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EXPLICIT SOLUTIONS OF THE BASIC BOUNDARY VALUE PROBLEMS OF STATICS OF THE ELASTIC MIXTURE THEORY FOR AN ANNULUS

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Abstract. Using the complex representation formulae of regular solutions of equations of statics of the theory of elastic mixtures, we construct the explicit solutions of the Dirichlet and Neumann type boundary value problems for an annulus in the form of absolutely and uniformly convergent series.

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1. Basic Equations and Some Auxiliary Questions

The basic homogeneous equations of statics of the elastic mixture theory are written in terms of displacement components as follows [1]:

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0; \end{aligned} \tag{1.1}$$

here Δ is a two-dimensional Laplace operator, $u' = (u'_1, u'_2)$ and $u'' = (u''_1, u''_2)$ are partial displacements,

$$\begin{aligned} &a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5,\\ &b_1 = \mu_1 + \lambda_1 + \lambda_5 - \frac{\alpha_2}{\rho_*}\rho_2, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \frac{\alpha_2}{\rho_*}\rho_1,\\ &d = \mu_3 + \lambda_3 - \lambda_5 - \frac{\alpha_2}{\rho_*}\rho_1 \equiv \mu_3 + \lambda_4 - \lambda_5\frac{\alpha_2}{\rho_*}\rho_2,\\ &\rho_* = \rho_1 + \rho_2, \quad \alpha_2 = \lambda_3 - \lambda_4, \end{aligned}$$

where μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5, ρ_1, ρ_2 are the constants characterizing the physical properties of a mixture and satisfying certain inequalities [2].

System (1.1) can be rewritten as

$$\begin{aligned} &a_1\Delta u' + c\Delta u'' + b_1 \operatorname{grad}\theta' + d \operatorname{grad}\theta'' = 0,\\ &c\Delta u' + a_2\Delta u'' + d \operatorname{grad}\theta' + b_2 \operatorname{grad} u'' = 0, \end{aligned} \tag{1.2}$$

where

$$\theta' = \frac{\partial u'_1}{\partial x_1} + \frac{\partial u'_2}{\partial x_2}, \quad \theta'' = \frac{\partial u''_1}{\partial x_1} + \frac{\partial u''_2}{\partial x_2}. \tag{1.3}$$

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Introducing the variables $z=x_1+ix_2$, $\overline{z}=x_1-ix_2$, where $x_1=\frac{z+\overline{z}}{2}$, $x_2=\frac{z-\overline{z}}{2i}$, we have

$$\begin{gathered}\frac{\partial}{\partial x_1}=\frac{\partial}{\partial z}+\frac{\partial}{\partial\overline{z}},\quad \frac{\partial}{\partial x_2}=i\Big(\frac{\partial}{\partial z}-\frac{\partial}{\partial\overline{z}}\Big),\\ \frac{\partial}{\partial z}=\frac{1}{2}\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big),\quad \frac{\partial}{\partial\overline{z}}=\frac{1}{2}\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big).\end{gathered}\tag{1.4}$$

By elementary calculations we obtain

$$\Delta=4\frac{\partial^2}{\partial z\,\partial\overline{z}},\quad \theta'=\frac{\partial U_1}{\partial z}+\frac{\partial\overline{U}_1}{\partial\overline{z}},\quad \theta''=\frac{\partial U_2}{\partial z}+\frac{\partial\overline{U}_2}{\partial\overline{z}}\,,\tag{1.5}$$

where

$$\begin{gathered}U_1=u_1+iu_2,\quad U_2=u_3+iu_4,\\ u_1=u_1',\quad u_2=u_2',\quad u_3=u_1'',\quad u_4=u_2''.\end{gathered}\tag{1.6}$$

Taking into account formulas (1.4), (1.5), and (1.6), we rewrite system (1.2) in the complex form

$$\begin{gathered}2a_1\,\frac{\partial^2 U_1}{\partial z\,\partial\overline{z}}+2c\,\frac{\partial^2 U_2}{\partial z\,\partial\overline{z}}+b_1\,\frac{\partial\theta'}{\partial\overline{z}}+d\,\frac{\partial\theta''}{\partial\overline{z}}=0,\\ 2c\,\frac{\partial^2 U_1}{\partial z\,\partial\overline{z}}+2a_2\,\frac{\partial^2 U_2}{\partial z\,\partial\overline{z}}+d\,\frac{\partial\theta'}{\partial\overline{z}}+b_2\,\frac{\partial\theta''}{\partial\overline{z}}=0.\end{gathered}$$

