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ADDITIVE SURJECTIONS PRESERVING RANK ONE AND APPLICATIONS

CHONG-GUANG CAO AND XIAN ZHANG

Abstract. We characterize additive rank-one preserving surjections on the full matrix algebra over any field. As applications, invertibility preservers, determinant preservers and characteristic polynomial preservers are characterized.

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1. Introduction

In the recent decade, researching "linear preserver problems (LPPs)" has become an active topic in the matrix theory (see [6], [10], [11] and the references therein) due to its theoretical value and practical significance for solving some problems posed in the areas of differential equations, systems science and statistics (see [10]). After 1991, some authors began to investigate "additive preserver problems (APPs)". An additive preserver is an additive operator acting on matrix spaces that leaves certain subsets, relations, or functions invariant (see [1], [7], [8], [4], [13], [5], [14], [9] and the references therein). An APP is more difficult than the corresponding LPP since it has a weaker hypothetical condition.

By the objects acted on by additive maps, APPs can be classified into two categories, one is APPs on operator algebras on Banach spaces ([1], [7], [8] and the references therein), the other is APPs on matrix algebras over fields ([4], [5], [9], [13], [14] and the references therein). In general, when these two categories of APPs have the same invariance, the conclusions of the first category APPs can be directly used to solve the second category APPs over the complex/real number field (see, for example, [12]). However, when the basic field of the second category APPs is different from the complex/real number field, these problems cannot be solved by using the conclusions of the first category APPs with the same invariance since Banach spaces are defined only on the complex/real number field (for example, $3 = 0$ is true in [3, Theorem 2.1] when the characteristic of the basic field is 3.).

Additive rank one preservers have an important position in APPs. Omladič and Šemrl [12] characterized additive rank-one preserving surjections on the operator algebra of a complex Banach space; Bell and Sourour [2] characterized additive rank-one preserving surjections on upper triangular matrix algebras

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over any field; Kuzma [7] described all additive mappings decreasing rank one on the operator algebra on a real/complex Banach space; Cao and Zhang [4] characterized additive rank-one preserving surjections on the symmetric matrix algebra over a field of characteristic not 2 or 3. However, the additive rank one preserver problem on the full matrix algebra over any field remains still open. The purpose of this paper is to solve this open problem and to apply it to characterize invertibility preservers, determinant preservers and characteristic polynomial preservers. These applications show the important position of the additive rank one preserver problem.

For convenience, we give the following notation and definitions which will be used in the rest of the paper.

Suppose $\mathbf{F}$ is any field. Let $\mathbf{F}^*$ be the multiplicative group of $\mathbf{F}$ containing of all nonzero elements. We denote by $\mathbf{F}^n$, $M_n(\mathbf{F})$ and $GL_n(\mathbf{F})$ the set of all n-dimensional column vectors, the set of all $n \times n$ matrices and the set of all $n \times n$ nonsingular matrices over $\mathbf{F}$, respectively. For a vector/matrix X, let X^T be the transpose of X. E_{ij} denotes the element in $M_n(\mathbf{F})$ with 1 in the (i, j)-th position and 0 elsewhere, and e_i denotes the element in $\mathbf{F}^n$ with 1 in the i-th position and 0 elsewhere. For a nonzero vector $x \in \mathbf{F}^n$, we write the set $\left\{ xy^T \middle| y \in \mathbf{F}^n \right\}$ and the set $\left\{ yx^T \middle| y \in \mathbf{F}^n \right\}$ as $x\mathbf{F}^n$ and $\mathbf{F}^n x$, respectively. Let $\langle k \rangle$ be the set $\{1, 2, \ldots, k\}$.

A map from $M_n(\mathbf{F})$ to itself is called an additive map if $f(A + B) = f(A) + f(B)$ for every pair of $A, B \in M_n(\mathbf{F})$. Let Γ be the subset of $M_n(\mathbf{F})$ containing all rank one matrices. We say that an additive map from $M_n(\mathbf{F})$ to itself preserves rank one if $f(A) \in \Gamma$ whenever $A \in \Gamma$. We denote by Ω the set of all additive surjections from $M_n(\mathbf{F})$ to itself preserving rank one.

