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COMMON FIXED POINTS ITERATION PROCESSES FOR A FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. Let E be a uniformly convex Banach space which satisfies Opial's condition or its dual E^* has the Kadec–Klee property, C a nonempty closed convex subset of E, and $T_j : C \to C$ an asymptotically nonexpansive mapping for each $j = 1, 2, \dots, r$. Suppose $\{x_n\}$ is generated iteratively by

$$x_0 \in C, \quad x_{n+1} = (1 - \alpha_{n(r)})x_n + \alpha_{n(r)} \frac{1}{n+1} \sum_{i=0}^{n} T_r^i U_{n(r-1)} x_n, \quad n = 0, 1, 2, \dots,$$

where $U_{n(j)} = (1-\alpha_{n(j)})I + \alpha_{n(j)} \frac{1}{n+1} \sum_{i=0}^{n} T_j^i U_{n(j-1)}$, $j = 1, 2, \dots, r$, $U_{n(0)} = I$, I is the identity map and $\{\alpha_{n(j)}\}$ is a suitable sequence in $[0, 1]$. If the set $\cap_{j=1}^r F(T_j)$ of common fixed points of $\{T_j\}_{j=1}^r$ is nonempty, then weak convergence of $\{x_n\}$ to some $p \in \cap_{j=1}^r F(T_j)$ is obtained.

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E and let T be a mapping of C into itself. Then, we denote by $F(T)$ the set of fixed points of T. A mapping T of C into itself is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \tag{1.1}$$

for all x, $y \in C$ and a mapping T of C into itself is said to be *asymptotically nonexpansive with Lipschitz constants* $\{k_n\}$ if $\overline{\lim}_{n\to\infty} k_n \le 1$ and

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{1.2}$$

for all x, $y \in C$ and $n = 0, 1, 2, \dots$.

In 1998, Atsushiba and Takahashi [1] introduced an iteration procedure of Mann's type for approximating common fixed points of two nonexpansive mappings S and T as follows:

$$x_0 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n \frac{1}{(n+1)^2} \sum_{i,j=0}^{n} S^i T^j x_n, \quad n = 0, 1, 2, \dots, \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^\infty$ is a sequence in $[0, 1]$. More precisely, they proved the following theorem:

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Theorem A (see [1, Theorem 1]). *Let E be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. Let C be a nonempty closed convex subset of E. Let S and T be nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \varnothing$. Suppose that $x_0 \in C$ and $\{x_n\}_{n=0}^{\infty}$ is given by (1.3). If $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $0 < a \leq \alpha_n \leq 1$ for some constant a, then $\{x_n\}_{n=0}^{\infty}$ converges weakly to a common fixed point of S and T.*

Let C be a nonempty convex subset of a Banach space E. Let $T_j : C \to C$ be a given mapping for each $j = 1, 2, \ldots, r$. In this paper, we consider the following iteration scheme generated by $T_1, T_2, \ldots, T_r$:

$$
\begin{aligned}
U_{n(1)} &= (1-\alpha_{n(1)})I + \alpha_{n(1)} \frac{1}{n+1} \sum_{i=0}^{n} T_1^i U_{n(0)},\\
U_{n(2)} &= (1-\alpha_{n(2)})I + \alpha_{n(2)} \frac{1}{n+1} \sum_{i=0}^{n} T_2^i U_{n(1)},\\
&\vdots\\
U_{n(r)} &= (1-\alpha_{n(r)})I + \alpha_{n(r)} \frac{1}{n+1} \sum_{i=0}^{n} T_r^i U_{n(r-1)},
\end{aligned}
$$

$$
x_0 \in C, \quad x_{n+1} = (1-\alpha_{n(r)})x_n + \alpha_{n(r)} \frac{1}{n+1} \sum_{i=0}^{n} T_r^i U_{n(r-1)} x_n, \quad n = 0, 1, 2, \ldots, \tag{1.4}
$$

where $T_j^0 = U_{n(0)} := I$, I is the identity map, and $\{\alpha_{n(j)}\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ for each $j = 1, 2, \ldots, r$. Using this scheme we improve Theorem A by removing the assumption that $ST = TS$ and extending the two nonexpansiveness mappings to a finite family of asymptotically nonexpansive mappings. Since the duals of reflexive Banach spaces with a Fréchet differentiable norm have the Kadec–Klee property (see [4] or [5]), we consider our main theorems under the assumption that E is a uniformly convex Banach space such that its dual E^* has the Kadec–Klee property.