Using (1.5), we find

$$\begin{gathered}(2a_1+b_1)\,\frac{\partial^2 U_1}{\partial z\,\partial\overline{z}}+(2c+d)\,\frac{\partial^2 U_2}{\partial z\,\partial\overline{z}}+b_1\,\frac{\partial^2\overline{U}_1}{\partial\overline{z}^{\,2}}+d\,\frac{\partial^2\overline{U}_2}{\partial\overline{z}^{\,2}}=0,\\ (2c+d)\,\frac{\partial^2 U_1}{\partial z\,\partial\overline{z}}+(2a_2+b_2)\,\frac{\partial^2 U_2}{\partial z\,\partial\overline{z}}+d\,\frac{\partial^2\overline{U}_1}{\partial\overline{z}^{\,2}}+b_2\,\frac{\partial^2\overline{U}_2}{\partial\overline{z}^{\,2}}=0,\end{gathered}$$

from which we obtain

$$\frac{\partial^2 U}{\partial z\,\partial\overline{z}}+\varepsilon^T\,\frac{\partial^2\overline{U}}{\partial\overline{z}^{\,2}}\,,\tag{1.7}$$

where $U=(U_1,U_2)=(u_1+iu_2,u_3+iu_4)$, U_1 and U_2 are defined by (1.6),

$$\begin{aligned}&\varepsilon^T=\begin{bmatrix}\varepsilon_1, & \varepsilon_3\\ \varepsilon_2, & \varepsilon_4\end{bmatrix},\\ &\delta_0\varepsilon_1=2(a_2b_1-cd)+b_1b_2-d^2,\quad \delta_0\varepsilon_2=2(da_1-cb_1),\\ &\delta_0\varepsilon_3=2(da_2-cb_2),\quad \delta_0\varepsilon_4=2(a_1b_2-cd)+b_1b_2-d^2,\\ &\delta_0=(2a_1+b_1)(2a_2+b_2)-(2c+d)^2\equiv 4\Delta_0 d_1d_2>0,\\ &\Delta_0=m_1m_3-m_2^2,\ \ m_1=l_1+\frac{l_4}{2},\ \ m_2=l_2+\frac{l_5}{2},\ \ m_3=l_3+\frac{l_6}{2},\\ &d_1=(a_1+b_1)(a_2+b_2)-(c+d)^2>0,\quad d_2=a_1a_2-c^2>0,\\ &l_1=\frac{a_2}{d_2},\ \ l_2=-\frac{c}{d_2},\ \ l_3=\frac{a_1}{d_2},\ \ l_1+l_2=\frac{a_2+b_2}{d_1},\\ &l_2+l_5=-\frac{c+d}{d_1},\quad l_3+l_6=\frac{a_1+b_1}{d_1}.\end{aligned}\tag{1.8}$$

The expression for ε^T can be rewritten as

$$\varepsilon^T = -\frac{1}{2}\, lm^{-1}, \tag{1.9}$$

with

$$l = \begin{bmatrix} l_4, & l_5 \\ l_5, & l_6 \end{bmatrix}, \quad m = \begin{bmatrix} m_1, & m_2 \\ m_2, & m_3 \end{bmatrix}, \tag{1.10}$$

where l_j $(j = 4, 5, 6)$ and m_j $(j = 1, 2, 3)$ are defined by (1.8).

Equation (1.7) is the basic homogeneous equation of statics of the elastic mixture theory in the complex form.

Applying the results obtained in [1], the vector U can be represented as

$$U = m\varphi_*(z) + \frac{l}{2}\, z\, \overline{\varphi_*'(z)} + \overline{\psi_*(z)},$$

where m and l are defined by (1.10); $\varphi_*(z)$ and $\psi_*(z)$ are arbitrary analytic vectors.