This paper is organized as follows. The structure of elements in Ω is characterized in Section 2. As applications, invertibility preservers, determinant preservers and characteristic polynomial preservers are characterized in Section 3.

2. Main Results and Their Proofs

To characterize the set Ω, the following five lemmas are required.

Lemma 1. *Given $x, y, u, v \in \mathbf{F}^n$, then*

(i) *$xy^T = O$ if and only if $x = O$ or $y = O$;*
(ii) *$xy^T = uv^T \neq O$ if and only if there exists $c \in \mathbf{F}^*$ such that $u = cx \neq O$ and $y = cv \neq O$;*
(iii) *if $xy^T + uv^T \in \Gamma$, then either x and u are linearly dependent, or y and v are linearly dependent;*
(iv) *if $n \geq 2$, $u \neq O$ and $v \neq O$, then there exists $w \in \Gamma$ satisfying $w \notin u\mathbf{F}^n \cup \mathbf{F}^n v$.*

Proof. The proof is omitted since it is simple. □

Lemma 2. *Suppose f is an additive rank one preserver from $M_n(\mathbf{F})$ to itself. Then, for any $k \in \langle n \rangle$, the statements* (a) *or* (b) *below holds.*

(a) *There exist an additive group injective endomorphism* g_k *on* $\mathbf{F}^n$ *and a nonzero vector* $u_k \in \mathbf{F}^n$ *satisfying* $f(e_k z^T) = u_k g_k(z)^T$ *for any* $z \in \mathbf{F}^n$.

(b) *There exist an additive group injective endomorphism* $\tilde{g}_k$ *on* $\mathbf{F}^n$ *and a nonzero vector* $\tilde{u}_k \in \mathbf{F}^n$ *satisfying* $f(e_k z^T) = \tilde{g}_k(z)\tilde{u}_k^T$ *for any* $z \in \mathbf{F}^n$.

Proof. For arbitrary but fixed $k \in \langle n\rangle$, it is obvious that $f(e_k\mathbf{F}^n) \cap \Gamma \neq \varnothing$. Let $O \neq uv^T \in f(e_k\mathbf{F}^n)$, where $u, v \in \mathbf{F}^n$.

Case 1. Suppose $\alpha = l_\alpha u$ for any nonzero $\alpha\beta^T \in f(e_k\mathbf{F}^n)$. Then $\alpha\beta^T = l_\alpha u\beta^T = u\,(l_\alpha\beta)^T \in u\mathbf{F}^n$, and hence $f(e_k\mathbf{F}^n) \subseteq u\mathbf{F}^n$ or, equivalently, there exists an additive group endomorphism g_k on $\mathbf{F}^n$ such that $f(e_k z^T) = ug_k(z)^T$ for any $z \in \mathbf{F}^n$.

Now we prove that g_k is injective. Indeed, if there exists a pair of vectors z_1 and z_2 such that $g_k(z_1) = g_k(z_2)$, then $f(e_k z_1^T) = f(e_k z_2^T)$, i.e., $f\left(e_k\,(z_1 - z_2)^T\right) = O$. This, together with $e_k\,(z_1 - z_2)^T \in \Gamma \cup \{O\}$ and the definition of f, implies $z_1 = z_2$. Therefore g_k is injective.

Let $u_k = u$. Then (a) holds.

Case 2. Suppose there exists nonzero $\alpha_0\beta_0^T \in f(e_k\mathbf{F}^n)$ such that α_0 and u are linearly independent. Then, by $\alpha_0\beta_0^T + uv^T \in f(e_k\mathbf{F}^n) \subseteq \Gamma \cup \{O\}$ and (ii) of Lemma 1, we have $\alpha_0\beta_0^T + uv^T \in \Gamma$. This, together with (iii) of Lemma 1 and the linear independence of α_0 and u, shows that β_0 and v are linearly dependent, i.e., there exists $l \in \mathbf{F}^*$ satisfying $\beta_0 = lv$, which implies $\alpha_0\beta_0^T = \alpha_0\,(lv)^T = (l\alpha_0)\,v^T$.

