_\lambda[\sigma(t), T]Q^{(1/\alpha)-1}(t)c(t),$$

$$Q(t) = \rho(t)\int_t^\infty q(s)G(c^*w_0^*[g(s), T])ds,$$

$$P(t) = \rho(t)q(t)G(c^*w_0^*[g(t), T]) + \frac{1}{2}\left(\frac{\rho'(t)}{\rho(t)Q^*(t)}\right)' - \frac{1}{4Q^*(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2$$

c^* is any positive constant and

$$c(t) = \begin{cases} c_1, \ c_1 > 0 \ is \ any \ constant, \ when \ \beta > \alpha, \\ 1, \ when \ \beta = \alpha, \\ c_2 \left(w_0^*[\lambda\sigma(t), T] \right)^{(\beta/\alpha)-1}, \ c_2 > 0 \ is \ any \ constant, \ when \ \beta < \alpha. \end{cases}$$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Define the function $w(t)$ as in the proof of Theorem 3.1 and obtain (3.4) – (3.8) for $t \geq t_4$. Now, one can easily obtain

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$
$$- \lambda\beta\sigma'(t)\rho^{-1/\alpha}(t)w_\lambda[\sigma(t), t_3]w^2(t)w^{(1/\alpha)-1}(t)x^{(\beta/\alpha)-1}[\lambda\sigma(t)]. \quad (3.18)$$

As a result

$$\left(\frac{w(t)}{\rho(t)} \right)' = \frac{1}{\rho(t)} \left[w'(t) - \frac{\rho'(t)}{\rho(t)}w(t) \right] \leq -q(t)G(bw_0^*[g(t), t_2]), \ t \geq t_3. \quad (3.19)$$

Integrating (3.19) from $t \geq t_3$ to u and letting $u \to \infty$, we have

$$w(t) \geq Q(t) \quad for \quad t \geq t_3. \quad (3.20)$$

Using (3.20) in (3.18), we obtain

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$
$$- \frac{Q^*(t)}{c(t)}x^{(\beta/\alpha)-1}[\lambda\sigma(t)]w^2(t) \quad for \quad t \geq t_4. \quad (3.21)$$

The rest of the proof is similar to that of Theorem 3.1 and hence omitted. $\square$

The following corollaries are immediate.

Corollary 3.1. *Let* $\alpha \geq 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one (and* G *as described in Lemma 2.4). Moreover, assume that the conditions (i)–(v), (1.3), (3.1) and (3.2) hold. If for some constant* λ, $0 < \lambda < 1$, *and all large* $T \geq t_0$ *with* $\sigma(t) > T/\lambda$, *the second order linear equation*

$$\left(\frac{1}{a(t)}y'(t) \right)' + p(t)y(t) = 0 \quad (3.22)$$

is oscillatory, where

$$p(t) = q(t)G(cw_0^*[g(t), T]), \quad c \ is \ any \ positive \ constant,$$
$$a(t) = \beta w^*(t)c(t)$$

and

$$c(t) = \begin{cases} c_1, \ c_1 > 0 \ is \ any \ constant, \ when \ \beta > \alpha, \\ 1, \ when \ \beta = \alpha, \\ c_2 \left(w_0^*[\lambda\sigma(t), T] \right)^{\beta-\alpha}, \ c_2 > 0 \ is \ any \ constant, \ when \ \beta < \alpha, \end{cases}$$

then equation (1.1) *is oscillatory.*

Corollary 3.2. *Let* $0 < \alpha \le 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \ge 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one* (*and* G *as described in Lemma* 2.4). *In addition, assume that the conditions* (i)–(v), (1.3), (3.1) *and* (3.2) *hold. Equation* (1.1) *is oscillatory if for some constant* λ, $0 < \lambda < 1$, *and all large* $T \ge t_0$ *with* $\sigma(t) > T/\lambda$, *the second order linear equation*

$$\left(\frac{1}{\overline{a}(t)} z'(t)\right)' + p(t)z(t) = 0 \tag{3.23}$$

is oscillatory, where $p(t)$ *is as in equation* (3.22),

$$\overline{a}(t) = \lambda\beta\sigma'(t)w_\lambda[\sigma(t), T] \left(\int\limits_t^\infty p(s)ds\right)^{(1/\alpha)-1} \overline{c}(t)$$

and

$$\overline{c}(t) = \begin{cases} c_1, \ c_1 > 0 \text{ is any constant, when } \beta > \alpha, \\ 1, \text{ when } \beta = \alpha, \\ c_2 \left(w_0^*[\lambda\sigma(t), T]\right)^{(\beta/\alpha)-1}, \ c_2 > 0 \text{ is any constant, when } \beta < \alpha. \end{cases}$$

The proof of Corollaries 3.1 and 3.2 can be obtained from that of Theorems 3.1 and 3.2, respectively, by letting $\rho(t) = 1$. The details are omitted.