For the stress vector TU we have

$$\begin{aligned} TU &= \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} \\ &= \frac{\partial}{\partial s(x)} \big[(A - 2E)\varphi_*(z) + Bz\, \overline{\varphi_*'(z)} + 2\mu\, \overline{\psi_*(z)}\big], \end{aligned} \tag{1.11}$$

where

$$A = 2\mu m, \quad B = \mu l, \quad \mu = \begin{bmatrix} \mu_1, & \mu_3 \\ \mu_3, & \mu_2 \end{bmatrix}, \quad \frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1},$$

$n = (n_1, n_2)$ is an arbitrary unit vector.

The main goal of this investigation is to construct explicitly, in the form of absolutely and uniformly convergent series, the solutions of the basic boundary value problems (BVP) of the linear theory of elastic mixtures for an annulus. In particular, we will consider the so-called the first (i.e., the Dirichlet type) and the second (i.e., the Neumann type) BVPs with prescribed displacement and stress vectors on the boundary of the annulus, respectively. Similar problems in the classical theory of elasticity have been studied by Muskhelishvili in [4].

2. Solution of the First Boundary Value Problem for an Annulus

Let (ρ, φ) be the polar coordinates in the plane Ox_1x_2. Denote by S_j $(j = 1, 2)$ the circumference with center at the origin and radius R_j, $R_2 > R_1$.

We are to find a regular solution of equation (1.7) in an annulus $R_1 < \rho < R_2$ satisfying on S_1 and S_2 the Dirichlet type conditions:

$$(U)_{\rho=R_1} = f(\varphi), \quad (U)_{\rho=R_2} = F(\varphi), \tag{2.1}$$

where f and F are known vectors having the definite smoothness.

We seek for a solution of the first (Dirichlet type) boundary value problem posed in the following form:

$$
\begin{aligned}
U = {} & h_0 + \sum_{n=1}^{\infty} \Big(\frac{R_1}{\rho}\Big)^n \big(e^{-in\varphi} h_n + e^{in\varphi} h_{-n}\big) \\
& - \varepsilon^T \frac{\rho^2 - R_1^2}{\rho^2} \sum_{n=1}^{\infty} n \Big(\frac{R_1}{\rho}\Big)^n e^{i(n+2)\varphi} \overline{h}_n \\
& + g_0 + \sum_{n=1}^{\infty} \Big(\frac{\rho}{R_2}\Big)^n \big(e^{-in\varphi} g_n + e^{in\varphi} g_{-n}\big) \\
& - \varepsilon^T \frac{R_2^2 - \rho^2}{R_2^2} \sum_{n=0}^{\infty} (n+2) \Big(\frac{\rho}{R_2}\Big)^n e^{-in\varphi} g_{-(n+2)} \\
& + X \ln \frac{\rho}{R_2} - \frac{\varepsilon^T}{2} e^{2i\varphi} \overline{X},
\end{aligned}
\tag{2.2}
$$

where h_k, g_k and X are the sought for constant vectors.

The some of the first three terms in (2.2) is a regular solution of the first boundary value problem outside the circle of radius R_1, the sum of the next three terms is a regular solution inside the circle of radius R_2, while the last two terms are helpful in obtaining a solution of the first boundary value problem inside the annulus $R_1 < \rho < R_2$.

Introduce the notation

$$
\lambda = \frac{R_1}{R_2}. \tag{2.3}
$$

We have $0 < \lambda < 1$, $\ln \lambda < 0$.

Passing to the limit in (2.2) when $\rho \to R_1$ and $\rho \to R_2$, we have

$$
\begin{aligned}
& h_0 + \sum_{n=1}^{\infty} \big(e^{-in\varphi} h_n + e^{in\varphi} h_{-n}\big) + g_0 + \sum_{n=1}^{\infty} \lambda^n \big(e^{-in\varphi} g_n + e^{in\varphi} g_{-n}\big) \\
& \quad - \varepsilon^T (1-\lambda^2) \sum_{n=1}^{\infty} (n+2) \lambda^n e^{-2n\varphi} \overline{g}_{-(n+2)} \\
& \quad - 2\varepsilon^T (1-\lambda^2) \overline{g}_{-2} + X \ln \lambda - \frac{\varepsilon^T}{2} e^{2i\varphi} \overline{X} = f(\varphi),
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
& h_0 + \sum_{n=1}^{\infty} \lambda^n \big(e^{-in\varphi} h_n + e^{in\varphi} h_{-n}\big) + g_0 + \sum_{n=1}^{\infty} \big(e^{-in\varphi} g_n + e^{in\varphi} g_{-n}\big) \\
& \quad - \varepsilon^T (1-\lambda^2) \sum_{n=1}^{\infty} n \lambda^n e^{i(n+2)\varphi} \overline{h}_n - \frac{\varepsilon^T}{2} e^{2i\varphi} \overline{X} = F(\varphi).
\end{aligned}
\tag{2.5}
$$