For any nonzero $xy^T \in f(e_k\mathbf{F}^n)$, it is clear that

$$\begin{aligned} &xy^T + (l\alpha_0)\,v^T \in f(e_k\mathbf{F}^n) \subseteq \Gamma \cup \{O\}, \\ &xy^T + uv^T \in f(e_k\mathbf{F}^n) \subseteq \Gamma \cup \{O\}. \end{aligned} \tag{1}$$

If y and v are linearly independent, then, by assertions (ii) and (iii) of Lemma 1 and (1), the vectors x and $l\alpha_0$ are linearly dependent and so are the vectors x and u. This, together with $x \neq O$ and $l \neq 0$, shows that u and α_0 are linearly dependent, which is impossible. Therefore, y and v are linearly dependent, i.e., there exists $l_y \in \mathbf{F}$ satisfying $y = l_y v$, which implies $xy^T = x\,(l_y v)^T = (l_y x)\,v^T \in \mathbf{F}^n v$. This is equivalent to saying that $\beta = l_\beta v$ for any nonzero $\alpha\beta^T \in f(e_k\mathbf{F}^n)$. By an argument similar to Case 1, we can deduce that (b) holds. □

Lemma 3. *Suppose* f *is an additive rank one preserver on* $M_n(\mathbf{F})$. *Then, for any* $k \in \langle n\rangle$, *the statement* (c) *or* (d) *below holds.*

(c) *There exist an additive group injective endomorphism* h_k *on* $\mathbf{F}^n$ *and a nonzero vector* $v_k \in \mathbf{F}^n$ *satisfying* $f(ze_k^T) = h_k(z)v_k^T$ *for any* $z \in \mathbf{F}^n$.

(d) *There exist an additive group injective endomorphism* $\tilde{h}_k$ *on* $\mathbf{F}^n$ *and a nonzero vector* $\tilde{v}_k \in \mathbf{F}^n$ *satisfying* $f(ze_k^T) = \tilde{v}_k\tilde{h}_k(z)^T$ *for any* $z \in \mathbf{F}^n$.

Proof. The proof is omitted since it is similar to that of Lemma 2. □

Lemma 4. *Suppose* $f \in \Omega$. *Further, let* u_1, g_1, h_1, v_1, $\tilde{u}_1$, $\tilde{g}_1$, $\tilde{h}_1$ *and* $\tilde{v}_1$ *be defined in Lemmas* 2 *and* 3. *Then either*

$$f(e_1 z^T) = u_1 g_1(z)^T, \quad f(ze_1^T) = h_1(z)v_1^T, \quad \forall z \in \mathbf{F}^n, \tag{2}$$

or

$$f(e_1 z^T) = \tilde{g}_1(z)\tilde{u}_1^T, \quad f(ze_1^T) = \tilde{v}_1\tilde{h}_1(z)^T, \quad \forall z \in \mathbf{F}^n. \tag{3}$$

Proof. It suffices to show by Lemmas 2 and 3 that i) (a) and (d) cannot hold simultaneously, and ii) neither can (b) and (c). Since the proofs of i) and ii) are similar, we will prove only i) by the reduction to absurdity.

Suppose i) does not hold, i.e., (a) and (d) hold simultaneously. Then $u_1 g_1(e_1)^T = f(e_1 e_1^T) = \tilde{v}_1\tilde{h}_1(e_1)^T$. This, together with (ii) of Lemma 1, implies that there exists $c \in \mathbf{F}^*$ satisfying $\tilde{v}_1 = cu_1$, and thus $f(ze_1^T) = \tilde{v}_1\tilde{h}_1(z)^T = (cu_1)\,\tilde{h}_1(z)^T = u_1\left(c\tilde{h}_1(z)\right)^T$ for any $z \in \mathbf{F}^n$.

Case 1. Suppose $f(\Gamma) \subseteq u_1\mathbf{F}^n$. Then $f(M_n(\mathbf{F})) \subseteq u_1\mathbf{F}^n$, which, by (iv) of Lemma 1, contradicts the surjectivity of that f.