Corollary 3.3. *Let* $\alpha = 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \ge 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one* (*and* G *as described in Lemma* 2.4). *Moreover, assume that the conditions* (i)–(v), (1.3), (3.1) *and* (3.2) *hold. Equation* (1.1) *is oscillatory if there exist a function* $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ *and a constant* λ, $0 < \lambda < 1$, *such that for all large* $T \ge t_0$ *with* $\sigma(t) > T/\lambda$, *the second order linear equation*

$$\left(\frac{1}{r(t)} y'(t)\right)' + \overline{p}(t)y(t) = 0 \tag{3.24}$$

is oscillatory, where

$$r(t) = \lambda\beta\frac{\sigma'(t)}{\rho(t)}w_\lambda[\sigma(t), T]c(t),$$

$$\overline{p}(t) = \rho(t)q(t)G(c^*w_0^*[g(t), T]) + \frac{1}{2}\left(\frac{\rho'(t)}{r(t)\rho(t)}\right)' - \frac{1}{4r(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2,$$

$c^* > 0$ *is any constant, and*

$$c(t) = \begin{cases} c_1, \ c_1 > 0 \text{ is any constant, when } \beta > 1, \\ 1, \text{ when } \beta = 1, \\ c_2 \left(w_0^*[\lambda\sigma(t), T]\right)^{\beta-1}, \ c_2 > 0 \text{ is any constant, when } \beta < 1. \end{cases}$$

Corollary 3.4. *Let* $\alpha = 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one (and* G *as described in Lemma* 2.4*). In addition, suppose that the conditions* (i)–(v), (1.3), (3.1) *and* (3.2) *hold. Equation* (1.1) *is oscillatory if for some constant* λ, $0 < \lambda < 1$, *and all large* $T \geq t_0$ *with* $\sigma(t) > T/\lambda$, *the second order linear equation*

$$\left(\frac{1}{\overline{r}(t)}y'(t)\right)' + p(t)y(t) = 0 \tag{3.25}$$

is oscillatory, where $p(t)$ *is as in equation* (3.22),

$$\overline{r}(t) = \lambda\beta\sigma'(t)w_\lambda[\sigma(t), T]c(t)$$

and $c(t)$ *is as in Corollary* 3.3.

Remark 1. We note that the above results can be applied to equations of type (1.1) with f being any of the following functions:

(i) $f(x)$ is a nondecreasing function. In this case we let $f(x) = H(x)$ and $G(x) = 1$,

(ii) $f(x) = \dfrac{|x|^{\beta-1}x}{1 + |x|^\gamma}$, where β and γ are positive constants,

(iii) $f(x) = |x|^{\beta-1}x \, \exp(-|x|^\gamma)$, where β and γ are positive constants,

(iv) $f(x) = |x|^{\beta-1}x \, \mathrm{sech}\, x$, where β is a positive constant.

Remark 2. In all of the above results we require the oscillation of linear second order ordinary differential equations of the form

$$(a(t)x'(t))' + q(t)x(t) = 0, \tag{3.26}$$

where $a(t) \in C([t_0, \infty), \mathbb{R}^+)$ and $q(t) \in C([t_0, \infty), \mathbb{R})$, so, it would be of interest if we give some oscillation criteria for equation (3.26).

Each of the following conditions is sufficient for the oscillation of (3.26).

(I$_1$) $\displaystyle\int^\infty \frac{1}{a(s)}ds = \infty$ and $\displaystyle\int^\infty q(s)ds = \infty$,

(I$_2$) $\displaystyle\liminf_{t\to\infty} A(t)Q(t) > \frac{1}{4}$, where $A(t) = \displaystyle\int_{t_0}^t \frac{1}{a(s)}ds$ and $Q(t) = \displaystyle\int_t^\infty q(s)ds$,

(I$_3$) there exists a function $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ such that

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\rho(s)q(s) - \frac{(\rho'(s))^2}{4a(s)\rho(s)}\right] ds = \infty.$$

For more oscillation results for equation (3.26), we refer the reader to the monograph by Agarwal et. al. [4].