2. Preliminaries

Throughout this paper, E is a real Banach space and E^* is the dual space of E. We use the notation $\overline{\lim} = \limsup$ and $\underline{\lim} = \liminf$, denote by $\mathbb{N}$ the set of all nonnegative integers, denote $\max\{a, 0\}$ by $(a)_+$ for a real number a, and put $B_d = \{x \in E : \|x\| \leq d\}$ for $d > 0$.

A Banach space E is said to be *uniformly convex* if the modulus of convexity of E

$$
\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| \leq 1,\ \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\} > 0 \tag{2.1}
$$

for all $0 < \varepsilon \leq 2$. It is well-known that a uniformly convex Banach space is reflexive. We say that E satisfies *Opial's condition* [9] if for each sequence $\{x_n\}$

of E converging weakly to x, $x \neq y$ implies

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\|, \tag{2.2}$$

and E is said to have *the Kadec–Klee property* (*KK-property*) [8] if whenever $x_n \to x$ weakly with $\|x_n\| \to \|x\|$, it follows that $x_n \to x$ strongly. The norm $\|.\|$ of E is said to be *Fréchet differentiable* if for all $x \in S(E)$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.3}$$

exists uniformly for all $y \in S(E)$ where $S(E) = \{x \in E : \|x\| = 1\}$. Note that duals of reflexive Banach spaces with Fréchet differentiable norms have the KK-property (see [4] or [5]) and there exist uniformly convex Banach spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the KK-property (see [6, Example 3.1]).

In the sequel we shall need the following lemmas.

Lemma 2.1 (see [3, Lemma 3])**.** *Let $\{\psi_n\}_{n=0}^{\infty}$ and $\{\varphi_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$\psi_{n+1} \le (1+\varphi_n)\psi_n, \quad n = 0, 1, 2, \dots . \tag{2.4}$$

If $\sum_{n=0}^{\infty} \varphi_n < \infty$, then $\lim_{n\to\infty} \psi_n$ exists.

Lemma 2.2 (see [10, Lemma 3])**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be an asymptotically nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Then for each $r > 0$, there holds*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{x\in C\cap B_d} \left\| \frac{1}{n+1}\sum_{i=0}^{n} T^i x - T^m\Big(\frac{1}{n+1}\sum_{i=0}^{n} T^i x\Big) \right\| = 0. \tag{2.5}$$

Lemma 2.3 (see [6, Lemma 3.2])**.** *Let E be a uniformly convex Banach space such that its dual E^* has the KK-property. Suppose that $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence such that $\lim_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$ exists for all $t \in [0,1]$ and f_1, $f_2 \in \omega_w(x_n)$. Then $\omega_w(x_n)$ is a singleton. Here, $\omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$.*

Lemma 2.4 (see [2, Lemma 1.1])**.** *Let E be a uniformly convex Banach space, K be a nonempty bounded closed convex subset of E. Then there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that, for any Lipschitzian continuous mapping $V : K \to E$, x, $y \in K$ and $t \in [0,1]$, the following inequality holds*

$$\|V(tx+(1-t)y)-(tVx+(1-t)Vy)\| \le Lg^{-1}(\|x-y\|-L^{-1}\|Vx-Vy\|), \tag{2.6}$$

where $L \ge 0$ is the Lipschitz constant of V.

Lemma 2.5 (see [13, Theorem 2])**.** *Let E be a uniformly convex Banach space and $d > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{2.7}$$

for all $x,\ y \in B_d$ and $\lambda \in [0,1]$.

3. Main Results

For our main results, we need the following lemmas.