Case 2. Suppose $f(\Gamma) \subsetneq u_1\mathbf{F}^n$. For any $xy^T \in \Gamma$ satisfying $f(xy^T) = uv^T \notin u_1\mathbf{F}^n$, it is clear that u and u_1 are linearly independent. We have

$$\begin{aligned}
&f(xy^T) = uv^T \in \Gamma,\\
&f((x+e_1)y^T) = uv^T + u_1 g_1(y)^T \in \Gamma,\\
&f(x(y+e_1)^T) = uv^T + u_1\left(c\tilde{h}_1(x)\right)^T \in \Gamma,\\
&f((x+e_1)(y+e_1)^T) = uv^T + u_1\left[g_1(y) + c\tilde{h}_1(x) + g_1(e_1)\right]^T \in \Gamma.
\end{aligned}$$

This, together with (iii) of Lemma 1 and the linear independence of u and u_1, implies that there exists $d \in \mathbf{F}^*$ satisfying $v = dg_1(e_1)$, and hence $f(xy^T) = uv^T = u\,(dg_1(e_1))^T = (du)\,g_1(e_1)^T \in \mathbf{F}^n g_1(e_1)$. To summarize, $f(\Gamma) \subseteq u_1\mathbf{F}^n \cup \mathbf{F}^n g_1(e_1)$. Furthermore, $f(M_n(\mathbf{F})) \subseteq u_1\mathbf{F}^n \cup \mathbf{F}^n g_1(e_1)$, which, by (iv) of Lemma 1, contradicts the surjectivity of f.

Combining the above two cases, we obtain i), and hence the proof is completed. □

Lemma 5. *Given $f \in \Omega$ satisfying* (2). *Then*

$$f(e_i z^T) = u_i g_i(z)^T,\ f(ze_i^T) = h_i(z)v_i^T, \quad \forall i \in \langle n\rangle, \quad z \in \mathbf{F}^n, \tag{4}$$

where u_i, g_i, h_i and v_i are defined in Lemmas 2 *and* 3.

Proof. By (2) and Lemmas 2 and 3, it suffices to show that, for an arbitrary but fixed positive integer $k \geq 2$, (b) and (d) do not hold.

Now we prove that (b) does not hold by the reduction to absurdity. Suppose (b) holds. Then, by (2), $h_1(e_k)v_1^T = f\left(e_k e_1^T\right) = \tilde{g}_k(e_1)\tilde{u}_k^T$. This, together with (ii) of Lemma 1, implies that there exists $c \in \mathbf{F}^*$ satisfying $\tilde{u}_k = cv_1$. Therefore, $f\left(e_k z^T\right) = \tilde{g}_k(z)\tilde{u}_k^T = \tilde{g}_k(z)\,(cv_1)^T = (c\tilde{g}_k(z))\,v_1^T$ for any $z \in \mathbf{F}^n$.

For any $xy \in \Gamma$ satisfying $f\left(xy^T\right) = uv^T \notin \mathbf{F}^n v_1$, it is clear that v and v_1 are linearly independent. We have

$$\begin{aligned}
&f(xy^T) = uv^T \in \Gamma,\\
&f((x+e_k)y^T) = uv^T + (c\tilde{g}_k(y))\, v_1^T \in \Gamma,\\
&f(x(y+e_1)^T) = uv^T + h_1(x)v_1^T \in \Gamma,\\
&f((x+e_k)(y+e_1)^T) = uv^T + [h_1(x) + c\tilde{g}_k(y) + h_1(e_k)]\, v_1^T \in \Gamma.
\end{aligned}$$

This, together with (iii) of Lemma 1 and the linear independence of v and v_1, implies that $u = dh_1(e_k)$ for some $d \in \mathbf{F}^*$, and thus

$$f\left(xy^T\right) = uv^T = dh_1(e_k)v^T = h_1(e_k)\,(dv)^T \subseteq h_1(e_k)\mathbf{F}^n.$$

To summarize, $f\left(\Gamma\right) \subseteq h_1(e_k)\mathbf{F}^n \cup \mathbf{F}^n v_1$ or, equivalently, $f\left(M_n(\mathbf{F})\right) \subseteq h_1(e_k)\mathbf{F}^n \cup \mathbf{F}^n v_1$. This, together with (iv) of Lemma 1, implies that f is not surjective, which is impossible. Therefore (b) does not hold.

By an argument similar to the above, we can deduce that (d) does not hold, and hence the proof is completed. □

Based on the above lemmas, we can characterize the set Ω as follows.