4. Oscillation of the Odd Order Equation (1.1)

In this section, we shall present oscillation criteria of comparison type for equation (1.1) when n is odd.

Theorem 4.1. *Let $\alpha \geq 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, and let G and H be a pair of continuous components of f with H being the nondecreasing one (and G as described in Lemma 2.4). Moreover, suppose that the conditions (i)–(v), (1.3), (3.1) and (3.2) hold. Every unbounded solution of equation (1.1) is oscillatory if there exists a function $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ such that for all large $T \geq t_0$ with $\sigma(t) > T$ and all constant $c > 0$, the linear second order ordinary differential equation*

$$\left(\frac{1}{r(t)} y'(t)\right)' + p(t)y(t) = 0 \tag{4.1}$$

is oscillatory, where

$$r(t) = \beta \frac{\sigma'(t)}{\rho(t)} w[\sigma(t), T] \overline{w}^{\alpha-1}[\sigma(t), T] c(t),$$

$$p(t) = \rho(t)q(t)G(cw_0^*[g(t), T]) + \frac{1}{2}\left(\frac{\rho'(t)}{r(t)\rho(t)}\right)' - \frac{1}{4r(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2$$

and

$$c(t) = \begin{cases} c_1, & c_1 > 0 \text{ is any constant, when } \beta > \alpha, \\ 1, & \text{when } \beta = \alpha, \\ c_2\left(w_0^*[\sigma(t), T]\right)^{\beta-\alpha}, & c_2 > 0 \text{ is any constant, when } \beta < \alpha. \end{cases}$$

Proof. Let $x(t)$ be an unbounded eventually positive solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. By Lemma 2.1 there exist an integer $\ell \in \{2, 4, \ldots, n-1\}$ and a $t_1 \geq t_0$ such that (2.1) holds. As in the proof of Theorem 3.1, we obtain (3.5) for $t \geq t_3$. Define

$$w(t) = \rho(t) \frac{L_{n-1}^{\alpha} x(t)}{x^{\beta}[\sigma(t)]}, \quad t \geq t_3.$$

Then, for $t \geq t_3$ we have

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)} w(t)$$

$$- \beta \frac{\sigma'(t)}{\rho(t)} \frac{x'[\sigma(t)]}{L_{n-1}^{\alpha} x(t)} x^{\beta-1}[\sigma(t)] w^2(t). \tag{4.2}$$

By Lemma 2.2, there exists a $t_4 \geq t_3$ such that $\sigma(t) > t_3$ for $t \geq t_4$,

$$x[\sigma(t)] \geq \overline{w}[\sigma(t), t_3] L_{n-1} x(t), \quad t \geq t_4, \tag{4.3}$$

and

$$x'[\sigma(t)] \geq w[\sigma(t), t_3] L_{n-1} x(t), \quad t \geq t_4. \tag{4.4}$$

Using (4.3) and (4.4) in (4.2) for $t \geq t_4$, we have

$$w'(t) \leq -\rho(t)q(t)G(bw_0^*[g(t), t_2]) + \frac{\rho'(t)}{\rho(t)}w(t)$$

$$- \beta \frac{\sigma'(t)}{\rho(t)}w[\sigma(t), t_3]\overline{w}^{\alpha-1}[\sigma(t), t_3]x^{\beta-\alpha}[\sigma(t)]w^2(t).$$

The rest of the proof is similar to that of Theorem 3.1 and hence omitted. $\square$

Theorem 4.2. *Let* $0 < \alpha \leq 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one* (*and* G *as described in Lemma* 2.4). *Moreover, assume that the conditions* (i)–(v), (1.3), (3.1) *and* (3.2) *hold. Every unbounded solution of equation* (1.1) *is oscillatory if there exists a function* $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that for all large* $T \geq t_0$ *with* $\sigma(t) > T$, *the linear second order ordinary differential equation*

$$\left(\frac{1}{\overline{r}(t)}y'(t) \right)' + \overline{p}(t)y(t) = 0 \tag{4.5}$$

is oscillatory, where

$$\overline{r}(t) = \beta \sigma'(t)\rho^{-1/\alpha}(t)w[\sigma(t), T]Q^{(1/\alpha)-1}(t)c(t),$$

$$Q(t) = \rho(t) \int_t^\infty q(s)G(cw_0^*[g(s), T])ds, \quad c \text{ is any positive constant,}$$

$$\overline{p}(t) = \rho(t)q(t)G(cw_0^*[g(t), T]) + \frac{1}{2}\left(\frac{\rho'(t)}{\overline{r}(t)\rho(t)} \right)' - \frac{1}{4\overline{r}(t)}\left(\frac{\rho'(t)}{\rho(t)} \right)^2$$

and

$$c(t) = \begin{cases} c_1, \ c_1 > 0 \text{ is any constant, when } \beta > \alpha, \\ 1, \text{ when } \beta = \alpha, \\ c_2 \left(w_0^*[\sigma(t), T] \right)^{(\beta/\alpha)-1}, \ c_2 > 0 \text{ is any constant, when } \beta < \alpha. \end{cases}$$

The proof can be modelled on that of Theorem 3.2 and hence omitted.