Lemma 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $S_n : C \to C$ be Lipschitzian mapping with the Lipschitz constant $L_n \geq 1$ such that $\sum_{n=0}^{\infty}(L_n - 1) < \infty$ and $\cap_{n=0}^{\infty} F(S_n) \neq \varnothing$. Suppose that $\{x_n\}$ is given by $x_{n+1} = S_n x_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$ exists for all $f_1,\ f_2 \in \cap_{n=0}^{\infty} F(S_n)$ and $t \in [0,1]$.*

Proof. This is basically the proof of Lemma 2.2 of Tan and Xu [12]. For completeness and without the assumption that C is bounded, we sketch the details. By Lemma 2.1 of Tan and Xu [12], we know that $\lim_{n\to\infty} \|x_n - f\|$ exists for each $f \in \cap_{n=0}^{\infty} F(S_n)$. Therefore, there exists $d > 0$ such that $\sup_{n\in\mathbb{N}}\{\|x_n - f_1\|\} \leq d$. Take $K = \{x \in C : \|x - f_1\| \leq d\}$. Then K is a nonempty bounded closed convex subset of E. Set

$$V_{n,m} = S_{n+m-1} S_{n+m-2} \cdots S_n$$

for all $n,\ m \geq 0$. Then $V_{n,m} x_n = x_{n+m}$ and for all $x,\ y \in C$

$$\|V_{n,m} x - V_{n,m} y\| \leq M_n \|x - y\|$$

where $M_n = \prod_{j=n}^{\infty} L_j$. Since $\sum_{n=0}^{\infty}(L_n - 1) < \infty$, we have $M_n \to 1$ as $n \to \infty$. Setting

$$b_{n,m} = \|V_{n,m}(tx_n + (1-t)f_1) - (tV_{n,m}x_n + (1-t)V_{n,m}f_1)\|,$$

then it follows from Lemma 2.4 that

$$\begin{aligned} b_{n,m} &\leq M_n g^{-1}(\|x_n - f_1\| - M_n^{-1}\|V_{n,m}x_n - V_{n,m}f_1\|) \\ &= M_n g^{-1}(\|x_n - f_1\| - M_n^{-1}\|x_{n+m} - f_1\|). \end{aligned} \tag{3.1}$$

Since $\lim_{n\to\infty} \|x_n - f_1\|$ exists, fixing $m \geq 0$ and letting $n \to \infty$ in (3.1), we obtain $b_{n,m} \to 0$ as $n \to \infty$. Set $a_n(t) = \|tx_n + (1-t)f_1 - f_2\|$. Then we have

$$\begin{aligned} a_{n+m}(t) &\leq \|V_{n,m}(tx_n + (1-t)f_1) - f_2\| + b_{n,m} \\ &\leq M_n a_n(t) + b_{n,m}. \end{aligned} \tag{3.2}$$

Fixing n and then, letting $m \to \infty$ in (3.2), we get

$$\overline{\lim_{m\to\infty}} a_{n+m}(t) \leq M_n[a_n(t) + g^{-1}(\|x_n - f_1\| - M_n^{-1} \lim_{m\to\infty} \|x_{n+m} - f_1\|)] \tag{3.3}$$

and letting $n \to \infty$ in (3.3), we have

$$\overline{\lim_{n\to\infty}} a_n(t) \leq \underline{\lim}_{n\to\infty}\, a_n(t) + g^{-1}(0) = \underline{\lim}_{n\to\infty}\, a_n(t).$$

This completes the proof. $\square$

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Suppose that $\{x_n\}$ is in C such that $x_n \to x$ weakly and $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\|T^m x_n - x_n\| = 0$. Then $x = Tx$.*

Proof. Since $\{x_n\}$ converges weakly to $x \in C$, $\{x_n\}$ is a bounded sequence in C. Therefore, there exists $d > 0$ such that $\{x_n\} \subset K = C \cap B_d$. Then K is a nonempty bounded closed convex subset in C. Since $T : C \to C$ is asymptotically nonexpansive and thus $T : K \to C$ is also an asymptotically nonexpansive mapping. Then the rest of the proof follows as in the proof of Lemma 2.3 of Tan and Xu [11] and is therefore omitted. □

In the sequel, let

$$d_{n(j)} := \frac{1}{n+1}\sum_{i=0}^{n} k_{i(j)} \quad \text{and} \quad e_{n(j)} := \left\|\frac{1}{n+1}\sum_{i=0}^{n} T_j^i U_{n(j-1)} x_n - x_n\right\|$$

for each $j = 1, 2, \ldots, r$.