Theorem 1. *$f \in \Omega$ if and only if either*

(i) *there exist a field automorphism σ on $\mathbf{F}$ and a pair of matrices $P, Q \in GL_n(\mathbf{F})$ such that $f(A) = PA^\sigma Q$ for any $A = (a_{ij}) \in M_n(\mathbf{F})$, where $A^\sigma = (\sigma(a_{ij}))$, or*

(ii) *there exist a field automorphism σ on $\mathbf{F}$ and a pair of matrices $P, Q \in GL_n(\mathbf{F})$ such that $f(A) = P\left(A^\sigma\right)^T Q$ for any $A = (a_{ij}) \in M_n(\mathbf{F})$, where $A^\sigma = (\sigma(a_{ij}))$.*

Proof. The "if" part is obvious. In order to prove the "only if" part, it suffices to show by Lemma 4 that (2) and (3) imply (i) and (ii), respectively. Since the proof of deducing (ii) from (3) is very similar to that of deducing (i) from (2), in the following we prove only that (2) implies (i).

It follows from (2) and Lemma 5 that (4) holds, and hence

$$f(A) = f\left(\sum_{i=1}^n a_i e_i^T\right) = \sum_{i=1}^n f\left(a_i e_i^T\right) = \sum_{i=1}^n h_i\left(a_i\right) v_i^T,$$
$$\forall A = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \in M_n(\mathbf{F}).$$

This, together with the surjectivity of f, implies that $v_1, \dots, v_n$ are linearly independent and $h_1, \dots, h_n$ are additive group automorphisms on $\mathbf{F}^n$. Therefore there exists a matrix $Q_1 \in GL_n(\mathbf{F})$ satisfying

$$Q_1^T v_i = e_i, \quad \forall i \in \langle n \rangle. \tag{5}$$

Similarly, $g_1, \dots, g_n$ are additive group automorphisms on $\mathbf{F}^n$ and there exists a matrix $P_1 \in GL_n(\mathbf{F})$ satisfying

$$P_1 u_i = e_i, \quad \forall i \in \langle n \rangle. \tag{6}$$

Denote

$$f_1(X) = P_1 f(X) Q_1, \quad \forall X \in M_n(\mathbf{F}). \tag{7}$$

Then $f_1 \in \Omega$ and from (4), (5) and (6) we have that

$$\begin{cases} f_1\left(e_i z^T\right) = P_1 f\left(e_i z^T\right) Q_1 = P_1 u_i g_i(z)^T Q_1 = e_i \psi_i(z)^T \\ f_1\left(z e_i^T\right) = P_1 f\left(z e_i^T\right) Q_1 = P_1 h_i(z) v_i^T Q_1 = \phi_i(z) e_i^T \end{cases}, \ \forall i \in \langle n \rangle, \ z \in \mathbf{F}^n, \tag{8}$$

where $\psi_i(z) = Q_1^T g_i(z)$ and $\phi_i(z) = P_1 h_i(z)$. Since $g_1, \ldots, g_n,\ h_1, \ldots, h_n$ are additive group automorphisms on $\mathbf{F}^n$, so are $\psi_1, \ldots, \psi_n,\ \phi_1, \ldots, \phi_n$.

For any $c \in \mathbf{F}$ and $i, j \in \langle n \rangle$, it follows from (8) that $e_i \psi_i(c e_j)^T = f_1\left(e_i (c e_j)^T\right) = f_1\left((c e_i) e_j^T\right) = \phi_j(c e_i) e_j^T$. This, together with (ii) of Lemma 1, implies that there exists $\sigma_{ij}(c) \in \mathbf{F}$ such that $\psi_i(c e_j) = \sigma_{ij}(c) e_j$. Thus, σ_{ij} is an additive group automorphism on $\mathbf{F}$ and satisfies

$$f_1(c E_{ij}) = f_1\left(e_i (c e_j)^T\right) = e_i \psi_i(c e_j)^T = e_i \left(\sigma_{ij}(c) e_j\right)^T = \sigma_{ij}(c) E_{ij}. \tag{9}$$

