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ON HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PROPERTY A

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Abstract. We study oscillatory properties of solutions of a functional differential equation of the form

$$u^{(n)}(t) + F(u)(t) = 0,$$

where $n \geq 2$ and $F : C(R_+; R) \to L_{loc}(R_+; R)$ is a continuous mapping. Sufficient conditions for this equation to have the so-called Property **A** are established. In the case of ordinary differential equation the obtained results lead to an integral generalization of the well-known theorem by Kondrat'ev.

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1. Introduction

Let $\tau \in C(R_+; R_+)$, $\lim_{t\to+\infty} \tau(t) = +\infty$. Denote by $V(\tau)$ the set of continuous mappings $F : C(R_+; R) \to L_{\rm loc}(R_+, R)$ satisfying the condition: $F(x)(t) = F(y)(t)$ holds for any $t \in R_+$ and $x, y \in C(R_+; R)$ provided that $x(s) = y(s)$ for $s \geq \tau(t)$.

This work is dedicated to the study of oscillatory properties of solutions of a functional differential equation of the form

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $n \geq 2$ and $F \in V(\tau)$.

For any $t_0 \in R_+$ we denote by $H_{t_0,\tau}$ the set of all functions $u \in C(R_+; R)$ satisfying $u(t) \neq 0$ for $t \geq t_*$, where $t_* = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$.

Throughout the work whenever the notation $V(\tau)$ and $H_{t_0,\tau}$ occurs, it will be understood, unless specified otherwise, that the function τ satisfies the conditions stated above.

It will always be assumed that the condition

$$F(u)(t)u(t) \geq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau}, \tag{1.2}$$

is fulfilled.

Let $t_0 \in R_+$. A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of equation (1.1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive, $\sup\{|u(s)| : s \in [t, +\infty)\} > 0$ for $t \geq t_0$ and there exists a function $\overline{u} \in C(R_+; R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality $\overline{u}^{(n)}(t) + F(\overline{u})(t) = 0$ holds almost everywhere on $[t_0, +\infty)$. A proper

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solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonosillatory.

Definition 1.1 ([1])**.** We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$|u^{(i)}(t)| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.3}$$

when n is odd.

Oscillatory properties of ordinary differential equations have been studied for a long time. As early as 1893, A. Kneser [2] obtained sufficient conditions for the equation

$$u^{(n)}(t) + p(t)u(t) = 0 \tag{1.4}$$

with $p \in L_{\text{loc}}(R_+; R_+)$ to have Property **A**. Noteworthy results in this direction were obtained by A. Kondrat'ev [1], I. Kiguradze and T. Chanturia [3]. For a differential equation with deviating arguments, which is a special case of (1.1), similar problems were considered in [4]–[6] (see also the references therein). As to functional differential equations, they are studied well enough in [7]–[9] both in linear and nonlinear cases. In the present paper we give sufficient conditions for equation (1.1) to have Property **A**. The results obtained in this work improve to a certain extent the results obtained in § 7 of Chapter 2 [9].

2. Some Auxiliary Lemmas

In this section we give auxiliary statements concerning some properties of monotone functions. In the sequel we denote by $\widetilde{C}^{n-1}_{\text{loc}}([t_0, +\infty))$ the set of those functions $u : [t_0, +\infty) \to R$ which are absolutely continuous on any finite subsegment of $[t_0, +\infty)$ along with their derivatives up to order $n-1$ inclusive.

Lemma 2.1 ([3])**.** *Let $u \in \widetilde{C}^{n-1}_{\text{loc}}([t_0, +\infty))$, $u(t) > 0$, $u^{(n)}(t) \le 0$ for $t \ge t_0$, and $u^{(n)}(t) \not\equiv 0$ in any neighborhood of $+\infty$. Then there exist $t_1 \ge t_0$ and $l \in \{0, \dots, n-1\}$ such that $l + n$ is odd and*

$$\begin{aligned} & u^{(i)}(t) > 0 \quad \textit{for} \quad t \ge t_1 \quad (i = 0, \dots, l-1), \\ & (-1)^{i+l} u^{(i)}(t) > 0 \quad \textit{for} \quad t \ge t_1 \quad (i = l, \dots, n-1), \\ & u^{(n)}(t) \le 0 \quad \textit{for} \quad t \ge t_1. \end{aligned} \tag{2.1$_l$}$$

Remark 2.1. If n is odd and $l = 0$, then in (2.1_l) it is meant that only the second and the third inequality are fulfilled.

Now we prove the following lemmas describing the behavior of non-oscillatory functions.