Next, we are concerned with the oscillatory behavior of all bounded solutions of equation (1.1).

For any function $g(t) \in C([t_0, \infty), \mathbb{R})$, we put

$$\mathcal{R}_g = \{t \in [t_0, \infty), \ g(t) < t\}.$$

We also introduce the notation: $\tau(t) = \max\{\min\{s, g(s)\} : t_0 \leq s \leq t\}$ and note that the following inequality holds $g(s) \leq \tau(t)$ for $\tau(t) < s < t$.

Now, we present the following results.

Theorem 4.3. *Let* $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one* (*and* G *as described in Lemma* 2.4). *In addition, suppose that the conditions* (i)–(v), (1.3),

(3.1) *and* (3.2) *hold. Every bounded solution of equation* (1.1) *is oscillatory if for every constant* $\xi > 0$ *either*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s)w_0^{\alpha}[\tau(t), g(s)]ds > \xi \quad \text{when } \alpha = \beta, \tag{4.6}$$

or

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s)w_0^{\beta}[\tau(t), g(s)]ds > 0 \quad \text{when } \alpha > \beta. \tag{4.7}$$

Remark 3. Of course in Theorem 4.3 for (4.6) to hold we need the left hand side of the inequality in (4.6) to be infinity. However we can improve (4.6) if we know a little more about G. For example if G is a constant (say $G(x) = a_0 > 0$) then it is enough to assume (4.6) only for $\xi = \frac{1}{a_0}$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. By Lemma 2.1, there exists a $t_1 \geq t_0$ such that $\inf\{g(t) : t \geq t_1\} > t_0$ and

$$(-1)^i L_j x(t) > 0, \ 0 \leq j \leq n \quad \text{on} \quad [t_1, \infty). \tag{4.8}$$

By Lemma 2.3 there exists a $t_2 \geq t_1$ such that

$$x[g(s)] \geq w_0[\tau(t), g(s)]L_{n-1}x[\tau(t)] \quad \text{for} \quad t \geq s \geq t_2. \tag{4.9}$$

Since $x(t)$ is bounded on $[t_0, \infty)$, there exist a constant $b > 0$ and a $t_3 \geq t_2$ such that

$$x[g(t)] \leq b \quad \text{for} \quad t \geq t_3. \tag{4.10}$$

Using condition (3.2), (4.9) and (4.10) in equation (1.1), we have

$$-\frac{d}{ds}(L_{n-1}x(s))^{\alpha} = q(s)f(x[g(s)]) = q(s)G(x[g(s)])H(x[g(s)])$$

$$\geq q(s)G(b)x^{\beta}[g(s)] \geq q(s)G(b)w_0^{\beta}[\tau(t), g(s)]L_{n-1}^{\beta}x[\tau(t)]$$
$$\text{for } t \geq s \geq t_3.$$

Integrating both sides of the above inequality from $\tau(t)$ to t, we have

$$L_{n-1}^{\alpha}x[\tau(t)] \geq L_{n-1}^{\beta}x[\tau(t)] \int_{\tau(t)}^{t} q(s)w_0^{\beta}[\tau(t), g(s)]G(b)ds. \tag{4.11}$$

Now, we consider the following two cases:
Case 1. $\alpha = \beta$. In this case (4.11) reduces to

$$L_{n-1}^{\alpha}x[\tau(t)]\left[\int_{\tau(t)}^{t} q(s)w_0^{\alpha}[\tau(t), g(s)]G(b)ds - 1\right] \leq 0 \quad \text{for} \quad t \geq t_1.$$

But this is inconsistent with (4.6).