Lemma 3.3. *Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $T_j : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_{n(j)}\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty}(\frac{1}{n+1}\sum_{i=0}^{n} k_{i(j)} - 1)_+ < \infty$ for each $j = 1, 2, \ldots, r$ and $\cap_{j=1}^{r} F(T_j) \neq \varnothing$. Suppose that $x_0 \in C$ and $\{x_n\}_{n=0}^{\infty}$ is given by (1.4). Then, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \cap_{j=1}^{r} F(T_j)$.*

Proof. Let $p \in \cap_{j=1}^{r} F(T_j)$. Then, we have

$$\begin{aligned}
&\|x_{n+1} - p\| = \|U_{n(r)} x_n - p\| \\
&= \left\|(1-\alpha_{n(r)})(x_n - p) + \alpha_{n(r)}\Big(\frac{1}{n+1}\sum_{i=0}^{n} T_r^i U_{n(r-1)} x_n - p\Big)\right\| \\
&\leq (1-\alpha_{n(r)})\|x_n - p\| + \alpha_{n(r)} d_{n(r)} \|U_{n(r-1)} x_n - p\| \\
&\vdots \\
&\leq \left[1 + \alpha_{n(r)}(d_{n(r)} - 1)_+ + \sum_{j=1}^{r-1}\Big(\prod_{l=j}^{r}\alpha_{n(l)}\Big)\Big(\prod_{l=j+1}^{r} d_{n(l)}\Big)(d_{n(j)} - 1)_+\right]\|x_n - p\|, \quad (3.4)
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $(d_{n(j)} - 1)_+ \to 0$ as $n \to \infty$ for each $j = 1, 2, \ldots, r$, it implies that each j, $\{d_{n(j)}\}_{n=0}^{\infty}$ is bounded. Thus, there exists $D > 0$ such that

$$\max_{1\leq j\leq r}\left\{\sup_{n\in\mathbb{N}}\{d_{n(j)}\}\right\} \leq D.$$

So, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \left[1 + (d_{n(r)} - 1)_+ + \sum_{j=1}^{r-1} D^{r-j}(d_{n(j)} - 1)_+\right]\|x_n - p\| \\
&= (1 + \varphi_n)\|x_n - p\|, \qquad (3.5)
\end{aligned}$$

for all $n \in \mathbb{N}$ where $\varphi_n := (d_{n(r)} - 1)_+ + \sum_{j=1}^{r-1} D^{r-j}(d_{n(j)} - 1)_+$. Since $\sum_{n=0}^{\infty}(d_{n(j)} - 1)_+ < \infty$ for each $j = 1, 2, \ldots, r$, we have $\sum_{n=0}^{\infty}\varphi_n < \infty$. Thus, $\lim_{n\to\infty}\|x_n - p\|$ exists by Lemma 2.1. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T_j : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_{n(j)}\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty}(\frac{1}{n+1}\sum_{i=0}^{n} k_{i(j)} - 1)_+ < \infty$ for each $j = 1, 2, \ldots, r$ and $\cap_{j=1}^{r} F(T_j) \neq \varnothing$. Let $\{\alpha_{n(j)}\}_{n=0}^{\infty}$ be a sequence in $[0, 1]$ satisfying the following conditions:*

$$0 < a \leq \alpha_{n(r)} \leq 1; \quad 0 < b \leq \alpha_{n(j)} \leq c < 1$$

for all $j = 1, 2, \ldots, r-1$ and some constants a, b, and c.