Lemma 2.2. *Let $t_0 \in (0, +\infty)$, $u \in \widetilde{C}_{\text{loc}}([t_0, +\infty))$ and (2.1_l) be fulfilled for some $l \in \{1, \dots, n-1\}$ with $l + n$ odd. Then*

$$\int\limits_{t_0}^{+\infty} t^{n-l-1} |u^{(n)}(t)| dt < +\infty, \tag{2.2}$$

$$\int\limits_{t_0}^{+\infty} s^{-2} \int\limits_{t_0}^{s} \xi^{n-l} |u^{(n)}(\xi)| d\xi ds < +\infty, \tag{2.3}$$

$$\lim_{t\to+\infty} \frac{1}{t} \int\limits_{t_0}^{t} s^{n-l} |u^{(n)}(s)| ds = 0, \tag{2.4}$$

$$u^{(i)}(t) \geq u^{(i)}(t_0) + \frac{1}{(l-i-1)!(n-l-1)!} \int\limits_{t_0}^{t} (t-s)^{l-i-1} \times \int\limits_{s}^{+\infty} (\xi - s)^{n-l-1} |u^{(n)}(\xi)| d\xi ds \quad \textit{for} \quad t \geq t_0 \quad (i = 0, \ldots, l-1). \tag{2.5}$$

If, moreover,

$$\int\limits_{t_0}^{+\infty} t^{n-l} |u^{(n)}(t)| dt = +\infty, \tag{2.6}$$

then there exists $t_* \geq t_0$ *such that*

$$u(t) \geq \frac{t^{l-1}}{l!} u^{(l-1)}(t) \quad \textit{for} \quad t \geq t_*, \tag{2.7}$$

$$\lim_{t\to+\infty} (u^{(l-1)}(t) - t u^{(l)}(t)) = +\infty, \tag{2.8}$$

$$\frac{u^{(i)}(t)}{t^{l-i}} \downarrow, \quad \frac{u^{(i)}(t)}{t^{l-i-1}} \uparrow \quad (i = 0, \ldots, l-1). \tag{2.9$_i$}$$

Proof. Taking into account (2.1_l), we deduce (2.2) from the identity

$$\sum_{j=i}^{k-1} \frac{(-1)^j t^{j-i} u^{(j)}(t)}{(j-i)!} = \sum_{j=i}^{k-1} \frac{(-1)^j t_0^{j-i} u^{(j)}(t_0)}{(j-i)!} + \frac{(-1)^{k-1}}{(k-i-1)!} \int\limits_{t_0}^{t} s^{k-i-1} u^{(k)}(s) ds \tag{2.10$_{ik}$}$$

with $i = l$ and $k = n$.

Let $\varepsilon > 0$ be any positive number. According to (2.2), choose $T > t_0$ such that

$$\int\limits_{T}^{+\infty} s^{n-l-1} |u^{(n)}(s)| ds < \varepsilon.$$

Therefore we obtain

$$\limsup_{t\to+\infty}\frac{1}{t}\int\limits_{t_0}^{t}s^{n-l}|u^{(n)}(s)|ds$$
$$=\limsup_{t\to+\infty}\left(\frac{1}{t}\int\limits_{t_0}^{T}s^{n-l}|u^{(n)(s)}|ds+\frac{1}{t}\int\limits_{T}^{t}s^{n-l}|u^{(n)}(s)|ds\right)$$
$$\leq\int\limits_{T}^{+\infty}s^{n-l-1}|u^{(n)}(s)|ds<\varepsilon.$$

Taking into account that ε is arbitrary, from the latter inequality we deduce the validity of condition (2.4).

On account of (2.1_l) and (2.2), from the identity

$$u^{(i)}(t)=\sum_{j=i}^{k-1}\frac{u^{(j)}(s)}{(j-i)!}(t-s)^{j-i}+\frac{1}{(k-i-1)!}\int\limits_{s}^{t}(t-\xi)^{k-i-1}u^{(k)}(\xi)d\xi \quad (2.11_{ik})$$

as $s\to+\infty$ we obtain

$$|u^{(i)}(t)|\geq\frac{1}{(n-i-1)!}\int\limits_{t}^{+\infty}(s-t)^{n-i-1}|u^{(n)}(s)|ds$$
$$\text{for}\quad t\geq t_0\quad(i=l,\dots,n-1). \qquad (2.12_i)$$

Analogously, from (2.11_{il}) as $s=t_0$ we have

$$u^{(i)}(t)\geq u^{(i)}(t_0)+\frac{1}{(l-i-1)!}\int\limits_{t_0}^{t}(t-\xi)^{l-i-1}u^{(l)}(\xi)d\xi$$
$$\text{for}\quad t\geq t_0\quad(i=0,\dots,l-1).$$

Hence by (2.12_l) we obtain (2.5).