Denote

$$f_2(X) = P_2 f_1(X) Q_2, \quad \forall X \in M_n(\mathbf{F}) \tag{10}$$

with

$$P_2 = \operatorname{diag}\left(\sigma_{11}(1)^{-1},\ \sigma_{21}(1)^{-1},\ \ldots,\ \sigma_{n1}(1)^{-1}\right),$$
$$Q_2 = \operatorname{diag}\left(1,\ \sigma_{11}(1)\sigma_{12}(1)^{-1},\ \ldots,\ \sigma_{11}(1)\sigma_{1n}(1)^{-1}\right).$$

Then $f_2 \in \Omega$ and, by (9), we can write

$$f_2(c E_{ij}) = \tau_{ij}(c) E_{ij} \tag{11}$$

and

$$\tau_{1k}(1) = \tau_{k1}(1) = 1, \quad \forall k \in \langle n \rangle. \tag{12}$$

Furthermore, τ_{ij} is an additive group automorphism on $\mathbf{F}$. Let $W = cE_{11} + cE_{1j} + E_{i1} + E_{ij}$. Then $W \in \Gamma$, and hence, by (11) and (12), $f_2(W) = \tau_{11}(c) E_{11} + \tau_{1j}(c) E_{1j} + E_{i1} + \tau_{ij}(1) E_{ij} \in \Gamma$, which implies

$$\tau_{1j}(c) = \tau_{ij}(1) \tau_{11}(c). \tag{13}$$

Choosing $c = 1$ in (13), we have $\tau_{ij}(1) = 1$ by virtue of (12). This, together with (13), gives

$$\tau_{1j}(c) = \tau_{11}(c). \tag{14}$$

By an argument similar to (14), it can be concluded that $\tau_{i1}(c) = \tau_{11}(c)$, and further

$$\tau_{pq}(c) = \tau_{11}(c), \quad \forall p, q \in \langle n \rangle. \tag{15}$$

Letting $\sigma = \tau_{11}$ and combining (11) and (15), we have

$$f_2(c E_{ij}) = \sigma(c) E_{ij}, \quad \forall i, j \in \langle n \rangle. \tag{16}$$

Based on (7), (10) and (16), in order to deduce (i), it suffices to show

$$\sigma(ab) = \sigma(a)\sigma(b), \quad \forall a, b \in \mathbf{F}. \tag{17}$$

Indeed, it follows from $E_{11} + aE_{12} + bE_{21} + abE_{22} \in \Gamma$ that $f_2(E_{11} + aE_{12} + bE_{21} + abE_{22}) \in \Gamma$. This, together with (12) and (16), implies (17), and hence the proof is completed. □

3. Applications

In this section, we assume that $\mathbf{F}$ is a field of characteristic not 2. The following lemma shows the relation between Γ and $GL_n(\mathbf{F})$.

Lemma 6. *Given nonzero $A \in M_n(\mathbf{F})$, then $A \in \Gamma$ if and only if for any $B \in GL_n(\mathbf{F})$ either $A + B \in GL_n(\mathbf{F})$ or $2A + B \in GL_n(\mathbf{F})$.*

Proof. The "if" part. It follows from $A \neq O$ that there exist a pair of matrices $P, Q \in GL_n(\mathbf{F})$ and a positive integer r satisfying $A = P\,(I_r \oplus O)\,Q$. If $r > 1$, then $A + B \notin GL_n(\mathbf{F})$ and $2A + B \notin GL_n(\mathbf{F})$ for $B = P\,(-2I_n + E_{11})\,Q \in GL_n(\mathbf{F})$, which contradicts the hypothesis. Therefore $r = 1$, i.e., $A \in \Gamma$.

The "only if" part. It follows from $A \in \Gamma$ that there exists a pair of matrices $U, V \in GL_n(\mathbf{F})$ satisfying $A = U\,(1 \oplus O)\,V$. For any $B \in GL_n(\mathbf{F})$, we can write

$$B = U \begin{bmatrix} c & \alpha \\ \beta & D \end{bmatrix} V, \quad c \in \mathbf{F}. \tag{18}$$

Thus

$$\begin{aligned} \det(A+B) &= \det(UV) \det \begin{bmatrix} 1+c & \alpha \\ \beta & D \end{bmatrix}, \\ \det(2A+B) &= \det(UV) \det \begin{bmatrix} 2+c & \alpha \\ \beta & D \end{bmatrix}. \end{aligned} \tag{19}$$

If $\det(A+B) = \det(2A+B) = 0$, then (19) implies $\det(UV) \det \begin{bmatrix} c & \alpha \\ \beta & D \end{bmatrix} = 0$. This, together with (18), gives $\det B = 0$, which contradicts the fact that $B \in GL_n(\mathbf{F})$. Therefore either $A + B \in GL_n(\mathbf{F})$ or $2A + B \in GL_n(\mathbf{F})$.