Suppose that $x_0 \in C$ and $\{x_n\}_{n=0}^{\infty}$ is given by (1.4). Then,

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \|T_j^m x_n - x_n\| = 0 \quad \text{for each } j = 1, 2, \ldots, r.$$

Proof. Let $p \in \cap_{j=1}^{r} F(T_j) \neq \varnothing$. By Lemma 3.3 and the hypotheses of Lemma 3.4 imply that $\{x_n\}_{n=0}^{\infty}$ and $\{d_{n(j)}\}_{n=0}^{\infty}$ are bounded for each $j = 1, 2, \ldots, r$. Then, there exists a constant $d > 0$ such that

$$\bigcup_{j=1}^{r} \Big\{ \frac{1}{n+1} \sum_{i=0}^{n} T_j^n U_{n(j-1)} x_n - p, U_{n(j)} x_n \Big\}_{n=0}^{\infty} \bigcup \Big\{ x_n - p \Big\}_{n=0}^{\infty} \subseteq B_d. \tag{3.6}$$

By Lemma 2.5, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$, and

$$\|\lambda x + (1-\lambda) y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda) g(\|x - y\|) \tag{3.7}$$

for all x, $y \in B_d$ and $\lambda \in [0, 1]$. Using (3.7), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 = \|U_{n(r)} x_n - p\|^2 \\
&= \left\| (1 - \alpha_{n(r)})(x_n - p) + \alpha_{n(r)} \Big(\frac{1}{n+1} \sum_{i=0}^{n} T_r^i U_{n(r-1)} x_n - p \Big) \right\|^2 \\
&\leq (1 - \alpha_{n(r)}) \|x_n - p\|^2 + \alpha_{n(r)} \left\| \frac{1}{n+1} \sum_{i=0}^{n} T_r^i U_{n(r-1)} x_n - p \right\|^2 \\
&\quad - \alpha_{n(r)}(1 - \alpha_{n(r)}) g\Big(\Big\| \frac{1}{n+1} \sum_{i=0}^{n} T_r^i U_{n(r-1)} x_n - x_n \Big\| \Big) \\
&\leq (1 - \alpha_{n(r)}) \|x_n - p\|^2 + \alpha_{n(r)} d_{n(r)}^2 \|U_{n(r-1)} x_n - p\|^2 - \alpha_{n(r)}(1 - \alpha_{n(r)}) g(e_{n(r)}) \\
&\ \vdots \\
&\leq \|x_n - p\|^2 \\
&\quad + \Big[\alpha_{n(r)} (d_{n(r)}^2 - 1)_+ + \sum_{j=1}^{r-1} \Big(\prod_{l=j}^{r} \alpha_{n(l)} \Big) \Big(\prod_{l=j+1}^{r} d_{n(l)}^2 \Big) (d_{n(j)}^2 - 1)_+ \Big] \|x_n - p\|^2 \\
&\quad - \Big[\alpha_{n(r)} (1 - \alpha_{n(r)}) g(e_{n(r)}) + \sum_{j=1}^{r-1} \Big(\prod_{l=j}^{r} \alpha_{n(l)} \Big) \Big(\prod_{l=j+1}^{r} d_{n(l)}^2 \Big) (1 - \alpha_{n(j)}) g(e_{n(j)}) \Big]
\end{aligned} \tag{3.8}$$

for all $n = 0, 1, 2, \ldots$. Since each j, $(d_{n(j)} - 1)_+ \to 0$ as $n \to \infty$, it implies that $\{d_{n(j)}\}_{n=0}^{\infty}$ is bounded and there exists a positive integer N_j such that $d_{n(j)}^2 \geq \frac{1}{2}$ for all $n \geq N_j$. Now, put

$$M \;:=\; \max_{1\leq j\leq r}\Big\{\sup_{n\in\mathbb{N}}\{d_{n(j)}+1,\ d_{n(j)}^2\}\Big\} < \infty$$

and

$$D \;:=\; \sup_{n\in\mathbb{N}}\{\|x_n - p\|\} < \infty.$$

Then, by (3.8), we have

$$\|x_{n+1}-p\|^2 \leq \|x_n - p\|^2 + \left[M(d_{n(r)}-1)_+ + \sum_{j=1}^{r-1} M^{r-j+1}(d_{n(j)}-1)_+\right] D$$

$$- \left[a(1-\alpha_{n(r)})g(e_{n(r)}) + a(1-c)\sum_{j=1}^{r-1}(b^{r-j})\Big(\frac{1}{2}\Big)^{r-j} g(e_{n(j)})\right] \quad (3.9)$$

for all $n \geq N$ where $N := \max_{1\leq j\leq r}\{N_j\}$. Hence by Lemma 3.3 and $(d_{n(j)} - 1)_+ \to 0$ as $n \to \infty$ for each $j = 1, 2, \ldots, r$, we have