Now assume that (2.6) holds. Then using (2.1_l), from (2.10_{l-1n}) we obtain

$$u^{(l-1)}(t)-tu^{(l)}(t)=\sum_{j=l-1}^{n-1}\frac{t^{j-l+1}}{(j-l+1)!}|u^{(j)}(t)|+\sum_{j=l-1}^{n-1}\frac{(-1)^{j+l-1}t_0^{j-l+1}u^{(j)}(t_0)}{(j-l+1)!}$$
$$+\frac{1}{(n-l)!}\int\limits_{t_0}^{t}s^{n-l}|u^{(n)}(s)|ds.$$

Hence by (2.6) we have (2.8). For any $t\geq t_0$ and $i\in\{1,\dots,l\}$ denote

$$\gamma_i(t)=iu^{(l-i)}(t)-tu^{(l-i+1)}(t)=-t^{i+1}(t^{-i}u^{(l-i)}(t))', \qquad (2.13)$$
$$r_i(t)=tu^{(l-i+1)}(t)-(i-1)u^{(l-i)}(t)=t^i(t^{1-i}u^{(l-i)}(t))'. \qquad (2.14)$$

According to (2.8), by L'Hôpital's rule we obtain

$$\lim_{t\to+\infty} t^{1-i}u^{(l-i)}(t) = +\infty \quad (i = 1,\dots,l).$$

Therefore, in view of (2.14), there exist $\alpha_l \geq \dots \geq \alpha_1$ such that $r_i(\alpha_i) > 0$ $(i = 1,\dots,l)$. Since $r_1(t) = tu^{(l)}(t) > 0$ and $r'_{i+1}(t) = r_i(t)$ for $t \geq t_0$ $(i = 1,\dots,l-1)$, we see that $r_i(t) > 0$ for $t \geq \alpha_i$. Analogously, by (2.8) we have $\gamma_1(t) \to +\infty$ for $t \to +\infty$ and $\gamma'_{i+1}(t) = \gamma_i(t)$. Therefore $\gamma_i(t) \to +\infty$ as $t \to +\infty$ $(i = 1,\dots,l)$. So, by (2.13) and (2.14), we obtain (2.9_i). On the other hand, by (2.13) we have

$$iu^{(l-i)}(t) \geq tu^{l-i+1}(t) \quad \text{for sufficiently large } t \ \ (i = 1,\dots,l).$$

Whence (2.7) follows. Now we will show that (2.3) is true. Indeed, if condition (2.6) is not fulfilled, then the validity of (2.3) is obvious. Therefore below we will assume that (2.6) is fulfilled. If we take into account (2.1_l) and (2.6), from the equality (2.10_{l-1n}) we see that there exits $t_1 > t_0$ such that

$$\frac{1}{(n-l)!}\int\limits_{t_1}^{t} s^{n-l}|u^{(n)}(s)|ds \leq u^{(l-1)}(t) - tu^{(l)}(t) \quad \text{for} \quad t \geq t_1.$$

Therefore, according to (2.9_{l-1}), we have

$$\frac{1}{(n-l)!}\int\limits_{t_1}^{+\infty} s^{-2}\int\limits_{t_1}^{s} \xi^{n-2}|u^{(n)}(\xi)|d\xi ds \leq -\int\limits_{t_1}^{+\infty}\Big(\frac{u^{l-1}(s)}{s}\Big)'ds < +\infty.$$

Hence (2.3) is fulfilled. The proof of the lemma is complete. □

Lemma 2.3. *Let $n \geq 2$, $l \in \{1,\dots,n-1\}$, $u_0 \in \widetilde{C}^{n-1}_{\rm loc}([t_0,+\infty))$. Then we have the equalities*

$$u_i^{(l)}(t) = (-1)^i t^i u_0^{(l+i)}(t) \quad (i = 1,\dots,n-l), \tag{2.15}$$

where

$$u_i(t) = (l+i-1)u_{i-1}(t) - tu'_{i-1}(t) \quad (i = 1,\dots,n-l). \tag{2.16}$$

Proof. In the case $i = 1$ the validity of (2.15) is obvious. Assume now that (2.15) is valid for some i $(1 \leq i < n-l)$ and show that it is valid for $i+1$ as well. Indeed, by (2.16), we have

$$\begin{aligned} u_{i+1}^{(l)}(t) &= ((l+i)u_i(t) - tu'_i(t))^{(l)} = iu_i^{(l)}(t) - tu_i^{(l+1)}(t) \\ &= (-1)^i it^i u_0^{(l+i)}(t) - (-1)^i t(t^i u_0^{(l+i)}(t))' = (-1)^i it^i u_0^{(l+i)}(t) \\ &\quad - (-1)^i t(it^{i-1}u_0^{(l+i)}(t) + t^i u_0^{(l+i+1)}(t)) = (-1)^{i+1}t^{i+1}u_0^{(l+i+1)}(t). \end{aligned}$$