For the sought for coefficients g_n, h_n, g_{-n}, h_{-n}, $n = 0, 1, 2, \dots$, (2.4) and (2.5) yield the following equations:

$$\begin{aligned} &h_0 + g_0 - X \ln\lambda - 2\varepsilon^T(1-\lambda^2)\overline{g}_{-2} = f_0, \quad h_0 + g_0 = F_0, \\ &h_{-2} + \lambda^2 g_{-2} - \frac{\varepsilon^T}{2}\overline{X} = f_{-2}, \quad \lambda^2 h_{-2} + g_{-2} - \frac{\varepsilon^T}{2}\overline{X} = F_{-2}, \end{aligned} \tag{2.6}$$

$$h_{-1} + \lambda g_{-1} = f_{-1}, \quad \lambda h_{-1} + g_{-1} = F_{-1}, \tag{2.7}$$

$$\begin{aligned} &h_n + \lambda^n g_n - \varepsilon^T(1-\lambda^2)(n+2)\lambda^n \overline{g}_{-(n+2)} = f_n, \\ &\lambda^n h_n + g_n = F_n, \qquad n \geq 1, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\lambda^n h_{-n} + g_{-n} - \varepsilon^T(1-\lambda^2)(n-2)\lambda^{n-2}\overline{h}_{n-2} = F_{-n}, \\ &h_{-n} + \lambda^n g_{-n} = f_{-n}, \qquad n \geq 3. \end{aligned} \tag{2.9}$$

Here f_n and F_n, $n = 0, \pm 1, \pm 2, \dots$, are the Fourier coefficients of the vectors f and F:

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{int}dt, \qquad F_n = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{int}dt.$$

Let us first consider system (2.6). By elementary calculations we obtain

$$\begin{aligned} (1-\lambda^4)h_{-2} &= \frac{\varepsilon^T}{2}(1-\lambda^2)\overline{X} + f_{-2} - \lambda^2 F_{-2}, \\ (1-\lambda^4)g_{-2} &= \frac{\varepsilon^T}{2}(1-\lambda^2)\overline{X} + F_{-2} - \lambda^2 f_{-2}. \end{aligned}$$

Substituting the value g_{-2} into (2.6), we have

$$\Big[E\ln\lambda - (\varepsilon^T)^2\frac{1-\lambda^2}{1+\lambda^2}\Big]X = f_0 - F_0 + 2\varepsilon^T\frac{\overline{F}_{-2} - \lambda^2 F_{-2}}{1+\lambda^2},$$

where

$$E = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}.$$

Introduce the notation

$$D_0 = E\ln\lambda - (\varepsilon^T)^2\frac{1-\lambda^2}{1+\lambda^2}.$$

It is easy to show that

$$\det D_0 = \Big(\ln\lambda - k_1^2\frac{1-\lambda^2}{1+\lambda^2}\Big)\Big(\ln\lambda - k_2^2\frac{1-\lambda^2}{1+\lambda^2}\Big),$$

where $k_1^2 < 1$, $k_2^2 < 1$, k_1 and k_2 are defined in [3] (formula (1.15)). It obviously follows that