$$\left[a(1-\alpha_{n(r)})g(e_{n(r)}) + a(1-c)\sum_{j=1}^{r-1}(b^{r-j})\Big(\frac{1}{2}\Big)^{r-j} g(e_{n(j)})\right]$$

$$\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \left[M(d_{n(r)}-1)_+ + \sum_{j=1}^{r-1} M^{r-j+1}(d_{n(j)}-1)_+\right] D \to 0$$

as $n \to \infty$. Thus,

$$(1-\alpha_{n(r)})g(e_{n(r)}) \to 0 \quad \text{as} \quad n \to \infty$$

and

$$g(e_{n(j)}) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each } i = 1, 2, \ldots, r-1.$$

Since g is a continuous and strictly increasing function with $g(0) = 0$, we have

$$\lim_{n\to\infty}(1-\alpha_{n(r)})\left\|\frac{1}{n+1}\sum_{i=0}^{n} T_j^i U_{n(r-1)}x_n - x_n\right\| = \lim_{n\to\infty}(1-\alpha_{n(r)})e_{n(r)} = 0 \quad (3.10)$$

and

$$\lim_{n\to\infty}\left\|\frac{1}{n+1}\sum_{i=0}^{n} T_j^i U_{n(j-1)}x_n - x_n\right\| = \lim_{n\to\infty} e_{n(j)} = 0 \quad (3.11)$$

for each $j = 1, 2, \ldots, r-1$.

Observe that for all $m = 1, 2, \ldots$

$$\|T_r^m x_{n+1} - x_{n+1}\| \leq \left\|T_r^m x_{n+1} - T_r^m\Big(\frac{1}{n+1}\sum_{i=0}^{n} T_r^i U_{n(r-1)}x_n\Big)\right\|$$

$$
\begin{aligned}
&+\left\|T_r^m\Big(\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\Big)-\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\right\|\\
&+\left\|\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n-x_{n+1}\right\|\\
&\le (k_{m(r)}+1)\left\|x_{n+1}-\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\right\|\\
&+\left\|T_r^m\Big(\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\Big)-\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\right\|\\
&= (k_{m(r)}+1)(1-\alpha_{n(r)})\left\|x_n-\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\right\|\\
&+\left\|T_r^m\Big(\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\Big)-\frac{1}{n+1}\sum_{i=0}^{n}T_r^iU_{n(r-1)}x_n\right\| \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\|T_j^mx_n-x_n\| &\le \left\|T_j^mx_n-T_j^m\Big(\frac{1}{n+1}\sum_{i=0}^{n}T_j^iU_{n(j-1)}x_n\Big)\right\|\\
&+\left\|T_j^m\Big(\frac{1}{n+1}\sum_{i=0}^{n}T_j^iU_{n(j-1)}x_n\Big)-\frac{1}{n+1}\sum_{i=0}^{n}T_j^iU_{n(j-1)}x_n\right\|\\
&+\left\|\frac{1}{n+1}\sum_{i=0}^{n}T_j^iU_{n(j-1)}x_n-x_n\right\|\\
&\le (k_{m(j)}+1)\left\|x_n-\frac{1}{n+1}\sum_{i=0}^{n}T_j^iU_{n(j-1)}x_n\right\|\\
&+\left\|T_j^m\Big(\frac{1}{n+1}\sum_{i=0}^{n}T_j^iU_{n(j-1)}x_n\Big)-\frac{1}{n+1}\sum_{i=0}^{n}T_j^iU_{n(j-1)}x_n\right\| \qquad (3.13)
\end{aligned}
$$

for each $j=1,2,\ldots,r-1$. Hence, by (3.10)−(3.13) and Lemma 2.2, we have

$$
\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\|T_j^mx_n-x_n\|=0 \qquad (3.14)
$$

for each $j=1,2,\ldots,r$. This completes the proof of Lemma 3.4. □

We will now prove our main theorems.