Hence, equalities (2.15) are valid for any $i \in \{1,\dots,n-l\}$. □

Lemma 2.4. *Let $t_0 \in (0,+\infty)$, $u \in \widetilde{C}_{\rm loc}([t_0,+\infty))$ and, for any $l \in \{1,\dots,n-1\}$, where $l+n$ is odd, conditions (2.1_l) and (2.6) be fulfilled. Then there exists $t_* > t_0$ such that*

$$u(t) \ge \frac{t^l}{(l-1)!(n-l-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-l-1} \times \int_{t_*}^{s}(s-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds \quad \textit{for} \quad t \ge t_*. \tag{2.17}$$

Proof. According to (2.1_l) and (2.5), we have

$$u(t) \ge \frac{1}{(l-1)!(n-l-1)!}\int_{t_0}^{t}(t-s)^{l-1} \times \int_s^{+\infty}(\xi-s)^{n-l-1}|u^{(n)}(\xi)|d\xi ds \quad \text{for} \quad t \ge t_0. \tag{2.18}$$

Denote

$$u_0(t) = \frac{t^l}{(l-1)!(n-l-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-l-1} \times \int_{t_0}^{s}(s-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds. \tag{2.19}$$

According to (2.3), it is obvious that the integral on the right side of (2.19) exists and equalities (2.16) are fulfilled, where

$$u_i(t) = \frac{t^{l+i}}{(l-1)!(n-l-i-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-l-i-1} \times \int_{t_0}^{s}(s-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds \quad (i=0,\dots,n-l-1), \tag{2.20}$$

$$u_{n-l}(t) = \frac{1}{(l-1)!}\int_{t_0}^{t}(t-s)^{l-1}s^{n-l}|u^{(n)}(s)|ds.$$

Therefore by virtue of Lemma 2.3 we have

$$(-1)^{n+l}u_0^{(n)}(t) = |u^{(n)}(t)| \quad \text{for} \quad t \ge t_0, \tag{2.21}$$

where $n+l$ is odd. According to (2.19), (2.21) and Lemma 2.1, there exists $l' \in \{0,\dots,n-1\}$ such that $l+l'$ is even and the conditions

$$\begin{aligned} &u_0^{(i)}(t) > 0 \quad (i = 0,\dots,l'),\\ &(-1)^{i+l'} u_0^{(i)}(t) > 0 \quad (i = l',\dots,n-1) \quad \text{for} \quad t \ge t_0 \end{aligned} \tag{$2.22_{l'}$}$$

are fulfilled. Now show that there exists $t_* > t_0$ such that the inequality

$$t^{l-1} \le u_0(t) \le t^l \quad \text{for} \quad t \ge t_* \tag{2.23}$$

is fulfilled. According to (2.3) and (2.19), the validity of the right inequality in (2.23) is obvious. By virtue of (2.6), there exists $t_1 > t_0$ such that

$$\int_{t_0}^{t_1} s^{n-l} |u^{(n)}(s)| ds = c > (l-1)!(n-l-1)! 2^{n-1}.$$

Therefore from (2.19) we obtain

$$\begin{aligned} u_0(t) &\ge \frac{t^l}{(l-1)!(n-l-1)!} \int_{2t}^{+\infty} (s-t)^{n-l-1} s^{-n-1+l} \int_{t_0}^{t_1} (1-\xi)^{l-1} \xi^{n-l} |u^{(n)}(\xi)| d\xi ds\\ &\ge \frac{t^l}{2^{l-1}(l-1)!(n-l-1)!} \int_{t_0}^{t_1} \xi^{n-l} |u^{(n)}(\xi)| d\xi \int_{2t}^{+\infty} (s-t)^{n-l-1} s^{-n-1+l} ds\\ &\ge \frac{ct^l}{(l-1)!(n-l-1)! 2^{l-1}} \int_{2t}^{+\infty} (1-\frac{t}{s})^{n-l-1} s^{-2} ds\\ &\ge \frac{ct^l}{(l-1)!(n-l-1)! 2^{n-2}} \int_{2t}^{+\infty} s^{-2} ds > t^{l-1} \text{ for } t \ge 2t_1. \end{aligned}$$

The latter inequality implies the existence of t_* such that inequality (2.23) is fulfilled. Therefore, according to (2.1_l) and ($2.22_{l'}$), since $l+l'$ is even, it follows that $l = l'$.