Case 2. $\alpha > \beta$. It follows from (4.11) that

$$L_{n-1}^{\alpha-\beta}x[\tau(t)] \geq \int_{\tau(t)}^{t} q(s)w_0^{\beta}[\tau(t), g(s)]G(b)ds. \tag{4.12}$$

Taking lim sup of both sides of (4.12) as $t \to \infty$, we see that the left-hand side approaches zero, which contradicts (4.7). This completes the proof. $\square$

Theorem 4.4. *Let the hypotheses of Theorem* 4.3 *hold with* $g(t) < t$ *for* $t \geq t_0$ *and conditions* (4.6) *and* (4.7) *are replaced, respectively, by*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} I_{n-2}(v, g(t); a_{n-2}, \ldots, a_1)a_{n-1}(v)\left(\int_v^t q(u)du\right)^{1/\alpha} dv > \xi,$$
$$when \quad \alpha = \beta \tag{4.13}$$

$\xi > 0$ *is any constant, or*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} I_{n-2}(v, g(t); a_{n-2}, \ldots, a_1)a_{n-1}(v)\left(\int_v^t q(u)du\right)^{1/\alpha} dv > 0$$
$$when \quad \alpha > \beta. \tag{4.14}$$

Then the conclusion of Theorem 4.3 *holds.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. There exists a $t_1 \geq t_0$ such that (4.8) holds for $t \geq t_1$. Choose a $t_2 > t_1$ such that $\inf\{g(t) : t \geq t_2\} > t_1$. By Taylor's formula with remainder it is easy to see that

$$x[g(t)] \geq \int_{g(t)}^{t} I_{n-2}(v, g(t); a_{n-2}, \ldots, a_1)a_{n-1}(v)(L_{n-1}x(v))dv$$

$$for \quad t > g(t) \geq t_2. \tag{4.15}$$

As in the proof of Theorem 4.3, we obtain (4.10) and from equation (1.1), we have

$$-\frac{d}{dt}(L_{n-1}^{\alpha}x(t)) \geq q(t)G(b)x^{\beta}[g(t)] \quad for \quad t \geq t_3 \geq t_2. \tag{4.16}$$

Integrating (4.16) from v to t, we obtain

$$L_{n-1}x(v) \geq \left(\int_v^t q(u)G(b)x^{\beta}[g(u)]du\right)^{1/\alpha} \quad for \quad t \geq v \geq t_3. \tag{4.17}$$

Substituting (4.17) in (4.15), we have

$$x[g(t)] \geq G(b)\int_{g(t)}^{t} I_{n-2}(v, g(t); a_{n-1}, \ldots, a_1)a_{n-1}(v)\left(\int_v^t q(u)du\right)^{1/\alpha} x^{\beta/\alpha}[g(t)]dv.$$

The rest of the proof is similar to that of Theorem 4.3 and hence omitted.   □

Theorem 4.5. *Let the hypotheses of Theorem* 4.3 *hold with* $0 < \beta < \alpha$ *and condition* (4.7) *is replaced by*

$$\int_{\mathcal{R}_g} q(s) w_0^\beta[s, g(s)] ds = \infty. \tag{4.18}$$

Then the conclusion of Theorem 4.3 *holds.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. There exists a $t_1 \geq t_0$ such that (4.8) holds for $t \geq t_1$. Choose a $t_2 \geq t_1$ such that $\inf\{g(t) : t \geq t_2\} > t_1$. As in the proof of Theorem 4.3, we obtain (4.10) and

$$x[g(t)] \geq w_0[t, g(t)] L_{n-1} x(t) \quad \text{for} \quad t \in \mathcal{R}_g \cap [t_2, \infty). \tag{4.19}$$

Using condition (3.2), (4.10) and (4.19) and letting $u(t) = L_{n-1} x(t) > 0$ on $\mathcal{R}_g \cap [t_2, \infty)$, we have

$$-\frac{du(t)}{dt} \geq q(t) G(b) w_0^\beta[t, g(t)] u^{\beta/\alpha}(t) \quad \text{for} \quad t \in \mathcal{R}_g \cap [t_2, \infty).$$

Choose $T \geq t_2$ arbitrarily. Dividing both sides of the above inequality by $u^{\beta/\alpha}(t)$ and integrating on $\mathcal{R}_g \cap [t_2, T]$, we find

$$\int_{\mathcal{R}_g \cap [t_2, T]} q(s) w_0^\beta[s, g(s)] ds \leq \frac{1}{G(b)} \int_T^{t_2} u^{-\beta/\alpha}(s) u'(s) ds = \frac{1}{G(b)} \int_{u(T)}^{u(t_2)} u^{-\beta/\alpha} du$$

$$= \frac{1}{G(b)} \left(\frac{\alpha}{\alpha - \beta} \right) \left[u^{1-(\beta/\alpha)}(t_2) - u^{1-(\beta/\alpha)}(T) \right].$$

Letting $T \to \infty$, we conclude that

$$\int_{\mathcal{R}_g \cap [t_2, \infty)} q(s) w_0^\beta[s, g(s)] ds \leq \frac{1}{G(b)} \left(\frac{\alpha}{\alpha - \beta} \right) u^{1-(\beta/\alpha)}(t_2) < \infty,$$

which contradicts condition (4.18). This completes the proof.   □

Now, we are ready to state oscillation theorems for the odd order equation (1.1).

