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# ON HOMOGENEOUS COVERINGS OF EUCLIDEAN SPACES

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**Abstract.** The notion of a homogeneous covering of a given set is introduced and examined. Some homogeneous coverings of a Euclidean space, consisting of pairwise congruent geometric figures (spheres, hyperplanes, etc.), are constructed using the method of transfinite induction.

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Let $E$ be a set and $\{X_j : j \in J\}$ be an injective family of nonempty subsets of $E$. We will say that $\{X_j : j \in J\}$ is a homogeneous covering of $E$ if there exists a cardinal number $\kappa$ such that the equality

$$\mathrm{card}(\{j \in J : x \in X_j\}) = \kappa$$

holds for all elements $x \in E$. In this case we also say that $\{X_j : j \in J\}$ is a $\kappa$-homogeneous covering of $E$ and $\kappa$ is the homogeneity index of $\{X_j : j \in J\}$.

**Example 1.** If $\kappa = 1$, then the $\kappa$-homogeneity of $\{X_j : j \in J\}$ simply means that $\{X_j : j \in J\}$ is a partition of the initial set $E$. More generally, let $\{\mathcal{P}_t : t \in T\}$ be a disjoint family of partitions of $E$ and let

$$\kappa = \mathrm{card}(T), \quad \mathcal{P} = \cup\{\mathcal{P}_t : t \in T\}.$$

It can be easily seen that $\mathcal{P}$ is a $\kappa$-homogeneous covering of $E$. In this case we say that $\mathcal{P}$ is a decomposable $\kappa$-homogeneous covering of $E$ and the family $\{\mathcal{P}_t : t \in T\}$ is a decomposition of $\mathcal{P}$.

There are many kinds of homogeneous coverings which are not decomposable (cf. Example 3 below).

As usual, we denote by $R^n$ the $n$-dimensional Euclidean space. This paper deals with some homogeneous coverings of $R^n$ $(n \geq 2)$ whose all elements are pairwise congruent geometric figures, e.g., spheres or hyperplanes in $R^n$.

**Example 2.** Recall that a Jordan curve is any homeomorphic image of the unit circumference $S_1 \subset R^2$. It can be shown that there exists no partition of the plane $R^2$ consisting of Jordan curves. On the other hand, there is a partition of the three-dimensional space $R^3$, whose all elements are circumferences in $R^3$ (the existence of such a partition can be established in the framework of the theory $ZF$, i.e., there is no need for the Axiom of Choice). Moreover, using the method of transfinite induction, one can construct a partition of $R^3$ consisting of pairwise congruent circumferences (see, e.g., [1]–[3]).

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The following purely set-theoretical statement will be useful for our further purposes.

**Theorem 1.** *Let $k$ be a natural number, $E$ be an infinite set and let $\mathcal{T}$ be a family of subsets of $E$ each of which contains at least two elements. Suppose that the following conditions are satisfied:*

1) *there exists a cardinal number $\lambda < \mathrm{card}(E)$ such that*

$$\mathrm{card}(X_1 \cap X_2 \cap \dots \cap X_k) \le \lambda$$

*for any pairwise distinct sets $X_1 \in \mathcal{T},\ X_2 \in \mathcal{T}, \dots,\ X_k \in \mathcal{T}$;*

2) *for each set $Y \subset E$ with $\mathrm{card}(Y) < \mathrm{card}(E)$ and for each element $x \in E \setminus Y$ there exists a set $X \in \mathcal{T}$ such that $x \in X$ and $X \cap Y = \varnothing$.*

*Then $\mathcal{T}$ contains a $k$-homogeneous covering of $E$.*

*Proof.* For every element $z \in E$ and for every family $\mathcal{T}' \subset \mathcal{T}$, denote

$$i(z, \mathcal{T}') = \mathrm{card}(\{Z \in \mathcal{T}' : z \in Z\}).$$

Also, denote by $\alpha$ the least ordinal number such that $\mathrm{card}(\alpha) = \mathrm{card}(E)$. Fix an enumeration $\{x_\xi : \xi < \alpha\}$ of all elements of $E$. We are going to construct, by the method of transfinite recursion, a certain family of sets $\{X_\xi : \xi < \alpha\} \subset \mathcal{T}$.

Suppose that, for an ordinal $\beta < \alpha$, the partial family $\{X_\xi : \xi < \beta\} = \mathcal{T}^* \subset \mathcal{T}$ has already been defined so that the inequality $i(z, \mathcal{T}^*) \le k$ holds for all elements $z \in E$. Put

$$Y = \cup\{X_{\xi_1} \cap X_{\xi_2} \cap \dots \cap X_{\xi_k} : \xi_1 < \xi_2 < \dots < \xi_k < \beta\}.$$

In view of condition 1), we readily obtain that $\mathrm{card}(Y) \le C^k_\beta \cdot \lambda$ if $\beta$ is finite, and $\mathrm{card}(Y) \le \mathrm{card}(\beta) \cdot \lambda$ if $\beta$ is infinite. Hence, in these both cases, we get $\mathrm{card}(Y) < \mathrm{card}(E)$. Now, let $\zeta < \alpha$ be the smallest ordinal number such that $i(x_\zeta, \mathcal{T}^*) < k$. According to condition 2) there exists a set $X \in \mathcal{T}$ for which we have $x_\zeta \in X$ and $X \cap Y = \varnothing$. We may assume without loss of generality that $X \ne X_\xi$ for all ordinals $\xi < \beta$. Now, we define $X_\beta = X$.