$$\det D_0 > 0.$$

From (2.6) we have

$$\begin{aligned}
\overline{h}_{-2} &= \frac{1}{1-\lambda^4}\Bigg\{\overline{f}_{-2} - \lambda^2\overline{F}_{-2} \\
&\quad + \frac{1-\lambda^2}{2}\,\frac{\left[\varepsilon^T\ln\lambda - k_1k_2\begin{bmatrix}\varepsilon_4, & -\varepsilon_3\\ -\varepsilon_2, & \varepsilon_1\end{bmatrix}\frac{1-\lambda^2}{1+\lambda^2}\right](f_0-F_0)}{\det D_0}\Bigg\}, \\
\overline{g}_{-2} &= \frac{1}{1-\lambda^4}\Bigg\{\overline{F}_{-2} - \lambda^2\overline{f}_{-2} \\
&\quad + \frac{1-\lambda^2}{2}\,\frac{\left[\varepsilon^T\ln\lambda - k_1k_2\begin{bmatrix}\varepsilon_4, & -\varepsilon_3\\ -\varepsilon_2, & \varepsilon_1\end{bmatrix}\frac{1-\lambda^2}{1+\lambda^2}\right](f_0-F_0)}{\det D_0}\Bigg\}.
\end{aligned} \tag{2.10}$$

Thus system (2.6) is solved.

Further, from (2.7) we obtain

$$g_{-1} = \frac{F_{-1}-\lambda f_{-1}}{1-\lambda^2}, \quad h_{-1} = \frac{f_{-1}-\lambda F_{-1}}{1-\lambda^2} \tag{2.11}$$

From (2.8) we have

$$\left.\begin{aligned}
(1-\lambda^{2n})h_n - \varepsilon^T(1-\lambda^2)(n+2)\lambda^n\,\overline{g}_{-(n+2)} &= f_n - \lambda^n F_n \\
(1-\lambda^{2n})g_n + \varepsilon^T(1-\lambda^2)(n+2)\lambda^{2n}\,\overline{g}_{-(n+2)} &= F_n - \lambda^n f_n
\end{aligned}\right\} \quad n \geq 1, \tag{2.12}$$

while (2.9) implies

$$\left.\begin{aligned}
(1-\lambda^{2n+4})h_{-(n+2)} &+ \varepsilon^T(1-\lambda^2)n\lambda^{2n-2}\overline{h}_n \\
&= f_{-(n+2)} - \lambda^{n+2}F_{-(n+2)} \\
(1-\lambda^{2n+4})g_{-(n+2)} &- \varepsilon^T(1-\lambda^2)n\lambda^{n}\overline{h}_n \\
&= F_{-(n+2)} - \lambda^{n+1}f_{-(n+2)}
\end{aligned}\right\} \quad n \geq 1. \tag{2.13}$$

By the second equation we define $\overline{g}_{-(n+2)}$ and substitute it into the first equation in (2.12). We obtain

$$\begin{aligned}
&\left[(1-\lambda^{2n})(1-\lambda^{2n+4})E - (\varepsilon^T)^2(1-\lambda^2)^2 n(n+2)\lambda^{2n}\right]h_n \\
&\quad = (1-\lambda^{2n+4})(f_n - \lambda^n F_n) \\
&\qquad + \varepsilon^T(1-\lambda^2)(n+2)(\overline{E}_{-(n+2)} - \lambda^{n+2}\overline{f}_{-(n+2)}), \quad n \geq 1.
\end{aligned} \tag{2.14}$$

Denote

$$D_n = (1-\lambda^{2n})(1-\lambda^{2n+4})E - (\varepsilon^T)^2(1-\lambda^2)^2 n(n+2)\lambda^{2n}.$$

After some calculations we find

$$\begin{aligned}
\det D_n &= \left[(1-\lambda^{2n})(1-\lambda^{2n+4}) - k_1^2(1-\lambda^2)^2 n(n+2)\lambda^{2n}\right] \\
&\quad \times \left[(1-\lambda^{2n})(1-\lambda^{2n+4}) - k_2^2(1-\lambda^2)^2 n(n+2)\lambda^{2n}\right] > 0, \quad n \geq 1.
\end{aligned}$$

Moreover, $\lim\limits_{n\to+\infty} D_n = 1$.