Theorem 4.6. *Let* $\alpha \geq 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one (and* G *as described in Lemma* 2.4*). Moreover, assume that conditions* (i)–(v), (1.3), (3.1) *and* (3.2) *hold with* $0 < \beta \leq \alpha$. *Equation* (1.1) *is oscillatory if there exists a function* $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that for all large* $T \geq t_0$ *with* $\sigma(t) > T$ *and all positive constants* c *and* ξ, *the linear equation* (4.1) *is oscillatory where* $r(t)$ *and* $p(t)$ *are as in Theorem* 4.1 *and*

$$c(t) = \begin{cases} 1 \text{ when } \alpha = \beta, \\ c_2 \left(w_0^*[\sigma(t), T] \right)^{\beta-\alpha}, \text{ where } c_2 > 0 \text{ is any constant, when } \alpha > \beta \end{cases}$$

and condition (4.6) or (4.13) holds when $\alpha = \beta$ or condition (4.7) or (4.14) or (4.18) holds when $\alpha > \beta$.

Theorem 4.7. *Let $\alpha \leq 1$, $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, and let G and H be a pair of continuous components of f with H being the nondecreasing one (and G as described in Lemma 2.4). In addition, assume that conditions (i)–(v), (1.3), (3.1) and (3.2) hold with $0 < \beta \leq \alpha$. Equation (1.1) is oscillatory if there exists a function $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ such that for all large $T \geq t_0$ with $\sigma(t) > T$ and all positive constants c and ξ, the linear equation (4.5) is oscillatory, where $\bar{r}(t)$ and $\bar{p}(t)$ are as in Theorem 4.2 and*

$$c(t) = \begin{cases} 1 \ \text{when} \ \alpha = \beta, \\ c_2 \left(w_0^*[\sigma(t), T]\right)^{(\beta/\alpha)-1}, \ \text{where} \ c_2 > 0 \ \text{is any constant, when} \ \alpha > \beta \end{cases}$$

and either condition (4.6) or (4.13) holds when $\alpha = \beta$ or condition (4.7) or (4.14) or (4.18) holds when $\alpha > \beta$.

Next, we state an interesting special case of Theorem 4.6.

Corollary 4.1. *Let $\alpha = 1$, conditions (i)–(v), (1.3) and (3.1) hold and f satisfies the condition*

$$f(x)\,sgn\,x \geq |x|^\beta \ \text{for} \ x \neq 0,$$

where β is the quotient of two positive odd integers. If for all large $T \geq t_0$ with $\sigma(t) > T$, the linear equation

$$\left(\frac{1}{a(t)}y'(t)\right)' + p(t)y(t) = 0$$

is oscillatory, where

$$a(t) = \beta\sigma'(t)w[\sigma(t), T]c(t), \ p(t) = q(t)$$

and

$$c(t) = \begin{cases} 1 \ \text{when} \ \beta = 1, \\ c_2 \left(w_0^*[\sigma(t), T]\right)^{\beta-1}, \ \text{where} \ c_2 > 0 \ \text{is any constant, when} \ \beta < 1. \end{cases}$$

Moreover, assume that either

$$\limsup_{t \to \infty} \int_{\tau(t)}^{t} q(s)w_0^\alpha[\tau(t), g(s)]ds \ > \ 1,$$

or $g(t) < t$ for $t \geq t_0$, and

$$\limsup_{t \to \infty} \int_{g(t)}^{t} I_{n-2}(v, g(t); a_{n-2}, \ldots, a_1)a_{n-1}(v)\left(\int_{v}^{t} q(s)ds\right)^{1/\alpha} ds > 1 \ \text{when} \ \beta = 1,$$

or condition (4.7) or (4.14) or (4.18) holds when $0 < \beta < 1$, then equation (1.1) is oscillatory.