Notice that the extended partial family $\{X_\xi : \xi \le \beta\} = \mathcal{T}^{**}$ possesses the same property: $i(z, \mathcal{T}^{**}) \le k$ for all elements $z \in E$.

Proceeding in this manner, we are able to construct the required family of sets $\{X_\xi : \xi < \alpha\} \subset \mathcal{T}$. Denote it by $\mathcal{T}_0$. It turns out that $\mathcal{T}_0$ is a $k$-homogeneous covering of $E$. Indeed, it follows directly from our construction that $i(x, \mathcal{T}_0) \le k$ for every element $x \in E$. Moreover, we can even assert that $i(x, \mathcal{T}_0) = k$. Namely, the latter equality is valid since $x = x_\xi$ for some ordinal $\xi < \alpha$ and since

$$\mathrm{card}(\{x_\zeta : \zeta \le \xi\}) \le \mathrm{card}(\xi) + 1 < \mathrm{card}(\alpha) = \mathrm{card}(E).$$

This completes the proof of the theorem. □

The next statement can be proved in a similar way.

**Theorem 2.** *Let $k$ be a natural number, $E$ be an infinite set and let $\mathcal{T}$ be a family of subsets of $E$ each of which contains at least two elements. Suppose that the following conditions are satisfied:*

1) $\mathrm{card}(E)$ *is a regular cardinal number;*

2) $\text{card}(X_1 \cap X_2 \cap \cdots \cap X_k) < \text{card}(E)$ *for any pairwise distinct sets* $X_1 \in \mathcal{T},\ X_2 \in \mathcal{T}, \ldots,\ X_k \in \mathcal{T}$;

3) *for each set* $Y \subset E$ *with* $\text{card}(Y) < \text{card}(E)$ *and for each element* $x \in E \setminus Y$, *there exists a set* $X \in \mathcal{T}$ *such that* $x \in X$ *and* $X \cap Y = \varnothing$.

*Then* $\mathcal{T}$ *contains a* $k$*-homogeneous covering of* $E$.

Let us give several geometric applications of Theorem 1.

**Example 3.** Fix a natural number $k \geq 2$, put $E = R^2$ and take as $\mathcal{T}$ the family of all circumferences in $E$ which are congruent to $S_1$. It is not difficult to verify that conditions 1) and 2) of Theorem 1 are valid in this situation. Consequently, there exists a $k$-homogeneous covering of $R^2$ whose all elements are pairwise congruent circumferences. In view of Example 2, no such covering is decomposable.

**Example 4.** Fix a natural number $k \geq 3$, put $E = R^3$ and take as $\mathcal{T}$ the family of all spheres in $E$ which are congruent to the two-dimensional unit sphere $S_2 \subset E$. Again, it can be verified that conditions 1) and 2) of Theorem 1 are satisfied in this situation. Consequently, there exists a $k$-homogeneous covering of $R^3$ consisting of pairwise congruent two-dimensional spheres.

**Example 5.** If $E = R^4$, then it is not difficult to show that there are four pairwise congruent three-dimensional spheres in $E$ whose intersection is a circumference and, hence, is of cardinality continuum. Therefore, a straightforward application of Theorem 1 is impossible in this case. However, a slight modification of the argument used in the proof of Theorem 1 yields the corresponding result for $R^4$ as well. Namely, we can assert that, for any natural number $k \geq 4$, there exists a $k$-homogeneous covering of $R^4$ consisting of pairwise congruent three-dimensional spheres. An analogous method works for the space $R^n$, where $n > 4$, and we obtain that, for any natural number $k \geq n$, there exists a $k$-homogeneous covering of $R^n$ whose all elements are pairwise congruent $(n-1)$-dimensional spheres.

**Example 6.** Put again $E = R^2$. Let $l$ be a one-dimensional vector subspace of $E$ and let $e$ be a nonzero vector in $E$ orthogonal to $l$. For each integer $m$, consider the family $\mathcal{S}_m$ of all those circumferences in $E$ which have diameter $\|e\|$ and are contained in the strip determined by the two parallel straight lines $l + me$ and $l + (m+1)e$. Finally, let $\mathcal{S}$ denote the union of families $\mathcal{S}_m$ where $m$ ranges over the set of all integers. It can be easily checked that $i(x, \mathcal{S}) = 2$ for all points $x \in E$, i.e. $\mathcal{S}$ turns out to be a 2-homogeneous covering of $E$ by pairwise congruent circumferences. Starting with this fact and iterating the above construction $k \geq 1$ times, we come to the $2k$-homogeneous covering of $E$ by pairwise congruent circumferences. Let us emphasize that this construction is carried out in the framework of the theory $ZF$, i.e., does not need the Axiom of Choice.