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or its dual E^* has the KK-property and $T_j : C\to C$ be an asymptotically nonexpansive mapping with sequence $\{k_{n(j)}\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty}(\frac{1}{n+1}\sum_{i=0}^{n}k_{i(j)}-1)_+<\infty$ for each $j=1,2,\ldots,r$ and $\cap_{j=1}^{r}F(T_j)\neq\varnothing$. Let $\{\alpha_{n(j)}\}_{n=0}^{\infty}$ be a sequence in $[0,1]$*

satisfying the following conditions:

$$0 < a \leq \alpha_{n(r)} \leq 1; \quad 0 < b \leq \alpha_{n(j)} \leq c < 1$$

for all $j = 1, 2, \ldots, r-1$ *and some constants* a, b, *and* c.

Suppose that $x_0 \in C$ *and* $\{x_n\}_{n=0}^{\infty}$ *is given by* (1.4). *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges weakly to a common fixed point of* $T_1, T_2, \ldots, T_r$.

Proof. Define a mapping $S_n : C \to C$ by $S_n x = U_{n(r)}x$, $x \in C$. Then, $x_{n+1} = S_n x_n$ and $\cap_{j=1}^{r} F(T_j) \subseteq F(S_n)$. Moreover, for all $x, y \in C$, we have

$$\begin{aligned}
\|S_n x - S_n y\| &= \|U_{n(r)}x - U_{n(r)}y\| \\
&\leq (1-\alpha_{n(r)})\|x-y\| + \alpha_{n(r)} \frac{1}{n+1} \sum_{i=0}^{n} \|T_r^i U_{n(r-1)}x - T_r^i U_{n(r-1)}y\| \\
&\leq (1-\alpha_{n(r)})\|x-y\| + \alpha_{n(r)} d_{n(r)} \|U_{n(r-1)}x - U_{n(r-1)}y\| \\
&\vdots \\
&\leq \left[1 + \alpha_{n(r)}(d_{n(r)}-1)_+ + \sum_{j=1}^{r-1} \Big(\prod_{l=j}^{r} \alpha_{n(l)}\Big)\Big(\prod_{l=j+1}^{r} d_{n(l)}\Big)(d_{n(j)}-1)_+\right] \|x-y\| \\
&\leq (1+\varphi_n)\|x-y\|, \qquad (3.15)
\end{aligned}$$

where φ_n is as in the proof of Lemma 3.3 and $\sum_{n=0}^{\infty} \varphi_n < \infty$. Thus, S_n is Lipschitzian with the Lipschitz constant $L_n := 1 + \varphi_n$ such that $\sum_{n=0}^{\infty}(L_n - 1) = \sum_{n=0}^{\infty} \varphi_n < \infty$ and $\cap_{n=0}^{\infty} F(S_n) \supseteq \cap_{j=1}^{r} F(T_j) \neq \varnothing$.

It follows from Lemmas 3.4 and 3.2 that $\omega_w(x_n) \subseteq \cap_{j=1}^{r} F(T_j)$. So to show that $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \ldots, T_r$, it suffices to show that $\omega_w(x_n)$ consists of just one point. In case E^* has the KK-property, it is easy to see from Lemmas 3.1 and 2.3 that the theorem is true. So, we suppose next that E satisfies Opial's condition. This follows basically as in the proof of Theorem 1 of [1] using Lemmas 3.3, 3.4, and 3.2. This completes the proof of Theorem 3.5. □

As a consequence of Theorem 3.5, we obtain the following result.

Theorem 3.6. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E *which satisfies Opial's condition or its dual* E^* *has the KK-property and* $T_j : C \to C$ *be nonexpansive mapping for each* $j = 1, 2, \ldots, r$ *such that* $\cap_{j=1}^{r} F(T_j) \neq \varnothing$. *Let* $\{\alpha_{n(j)}\}_{n=0}^{\infty}$ *be a sequence in* $[0,1]$ *satisfying the following conditions:*

$$0 < a \leq \alpha_{n(r)} \leq 1; \quad 0 < b \leq \alpha_{n(j)} \leq c < 1$$

for all $j = 1, 2, \ldots, r-1$ *and some constants* a, b, *and* c.

Suppose that $x_0 \in C$ *and* $\{x_n\}_{n=0}^{\infty}$ *is given by* (1.4). *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges weakly to a common fixed point of* $T_1, T_2, \ldots, T_r$.

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