Now show that

$$\lim_{t\to+\infty} t^i u_0^{(l+i)}(t) = 0 \quad (i = 0,\dots,n-l-1). \tag{2.24}$$

Indeed, using (2.20) we have

$$u_i^{(l)}(t) = \frac{1}{(l-1)!(n-l-i-1)!} \sum_{j=0}^{l} C_l^j (t^{l+i})^{(l-j)} \times \left(\int_t^{+\infty} s^{-n} (s-t)^{n-l-i-1} \int_{t_0}^{s} (s-\xi)^{l-1} \xi^{n-l} |u^{(n)}(\xi)| d\xi ds \right)^{(j)}. \tag{2.25}$$

Let $j \le n-l-i-1$. Then according to (2.3) we obtain

$$\rho_{ji}(t) = (t^{l+i})^{(l-j)}\bigg|\bigg(\int\limits_t^{+\infty} s^{-n}(s-t)^{n-l-i-1}\int\limits_{t_0}^{s}(s-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds\bigg)^{(j)}\bigg|$$
$$\le (l+i)!(n-l-i-1)!t^{i+j}$$
$$\times \int\limits_t^{+\infty} s^{-n}(s-t)^{n-l-i-j-1}\int\limits_{t_0}^{s}(s-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds$$
$$\le (l+i)!(n-l-i-1)!t^{i+j}\int\limits_t^{+\infty} s^{-i-j-2}\int\limits_{t_0}^{s}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds$$
$$\le (l+i)!(n-l-i-1)!\int\limits_t^{+\infty} s^{-2}\int\limits_{t_0}^{s}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds \to 0$$
$$\text{for} \quad t\to+\infty \quad (j=0,\dots,n-l-i-1). \tag{2.26}$$

Let $j \ge n-l-i$. Then we have

$$\rho_{ji}(t) \le (l+i)!(n-l-i-1)!t^{i+j}$$
$$\times\bigg(t^{-n}\int\limits_{t_0}^{t}(t-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi\bigg)^{(j+i+l-n)}$$
$$= (l+i)!(n-l-i-1)!t^{i+j}\sum_{k=0}^{j+i+l-n}C^k_{j+i+l-n}(t^{-n})^{(j+i+l-k)}$$
$$\times\bigg(\int\limits_{t_0}^{t}(t-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi\bigg)^{(k)}. \tag{2.27}$$

On the other hand, by virtue of (2.4) we have

$$t^{i+j}\bigg|(t^{-n})^{(j+i+l-k)}\bigg(\int\limits_{t_0}^{t}(t-s)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi\bigg)^{(k)}\bigg|$$
$$\le n!t^{i+j}t^{-j-i-l+k}\int\limits_{t_0}^{t}(t-\xi)^{(l-k-1)}\xi^{n-l}|u^{(n)}(\xi)|d\xi$$
$$\le n!(l-1)!t^{-1}\int\limits_{t_0}^{t}\xi^{n-l}|u^{(n)}(\xi)|d\xi \to 0$$
$$\text{for} \quad t\to+\infty \quad (j=n-l-i,\dots,l). \tag{2.28}$$

According to (2.25)–(2.28) and (2.15), it is obvious that conditions (2.24) are fulfilled. Therefore, by (2.21) we have

$$u_0^{(l)}(t) = \frac{1}{(n-l-1)!} \int\limits_t^{+\infty} (s-t)^{n-l-1} |u^{(n)}(s)| ds \quad \text{for} \quad t \geq t_0.$$

From the latter equality we obtain

$$u_0(t) = \sum_{i=0}^{l-1} \frac{(t-t_0)^i}{i!} u_0^{(i)}(t_0) + \frac{1}{(l-1)!(n-l-1)!} \int\limits_{t_0}^{t} (t-s)^{l-1} \int\limits_s^{+\infty} (\xi - s)^{n-l-1} |u^{(n)}(\xi)| d\xi ds \quad \text{for} \quad t \geq t_0.$$

Hence, if we take into account (2.18) and (2.19), we obtain

$$u(t) \geq \frac{t^l}{(l-1)!(n-l-1)!} \int\limits_t^{+\infty} s^{-n} (s-t)^{n-l-1} \times \int\limits_{t_0}^{s} (s-\xi)^{l-1} \xi^{n-l} |u^{(n)}(\xi)| d\xi ds - ct^{l-1} \quad \text{for} \quad t \geq t_0, \tag{2.29}$$

where $c = \sum_{i=0}^{l-1} \frac{u_0^{(i)}(t_0)}{i!}$.