From (2.14) we obtain

$$h_n = \frac{1}{\det D_n}\Big[(1-\lambda^{2n})(1-\lambda^{2n+4})E - \begin{bmatrix} \varepsilon_4, & -\varepsilon_3 \\ -\varepsilon_2, & \varepsilon_1 \end{bmatrix}(1-\lambda^2)^2 n(n+2)\lambda^{2n}\Big]\Big[(1-\lambda^{2n+4}(f_n-\lambda^n F_n) + \varepsilon^T(1-\lambda^2)(n+2)(\overline{F}_{-(n+2)} - \lambda^{n+2}\overline{f}_{-(n+2)})\Big] > 0, \quad n=1,2,\dots .$$

Substituting the value h_n into (2.13), we uniquely define $\overline{g}_{-(n+2)}$ and $\overline{h}_{-(n+2)}$.

Thus we have uniquely defined all the sought for coefficients. Let us substitute the values of these coefficients into (2.2). Then we obtain a solution of the first boundary value problem in the form of a series. For these series together with their first derivatives to be absolutely and uniformly convergent it is sufficient that the functions $f'(\varphi)$ and $F'(\varphi)$ satisfy the Hölder condition with an exponent $\alpha > \frac{1}{2}$. Solutions obtained under such conditions are regular in an annulus.

Thus we have proved the following

Theorem 1. *The Dirichlet type BVP* (1.7), (2.1) *is uniquely solvable in the class of regular vectors, and the solution is represented in the form of absolutely and uniformly convergent series* (2.2), *where the constant vectors h_k and g_k* ($k = 0, \pm 1, \dots$) *solve the system of equations* (2.6)–(2.9), *if the boundary data f and F are from the space $C^{1,\alpha}$ with $\alpha > \frac{1}{2}$.*

3. Solution of the Second Boundary Value Problem for an Annulus

In this paragraph we will construct an explicit solution of the boundary value problem for equation (1.7) when stresses are assumed to be given on the concentrical circumferences S_1 and S_2:

$$(TU)_{\rho=R_1} = f, \quad (TU)_{\rho=R_2} = F, \tag{3.1}$$

where the vector TU is defined by (1.11).

We seek the stress vector in the annulus $R_1 < \rho < R_2$ in the following form:

$$\begin{aligned} TU = {} & \frac{R_1}{\rho}h_0 + \sum_{n=1}^{\infty}\Big(\frac{R_1}{\rho}\Big)^{n+1}\big(e^{-in\varphi}h_n + e^{in\varphi}h_{-n}\big) \\ & + \frac{1}{\Delta_2}\begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix}\frac{\rho^2 - R_1^2}{\rho^2}\sum_{n=0}^{\infty}(n+2)\Big(\frac{R_1}{\rho}\Big)^{n+1} e^{i(n+1)\varphi}\overline{h}_n \\ & + \sum_{n=1}^{\infty}\Big(\frac{\rho}{R_2}\Big)^{n-1}\big(e^{-in\varphi}g_n + e^{in\varphi}h_{-n}\big) \\ & + \frac{1}{\Delta_2}\begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix}\frac{R_2^2 - \rho^2}{R_2^2}\sum_{n=1}^{\infty} n\Big(\frac{\rho}{R_2}\Big)^{n-1} e^{-in\varphi}\,\overline{g}_{-(n+2)}, \end{aligned} \tag{3.2}$$

where $\Delta_2 = \det(A - 2E) > 0$ and H_1, H_2, H_3, H_4 are the definite complex constants [3].

Here $h_0, h_n, h_{-n}, g_n, g_{-n}$ $(n = 1, 2, \dots)$ are the constant vectors to be found.
Using the boundary conditions (3.1), we have

$$\begin{aligned}
&h_0 + \sum_{n=1}^{\infty} \big(e^{-in\varphi} h_n + e^{in\varphi} h_{-n}\big) + \sum_{n=1}^{\infty} \lambda^{n-1} \big(e^{-in\varphi} g_n + e^{in\varphi} g_{-n}\big) \\
&\quad + \frac{1}{\Delta_2} \begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix} (1-\lambda^2) \sum_{n=1}^{\infty} n\lambda^{n-1} e^{-in\varphi} \overline{g}_{-(n+2)} = f(\varphi), \\
&\lambda h_0 + \sum_{n=1}^{\infty} \lambda^{n+1} \big(e^{-in\varphi} h_n + e^{in\varphi} h_{-n}\big) + \sum_{n=1}^{\infty} \big(e^{-in\varphi} g_n + e^{in\varphi} g_{-n}\big) \\
&\quad + \frac{1}{\Delta_2} \begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix} (1-\lambda^2) \sum_{n=1}^{\infty} (n+2) \lambda^{n+1} e^{i(n+2)\varphi} \overline{h}_n = F(\varphi),
\end{aligned}$$

where f and F are the given vectors.