5. APPLICATIONS

In this section, we shall extend our previous results to neutral equations of the type

$$L_n(x(t) + p(t)x[h(t)]) + q(t)f(x[g(t)]) = 0, \tag{5.1}$$

where $q(t), g(t), f(x)$ and L_n are as in equation (1.1), $p(t) \in C([t_0, \infty), \mathbb{R}^+ \cup \{0\})$ and $h(t) \in C([t_0, \infty), \mathbb{R})$, $h'(t) > 0$ for $t \geq t_0$ and $\lim_{t \to \infty} h(t) = \infty$.

Set

$$y(t) = x(t) + p(t)x[h(t)], \ t \geq t_0 \geq 0. \tag{5.2}$$

Then for $t \geq t_0$, we have

$$\frac{d}{dt}(L_{n-1}^\alpha y(t)) + q(t)f(x[g(t)]) = 0. \tag{5.3}$$

Now, if $x(t)$ is a nonoscillatory solution of equation (5.1), say, $x(t) > 0$, $x[h(t)] > 0$ and $x[g(t)] > 0$ for $t \geq t_0 \geq 0$. Then $y(t) > 0$ for $t \geq t_0$ and there exists a $t_1 \geq t_0$ such that either (I_1). $y'(t) > 0$, or (I_2). $y'(t) < 0$ for $t \geq t_1$.

(I_1). Suppose $y'(t) > 0$ for $t \geq t_1$. This case occurs when n is even, or when we are concerned with unbounded solutions of odd order equation (5.1).

We shall examine the following two cases for $p(t)$ and $h(t)$:

$$(I). \ \{0 \leq p(t) \leq 1, \ h(t) < t\} \text{ and } (II). \ \{p(t) \geq 1, \ h(t) > t\}.$$

For the case (I), we assume that

$$0 \leq p(t) \leq 1, \ h(t) < t \text{ and } p(t) \not\equiv 1 \text{ eventually.} \tag{5.4}$$

Now, we have for $t \geq t_1$,

$$x(t) = y(t) - p(t)x[h(t)] = y(t) - p(t)[y[h(t)] - p[h(t)]x[h \circ h(t)]]$$
$$\geq y(t) - p(t)y[h(t)] \geq (1 - p(t))y(t). \tag{5.5}$$

Next, if $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, then $f(x) = G(x)H(x)$. Now, if condition (3.2) holds, then equation (5.3) becomes

$$0 = L_n y(t) + q(t)f(x[g(t)])$$
$$= L_n y(t) + q(t)G(x[g(t)])H(x[g(t)])$$
$$\geq L_n y(t) + q(t)G(x[g(t)])(1 - p[g(t)])^\beta y^\beta[g(t)] \text{ for } t \geq t_1.$$

Using the fact that $x(t) \geq y(t)$, there exist a constant $b > 0$ and a $t_2 \geq t_1$ such that

$$y[g(t)] \leq bw_0^*[g(t), t_2] \text{ for } t \geq t_3 \text{ for some } t_3 \geq t_2.$$

Thus, we see that

$$L_n y(t) + q(t)(1 - p[g(t)])^\beta G(bw_0^*[g(t), t_2])y^\beta[g(t)] \leq 0 \text{ for } t \geq t_3. \tag{5.6}$$

Next, for the case (II), we assume that

$$p(t) \geq 1, \ \tau(t) > t \text{ and } p(t) \not\equiv 1 \text{ eventually.} \tag{5.7}$$

We also let

$$p^*(t) = \frac{1}{p[h^{-1}(t)]} \left(1 - \frac{1}{p[h^{-1} \circ h^{-1}(t)]}\right) \quad \text{for all large } t,$$

where h^{-1} is the inverse function of h. Now, we have

$$\begin{aligned}
x(t) &= \frac{1}{p[h^{-1}(t)]} \left(y[h^{-1}(t)] - x[h^{-1}(t)]\right) \\
&= \frac{y[h^{-1}(t)]}{p[h^{-1}(t)]} - \frac{1}{p[h^{-1}(t)]} \left(\frac{y[h^{-1} \circ h^{-1}(t)]}{p[h^{-1} \circ h^{-1}(t)]} - \frac{x[h^{-1} \circ h^{-1}(t)]}{p[h^{-1} \circ h^{-1}(t)]}\right) \\
&\geq \frac{y[h^{-1}(t)]}{p[h^{-1}(t)]} - \frac{y[h^{-1} \circ h^{-1}(t)]}{p[h^{-1}(t)]p[h^{-1} \circ h^{-1}(t)]} \\
&\geq \frac{1}{p[h^{-1}(t)]} \left(1 - \frac{1}{p[h^{-1} \circ h^{-1}(t)]}\right) y[h^{-1}(t)] \\
&= p^*(t)y[h^{-1}(t)] \quad \text{for } t \geq t_2.
\end{aligned} \tag{5.8}$$