By virtue of (2.6) there exists $t_1 > t_0$ such that

$$\int\limits_{t_0}^{t_1} s^{n-l} |u^{(n)}(s)| ds > 2^{n-1} (l-1)!(n-l-1)! c.$$

Hence from (2.29) we obtain

$$\begin{aligned} u(t) &\geq \frac{t^l}{(l-1)!(n-l-1)!} \int\limits_t^{+\infty} s^{-n} (s-t)^{n-l-1} \\ &\quad \times \left(\int\limits_{t_0}^{t_1} (s-\xi)^{l-1} \xi^{n-l} |u^{(n)}|(\xi) d\xi + \int\limits_{t_1}^{s} (s-\xi)^{l-1} \xi^{n-l} |u^{(n)}(\xi)| d\xi \right) ds - ct^{l-1} \\ &\geq \frac{t^l}{(l-1)!(n-l-1)!} \int\limits_t^{+\infty} s^{-n} (s-t)^{n-l-1} \int\limits_{t_1}^{s} (s-\xi)^{l-1} \xi^{n-l} |u^{(n)}(\xi)| d\xi ds \\ &\quad + \frac{t^l}{(l-1)!(n-l-1)!} \int\limits_{2t}^{+\infty} s^{-n+l-1} (s-t)^{n-l-1} \end{aligned}$$

$$\times \int_{t_0}^{t_1} \left(1-\frac{\xi}{s}\right)^{l-1} \xi^{n-l}|u^{(n)}(\xi)|d\xi ds - ct^{l-1}$$

$$\geq \frac{t^l}{(l-1)!(n-l-1)!} \int_t^{+\infty} s^{-n}(s-t)^{n-l-1} \int_{t_1}^{s} (s-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds$$

$$+ \frac{t^l}{2^{n-2}(l-1)!(n-l-1)!} \int_{t_0}^{t_1} \xi^{n-l}|u^{(n)}(\xi)|d\xi \int_{2t}^{+\infty} s^{-2}ds - ct^{l-1}$$

$$> \frac{t^l}{(l-1)!(n-l-1)!} \int_t^{+\infty} s^{-n}(s-t)^{n-l-1} \int_{t_1}^{s} (s-\xi)^{l-1}\xi^{n-l}|u^{(n)}(\xi)|d\xi ds$$

for $t \geq 2t_1$,

which proves the validity of inequality (2.17) with $t_* = 2t_1$. □

Lemma 2.5. *Let $t_0 \in R_+$, η, $\psi \in C([t_0,+\infty), (0,+\infty))$,*

$$\liminf_{t\to+\infty} \eta(t) = 0, \quad \psi(t) \uparrow +\infty \quad \textit{for} \quad t \uparrow +\infty, \tag{2.30}$$

$$\lim_{t\to+\infty} \psi(t)\widetilde{\eta}(t) = +\infty, \tag{2.31}$$

where $\widetilde{\eta}(t) = \min\{\eta(s) : s \in [t_0,t]\}$. Then there exists a sequence of points $\{t_k\}$ such that $t_k \uparrow +\infty$ as $k \uparrow +\infty$,

$$\widetilde{\eta}(t_k)\psi(t_k) \leq \widetilde{\eta}(s)\psi(s) \quad \textit{for} \quad s \geq t_k,$$
$$\widetilde{\eta}(t_k) = \eta(t_k) \quad (k = 1,2,\dots).$$

Proof. Denote by E_i $(i = 1,2)$ the sets

$$t \in E_1 \Leftrightarrow \widetilde{\eta}(t)\psi(t) \leq \widetilde{\eta}(s)\psi(s) \quad \text{for} \quad s \geq t,$$
$$t \in E_2 \Leftrightarrow \widetilde{\eta}(t) = \eta(t).$$

According to (2.30) and (2.31), it is obvious that

$$\sup E_i = +\infty \quad (i = 1,2). \tag{2.32}$$

Let $m \in N$. According to (2.32), there exist $t_m^i \in E_i$ $(i = 1,2)$ such that $m \leq t_m^2 \leq t_m^1$. Assume $t_m^1 \notin E_2$. Then there exists $t_m^* \in [t_m^2, t_m^1)$ such that $\widetilde{\eta}(t) = \widetilde{\eta}(t_m^1)$ when $t \in [t_m^*, t_m^1]$ and $\widetilde{\eta}(t_m^*) = \eta(t_m^*)$. On the other hand, since $t_m^1 \in E_1$, according to the second condition of (2.30) we have

$$\widetilde{\eta}(t_m^*)\psi(t_m^*) \leq \widetilde{\eta}(s)\psi(s) \quad \text{when} \quad s \geq t_m^*.$$

Therefore $t_m^* \in E_1 \cap E_2$. From the above reasoning it follows that $\sup E_1 \cap E_2 = +\infty$. This proves the validity of the lemma. □

Remark 2.2. Lemma 2.5 concerns some properties of nonmonotone positive functions. A lemma of different type likewise concerning some properties of nonmonotone functions can be found in [9, Lemma 7.7].