Using (3.1) and standard approach for the coefficients $h_0, h_n, h_{-n}, g_n, g_{-n}$ we obtain the algebraic equations

$$h_0 = f_0, \quad \lambda h_0 = F_0, \tag{3.3}$$

$$h_{-1} + g_{-1} = f_{-1}, \quad \lambda^2 h_{-1} + g_{-1} = F_{-1}, \tag{3.4}$$

$$\left.\begin{aligned} &h_n + \lambda^{n-1} g_n + \frac{1}{\Delta_2} \begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix} (1-\lambda^2) n \lambda^{n-1} \overline{g}_{-(n+2)} = f_n \\ &\lambda^{n+1} h_n + g_n = F_n \end{aligned}\right\} \quad n \geq 1, \tag{3.5}$$

$$\left.\begin{aligned} &\lambda^{n+1} h_{-n} + g_{-n} + \frac{n}{\Delta_2} \begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix} \lambda^{n-1} \overline{h}_{n-1} = F_n \\ &h_{-n} + \lambda^{n-1} g_{-n} = f_{-n} \end{aligned}\right\} \quad n \geq 2, \tag{3.6}$$

where f_n and F_n $(n = 0, \pm 1, \pm 2, \dots)$ are the Fourier coefficients of the functions f and F, respectively.

From (3.3) it follows that $\lambda f_0 = F_0$, i.e., $R_1 f_0 = R_2 F_0$, which fact coincides with the stress equilibrium condition (the principal stress vector is equal to zero). Therefore we can write

$$h_0 = \frac{F_0}{\lambda} = f_0.$$

Let us rewrite (3.4) as

$$h_{-1} + g_{-1} = f_{-1}, \quad R_1^2 h_{-1} + R_2^2 g_{-1} = R_2^2 F_{-1}.$$

Since the principal stress moment is to be equal to zero, i.e., $R_1^2 f_{-1} = R_2^2 F_{-1}$, we obtain

$$g_{-1} = 0, \quad h_{-1} = f_{-1} = \frac{R_2^2}{r_1^2} F_{-1}.$$

Thus we have defined g_{-1} and h_{-1}.

Using equation (3.6), we obtain

$$h_{-1}=\frac{1}{1-\lambda^4}\left[f_{-2}-\lambda F_{-2}+\frac{2}{\Delta_2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)\lambda^2 f_0\right],$$

$$g_{-1}=\frac{1}{1-\lambda^4}\left[F_{-2}-\lambda^3 f_{-2}-\frac{2}{\Delta_2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)\lambda f_0\right].$$

From (3.5) and (3.6) we have

$$\left.\begin{aligned}&(1-\lambda^{2n})h_n+\frac{n}{\Delta_2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)\lambda^{n-1}\,\overline{g}_{-(n+2)}=f_n-\lambda^{n-1}F_n\\ &(1-\lambda^{2n})g_n-\frac{n}{\Delta_2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)\lambda^{2n}\,\overline{g}_{-(n+2)}=F_n-\lambda^{n+1}f_n\end{aligned}\right\}\quad n\geq 1,\quad (3.7)$$

$$\left.\begin{aligned}&(1-\lambda^{2n})h_{-n}-\frac{n}{\Delta_2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)\lambda^{2n-2}\,\overline{h}_{n-2}=f_{-n}-\lambda^{n-1}F_{-n}\\ &(1-\lambda^{2n})g_{-n}-\frac{n}{\Delta_2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)\lambda^{n-1}\,\overline{h}_{n-1}=F_{-n}-\lambda^{n+1}f_{-n}\end{aligned}\right\}\quad n\geq 3.\quad (3.8)$$

From the second equation of (3.8) we can write

$$\left\{(1-\lambda^{2n})(1-\lambda^{2n+4})E-\frac{1}{\Delta_2^2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)^2 n(n+2)\lambda^{2n}\right\}h_n$$

$$=-\frac{1}{\Delta_2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)n\big(\overline{F}_{-(n+2)}-\lambda^{n+3}\overline{f}_{-(n+2)}\big)+(1-\lambda^{2n+4})(f_n-\lambda^{n-1}F_n),$$

where E is the unit matrix.