As we know (see Example 3), there also are $(2k+1)$-homogeneous coverings of $E = R^2$ consisting of pairwise congruent circumferences. However, the existence of such coverings was established with the aid of essentially non-constructive methods. At the present moment, it is unknown whether the existence of a $(2k+1)$-homogeneous covering of $E$ by pairwise congruent circumferences is provable in the framework of the theory $ZF$.

**Example 7.** Let $E = R^n$ where $n \geq 3$. If $L$ is an affine hyperplane in $E$, then the symbol $e(L)$ will denote the exterior normal of $L$.

Let $\mathcal{L}$ be a family of affine hyperplanes in $E$ satisfying the following conditions:

1) for any pairwise distinct hyperplanes $L_1 \in \mathcal{L}$, $L_2 \in \mathcal{L}$, $\dots$, $L_n \in \mathcal{L}$, the vectors $e(L_1)$, $e(L_2)$,$\dots$, $e(L_n)$ are linearly independent;

2) for each set $Y \subset E$ with $\mathrm{card}(Y) < \mathrm{card}(E)$ and for each point $x \in E \setminus Y$ there exists a hyperplane $L \in \mathcal{L}$ passing through $x$ and not intersecting $Y$.

Applying Theorem 1 to $E$ and $\mathcal{L}$, we conclude that, for any natural number $k \geq n$, the given family $\mathcal{L}$ contains a $k$-homogeneous covering of $E$.

**Example 8.** Let $E = R^2$ and let $\mathcal{L}$ be a family of straight lines in $E$ satisfying the condition

$$\mathrm{card}(\{l \in \mathcal{L} : x \in l\}) = \mathrm{card}(E)$$

for every point $x \in E$. Again, Theorem 1 is applicable in this situation. We thus claim that, for each natural number $k \geq 2$, the family $\mathcal{L}$ contains a $k$-homogeneous covering of $E$.

In connection with Example 8, the following problem of combinatorial geometry seems to be of interest.

**Problem.** Let $k \geq 2$ be a natural number and let $\mathcal{L}$ be a family of straight lines in the plane $R^2$. Find necessary and sufficient conditions under which $\mathcal{L}$ contains a $k$-homogeneous covering of $R^2$.

**Example 9.** Let $k \geq 2$ be a natural number, let $E = R^2$ and let $\mathcal{L}$ be a family of analytic curves in $E$. Clearly, for any pairwise distinct curves $L_1 \in \mathcal{L}$, $L_2 \in \mathcal{L}$, $\dots$, $L_k \in \mathcal{L}$, we have the inequality

$$\mathrm{card}(L_1 \cap L_2 \cap \dots \cap L_k) \leq \omega < \mathrm{card}(E),$$

where $\omega$ stands for the first infinite cardinal number. Suppose also that the following condition is satisfied: for each set $Y \subset E$ with $\mathrm{card}(Y) < \mathrm{card}(E)$ and for each point $x \in E \setminus Y$, there exists a curve $L \in \mathcal{L}$ passing through $x$ and not intersecting $Y$. Then, applying Theorem 1 to $E$ and $\mathcal{L}$, we obtain that $\mathcal{L}$ contains a $k$-homogeneous covering of $E$. Evidently, this example is a generalized version of Example 3.

**Example 10.** Let $E = S$ be a two-dimensional sphere in the space $R^3$ and let $L$ be a circumference on $S$ whose radius is smaller than that of $S$. Consider the family $\mathcal{L}$ of all those circumferences on $S$ which are congruent to $L$. Obviously, Theorem 1 can be applied to $S$ and $\mathcal{L}$. We thus get that, for any natural number $k \geq 2$, there exists a $k$-homogeneous covering of $S$ whose all elements are circumferences congruent to $L$.

The same result remains true for circumferences on $S$ whose radii are equal to the radius of $S$ (in this case, some additional technical details occur, but they are not difficult).

*Remark.* In the context of the results obtained in this paper, let us recall the following well-known statement due to Mazurkiewicz: there exists a subset $X$ of the plane $R^2$ such that, for every straight line $\ell \subset R^2$, the set $X \cap \ell$ is two-element. The proof is based on the method of transfinite recursion. Applying the same method, it can be established that, for any natural number $k \geq 2$, there exists a set $Y \subset R^2$ such that

$$(\forall \ell)\ (\ell \text{ is a straight line in } R^2 \Rightarrow \mathrm{card}\,(Y \cap \ell) = k).$$

Similarly, for any natural number $k \geq 3$, there exists a set $Z \subset R^2$ such that

$$(\forall S)\ (S \text{ is a circumference in } R^2 \Rightarrow \mathrm{card}\,(Z \cap S) = k).$$

The above statements can be regarded as dual analogues of some results presented in this paper.

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