Remark 2.3. Some estimations of Lemma 2.2, as well as of Lemmas 2.4 and 2.5, are given for the first time in the present paper.

3. On Solutions of Type (2.1_l)

In this section sufficient conditions will be given for equation (1.1) to have no solution of type (2.1_l), where $l \in \{1,\dots,n-1\}$ and $l+n$ is odd.

Let σ, $\overline{\sigma} \in C(R_+;R_+)$, $\sigma(t) \le \overline{\sigma}(t)$ for $t \in R_+$ and $\lim_{t\to+\infty}\sigma(t) = +\infty$. Denote by $M(\sigma,\overline{\sigma})$ the set of continuous mappings $\varphi : C(R_+;R_+) \to L_{\rm loc}(R_+;R_+)$ satisfying the conditions:

$$\varphi(x)(t) \ge \varphi(y)(t) \quad \text{for} \quad t\in R_+ \quad \text{and} \quad x(s)\ge y(s) \quad \text{for} \quad s\ge\sigma(t), \tag{3.1}$$

$$\varphi(xy)(t) \ge x(\overline{\sigma}(t))\varphi(y)(t) \quad \text{for} \quad t\in R_+ \quad \text{if} \quad x(t)\downarrow 0 \quad \text{as} \quad t\uparrow+\infty$$
$$\text{and} \quad x(s)>0,\ y(s)>0 \quad \text{for} \quad s\ge\sigma(t). \tag{3.2}$$

Theorem 3.1. *Let condition* (1.2) *be fulfilled,* $l \in \{1,\dots,n-1\}$*, where* $l+n$ *is odd, and for sufficiently large* $t_1 \in [t_0,+\infty)$ *there exist* $\varphi \in M(\sigma,\overline{\sigma})$ *such that*

$$|F(u)(t)| \ge \varphi(|u|)(t) \quad \textit{for} \quad u\in H_{t_1,\tau}, \quad t\ge t_1. \tag{3.3}$$

Moreover, let

$$\liminf_{t\to+\infty}\frac{t}{\overline{\sigma}(t)} > 0, \tag{3.4}$$

$$\int^{+\infty} t^{n-l-1}\varphi(c\Theta_l)(t)dt = +\infty \quad \textit{for any} \quad c>0, \tag{3.5$_l$}$$

with $\Theta_l(t)=t^l$*, and for some* $\gamma>1$ *and any* $\lambda\in(l-1,l]$

$$\liminf_{t\to+\infty} t^{l-1-\lambda}\int_0^t s^{n-l}\varphi(\Theta_\lambda)(s)ds > \gamma\prod_{i=0,\,i\ne l-1}^{n-1}|\lambda-i|, \tag{3.6$_l$}$$

where $\Theta_\lambda(t) = t^\lambda$*. Then equation* (1.1) *has no solution satisfying condition* (2.1_l).

Remark 3.1. For operators F from a quite a large class, (3.5_l) is a necessary condition for equation (1.1) to have a solution of type (2.1_l) (see [9, Lemma 5.1]).

Proof of Theorem 3.1. Assume the contrary, i.e., that equation (1.1) has a proper solution satisfying (2.1_l). First of all show that (2.6) is fulfilled. Indeed, by (2.1_l) there exist $c>0$ and $t_*\ge t_1$ such that $u(t)\ge ct^{l-1}$ for $t\ge t_*$. Therefore (1.1) and (3.1)–(3.5_l) imply that

$$\int_{t^*}^{+\infty} s^{n-l}|u^{(n)}(s)|ds \ge \int_{t^*}^{+\infty} s^{n-l}\varphi(c\Theta_{l-1})(s)ds \ge \int_{t^*}^{+\infty} s^{n-l-1}\varphi(c\Theta_l)(s)ds = +\infty,$$

where $t^* > t_*$ is a sufficiently large number. Therefore below we will assume that condition (2.6) is fulfilled. According to (2.6) the conditions of Lemmas 2.2 and 2.4 are fulfilled. Therefore the first condition of (2.9_0) leads to the relation

$$\lim_{t\to+\infty}\frac{u(t)}{t^l}=c_0\geq 0. \tag{3.7}$$

Show that $c_0=0$. Assume the contrary. Let $c_0>0$. Then by (3.7) there exists $t_2>t_1$ such that

$$u(t)\geq \frac{c_0}{2}t^l \quad \text{for} \quad t\geq t_2.$$

Therefore, according to (2.1_l), (2.12_l), (3.1) and (3.3), we have

$$\int\limits_{t_2}^{+\infty} s^{n-l-1}\varphi\Big(\frac{c_0}{2}\Theta_l\Big)(s)ds<+\infty,$$

where $\Theta_l(t)=t^l$. This contradicts condition (3.5_l). The obtained contradiction proves that

$$\frac{u(t)}{t^l}\downarrow 0 \quad \text{for} \quad t\uparrow+\infty. \tag{3.8}$$