Thus we have obtained one vector relation for defining h_n. Denote

$$D_n=(1-\lambda^{2n})(1-\lambda^{2n+4})E-\frac{1}{\Delta_2^2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}(1-\lambda^2)^2 n(n+2)\lambda^{2n}.$$

After some calculations we find

$$\det D_n=\Big[(1-\lambda^{2n})(1-\lambda^{2n+4})-(1-\lambda^2)n(n+2)\lambda^{2n}\Big]\Big[(1-\lambda^{2n})(1-\lambda^{2n+4})$$
$$-(1-\lambda^2)n(n+2)\lambda^{2n}-\frac{4\lambda_5(a_0b+b_0\Delta_1)(1-\lambda^2)^2n(n+2)\lambda^{2n}}{d_2d_1^2}\Big],$$

where $\lambda_5<0$, $a_0=a_1+a_2+2c\equiv\mu_1+\mu_2+2\mu_3>0$, $b_0=b_1+b_2+2d=b_1-\lambda_5+b_2-\lambda_5+2(d+\lambda_5)>0$, $\Delta_1=\mu_1\mu_2-\mu_3^2>0$, $d_2=a_1a_2-c^2=\Delta_1-\lambda_5a_0$, $d_1=(a_1+b_1)(a_2+b_2)-(c+d)^2\equiv\Delta_1+a+b$, $a=\mu_1(b_2-\lambda_5)+\mu_2(b_1-\lambda_5)-2\mu_3(d+\lambda_5)>0$, $b=(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2>0$.

We prove

$$(1-\lambda^{2n})(1-\lambda^{2n+4})-(1-\lambda^2)n(n+2)\lambda^{2n}>0,\quad n\geq 1,$$

and therefore $\det D_n>0$, $n\geq 1$. Moreover, $\lim_{n\to+\infty}D_n=1$.

Thus

$$h_n=\frac{1}{\det D_n}\left[(1-\lambda^{2n})(1-\lambda^{2n+4})E-\frac{1}{\Delta_2^2}\begin{bmatrix}H_1, & H_2\\ H_3, & H_4\end{bmatrix}^2(1-\lambda^2)^2n(n+2)\lambda^{2n}\right]$$

$$\times \left[-\frac{n}{\Delta_2} \begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix} (1-\lambda^2)^2 \lambda^{n+1} \big(\overline{F}_{-(n+2)} - \lambda^{n+3} \overline{f}_{-(n+2)}\big) \right.$$
$$+ (1-\lambda^{2n+4})(f_n - \lambda^{n-1} F_n).$$

Substituting the expression of h_n into (3.8), we can define g_{-n} and h_{-n} for any $n \geq 3$. Finally, using (3.7) we define g_n for $n \geq 1$.

Thus we have defined all the unknown coefficients g_n, h_n, g_{-n}, h_{-n}.

To conclude, note that all sought for coefficients are defined uniquely if the principal vector and the principal moment of stresses are equal to zero.

Substituting the defined coefficients into formula (3.1), we obtain the stress vector in the form of series for the annulus $R_1 < \rho < R_2$. Series (3.2) is absolutely and uniformly convergent if the given vectors f and F satisfy the Hölder condition with an exponent $\alpha > \frac{1}{2}$.

Thus we have proved the following

Theorem 2. *The Neumann type BVP* (1.7), (3.1) *is solvable and the corresponding stress vector is represented in the form of absolutely and uniformly convergent series* (3.2), *where the constant vectors h_k and g_k* $(k = 0, \pm 1, \dots)$ *solve the system of equations* (3.3)–(3.6), *if the boundary data f and F belong to the space $C^{0,\alpha}$ with $\alpha > \frac{1}{2}$ and the corresponding principal* (*resultant*) *vector and moment are equal to zero.*

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