The proof is completed. □

Based on Lemma 6 and Theorem 1, we have the following theorem which characterizes invertibility preservers on $M_n(\mathbf{F})$.

Theorem 2. *f is an additive map from $M_n(\mathbf{F})$ to itself satisfying*

$$f\,(GL_n(\mathbf{F})) = GL_n(\mathbf{F}), \quad f\,(M_n(\mathbf{F}) \setminus GL_n(\mathbf{F})) = M_n(\mathbf{F}) \setminus GL_n(\mathbf{F}) \tag{20}$$

if and only if f has one of the two forms in Theorem 1.

Proof. The "if" part is obvious. We will prove the "only if" part.

For any $A \in \Gamma$, it follows from Lemma 6 that for any $B \in GL_n(\mathbf{F})$ either $A + B \in GL_n(\mathbf{F})$ or $2A + B \in GL_n(\mathbf{F})$. Thus for any $B \in GL_n(\mathbf{F})$ either $f(A) + f(B) \in GL_n(\mathbf{F})$ or $2f(A) + f(B) \in GL_n(\mathbf{F})$. This, together with $f\,(GL_n(\mathbf{F})) = GL_n(\mathbf{F})$, implies that for any $D \in GL_n(\mathbf{F})$ either $f(A) + D \in GL_n(\mathbf{F})$ or $2f(A) + D \in GL_n(\mathbf{F})$.

Case 1. Suppose $f(A) \neq O$ for every $A \in \Gamma$. Then, by Lemma 6, $f(A) \in \Gamma$ for every $A \in \Gamma$. Therefore $f \in \Omega$. Using (20) and Theorem 1 shows that f has one of the two forms in Theorem 1.

Case 2. Suppose $f(A) = O$ for some $A \in \Gamma$. Then there exists a matrix $C \in M_n(\mathbf{F})$ such that $\det C = 0$ and $\det(A + C) \neq 0$, and hence by (20),

$$\det f(C) = 0 \tag{21}$$

and

$$\det f\,(A+C) \neq 0. \tag{22}$$

Combining $f(A) = O$, (22) and the additivity of f, we have $f(C) \neq 0$, which contradicts (21). □

Using Theorem 2, one can easily obtain the following two corollaries which respectively characterize determinant preservers and characteristic polynomial preservers on $M_n(\mathbf{F})$.

Corollary 1. *f is an additive surjection from $M_n(\mathbf{F})$ to itself satisfying* $\det(f(A)) = \det A$ *for any $A \in M_n(\mathbf{F})$ if and only if there exists a pair of matrices $P, Q \in GL_n(\mathbf{F})$ satisfying* $\det(PQ) = 1$ *such that either*

$$f(A) = PAQ, \quad \forall A \in M_n(\mathbf{F}),$$

or

$$f(A) = PA^TQ, \quad \forall A \in M_n(\mathbf{F}).$$

Corollary 2. *f is an additive surjection from $M_n(\mathbf{F})$ to itself such that $f(A)$ and A have the same characteristic polynomials for any $A \in M_n(\mathbf{F})$ if and only if there exists a matrix $P \in GL_n(\mathbf{F})$ such that either*

$$f(A) = PAP^{-1}, \quad \forall A \in M_n(\mathbf{F}),$$

or

$$f(A) = PA^TP^{-1}, \quad \forall A \in M_n(\mathbf{F}).$$

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Authors' addresses:

Chong-Guang Cao and Xian Zhang
Department of Mathematics
Heilongjiang University, Harbin 150080
P. R. China
E-mails: caochongguang@163.com
zhangx663@sohu.com

Xian Zhang
School of Mechanical and Manufacturing Engineering
The Queen's University of Belfast
Ashby Building, Stranmillis Road Belfast, BT9 5AH
U.K.
E-mail: x.zhang@qub.ac.uk