As before, one can easily obtain

$$L_n y(t) + q(t)f(p^*[g(t)])^\beta G(bw_0^*[g(t), t_2])y^\beta[h^{-1} \circ g(t)] \leq 0 \quad \text{for } t \geq t_3. \tag{5.9}$$

Now, it is easy to restate results similar to Theorems 3.1, 3.2, 4.1 and 4.2 for equation (5.1) when either (I), or (II) holds. Such formulations are left to the reader.

(I_2). Suppose $y'(t) < 0$ for $t \geq t_1$. This occurs when we are concerned with bounded solutions of odd order equation (5.1). Here, we can examine the two cases:

$$(III).\ \{0 \leq p(t) \leq 1,\ h(t) > t\} \text{ and } (IV).\ \{p(t) \geq 1,\ h(t) < t\}.$$

One can proceed as above and obtain similar results. Here, we omit the details.

Next, we shall consider a special case of equation (1.1), namely, the equation

$$\frac{d}{dt}\left(x^{(n-1)}(t)\right)^\alpha + q(t)f(x[g(t)]) = 0. \tag{5.10}$$

In this case one can easily compute the following: For all large $T \geq t_0$ and $t \geq s \geq T$, we have

$$w_\ell[t, T] = \frac{(t - T)^{n-2}}{(n - 2)\,(\ell - 2)!\,(n - \ell - 1)!}, \quad w_1[t, s] = \frac{(t - s)^{n-2}}{(n - 2)!}.$$

Now, it is easy to see that for some $\lambda,\ 0 < \lambda < 1,$

$$\begin{aligned}
w_\lambda[t, T] &= \min\{w_1[t, \lambda t],\ w_\ell[t, T],\ \ell \in \{1, 2, \ldots, n - 1\}\} \\
&\geq \frac{(\lambda(1 - \lambda))^{n-2}}{(n - 2)!}t^{n-2} \quad \text{for all large } t
\end{aligned}$$

(see also, Philos [11]). Inequalities (2.2), (2.3) and (2.4) in Lemma 2.2 take, respectively, the form

$$x(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} x^{(n-1)}(t), \qquad (2.2)'$$

$$x'(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} x^{(n-1)}(t) \qquad (2.3)'$$

and

$$x'[\lambda t] \geq \frac{(\lambda(1-\lambda))^{n-2}}{(n-2)!} t^{n-2} x^{(n-1)}(t). \qquad (3.4)'$$

Also, we find

$$w_0[t,s] = \frac{(t-s)^{n-1}}{(n-1)!}, \quad w_0^*[t,T] = \frac{(t-T)^{n-1}}{(n-1)!},$$

$$w_\ell[t,T] = \frac{(t-T)^{n-1}}{(n-1)(\ell-1)!(n-\ell-1)!} \quad \text{and}$$

$$\overline{w}[t,T] = \frac{\lambda}{(n-1)!} t^{n-1} \quad \text{for some} \quad \lambda,$$

$0 < \lambda < 1$ and all large t.

As an example, we restate Theorem 3.1 for equation (5.10). The formulation of other results for equation (5.10) can be done similarly.

Theorem 3.1′. *Let the hypotheses of Theorem 3.1 hold with* $a_i = 1$, $i = 1, \ldots, n-1$. *Equation (5.10) is oscillatory if equation (3.3) is oscillatory where*

$$p(t) = \rho(t)q(t)G(cg^{n-1}(t)) + \frac{1}{2}\left(\frac{\rho'(t)}{\rho(t)C(t)}\right)' - \frac{1}{4C(t)}\left(\frac{\rho'(t)}{\rho(t)}\right)^2,$$

c is any positive constant and

$$\frac{\rho(t)}{\beta\sigma'(t)}\sigma^{1-(n-1)\alpha}(t)C(t)$$

$$= \begin{cases} c_1, \ c_1 \ \text{is any positive constant, when} \ \beta > \alpha, \\ 1, \ \text{when} \ \beta = \alpha, \\ c_2\sigma^{(n-1)(\beta-\alpha)}(t), \ c_2 \ \text{is any positive constant, when} \ \beta < \alpha. \end{cases}$$

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