By to (1.1), (3.3) and Lemma 2.4, there exists $t_*>t_1$ such that the inequality

$$u(t)\geq \frac{t^l}{(l-1)!(n-l-1)!}\int\limits_t^{+\infty} s^{-n}(s-t)^{n-l-1} \times\int\limits_{t_*}^{s}(s-\xi)^{l-1}\varphi(u)(\xi)d\xi ds \quad \text{for} \quad t\geq t_* \tag{3.9_l}$$

is fulfilled.

On the other hand, according to (2.7)–(2.9_1) and (3.8) we have

$$\lim_{t\to+\infty}\frac{u(t)}{t^{l-1}}=+\infty, \quad \lim_{t\to+\infty}\frac{u(t)}{t^l}=0. \tag{3.10}$$

Denote by Δ the set of $\lambda\in(l-1,l]$ such that the condition

$$\liminf_{t\to+\infty}\frac{u(t)}{t^\lambda}=0 \tag{3.11}$$

is fulfilled. Assume $\lambda_0=\inf\Delta$. By virtue of (3.10) it is obvious that $\Delta\neq\varnothing$ and $\lambda_0\in[l-1,l]$. Taking into account (3.6_l) choose $\gamma_1\in(1,\gamma]$, $t^*>t_*$, $\lambda^*\in[\lambda_0,l]\cap(l-1,l]$ and $\varepsilon_0>0$ such that

$$\liminf_{t\to+\infty}\frac{u(t)}{t^{\lambda^*}}=0, \quad \lim_{t\to+\infty}\frac{u(t)}{t^{\lambda^*-\varepsilon_0}}=+\infty, \tag{3.12}$$

$$t^{l-1-\lambda^*}\int\limits_{t_*}^{t} s^{n-l}\varphi(\Theta_{\lambda^*})(s)ds>\gamma_1\prod_{i=0,i\neq l-1}^{n-1}|\lambda-i| \quad \text{for} \quad t\geq t^*, \tag{3.13_l}$$

$$\gamma_1 > \frac{\prod_{i=0}^{l-2}|\lambda^*-i|\prod_{i=l}^{n-1}|\lambda^*-i-\varepsilon_0|}{\left(\frac{c}{2}\right)^{\varepsilon_0}\prod_{i=0;i\neq l-1}^{n-1}|\lambda^*-i|}, \tag{3.14}$$

where

$$\begin{gathered}\theta_{\lambda^*}(t)=t^{\lambda^*}, \quad \liminf_{t\to+\infty}\frac{t}{\widetilde{\sigma}(t)}=c>0,\\ \widetilde{\sigma}(t)=\max\{\max(s,\overline{\sigma}(s)) : t_* \le s \le t\}\end{gathered} \tag{3.15}$$

(in the case $l=1$, in (3.14) we mean $\prod_{i=0}^{l-2}|\lambda^*-i|=1$).

Denote

$$\widetilde{u}(t)=\min\Big\{\frac{u(s)}{s^{\lambda^*}} : t_*\le s\le t\Big\}. \tag{3.16}$$

By (3.12) it is obvious that

$$\widetilde{u}(t)\downarrow 0 \quad \text{for} \quad t\uparrow+\infty, \quad \lim_{t\to+\infty} t^{\varepsilon_0}\widetilde{u}(t)=+\infty. \tag{3.17}$$

Therefore, if we take into account Lemma 2.5 with $\eta(t)=\frac{u(t)}{t^{\lambda^*}}$, $\psi(t)=t^{\varepsilon_0}$, we will see that there exists a sequence $\{t_k\}_{k=3}^{+\infty}$ such that $t_k\uparrow+\infty$ for $k\uparrow+\infty$ and

$$\widetilde{u}(\widetilde{\sigma}(t_k))(\widetilde{\sigma}(t_k))^{\varepsilon_0}\le \widetilde{u}(\widetilde{\sigma}(s))(\widetilde{\sigma}(s))^{\varepsilon_0} \quad \text{for} \quad s\ge\widetilde{\sigma}(t_k), \tag{3.18}$$

$$\widetilde{u}(\widetilde{\sigma}(t_k))=\frac{u(\widetilde{\sigma}(t_k))}{(\widetilde{\sigma}(t_k))^{\lambda^*}} \quad (k=3,4,\dots). \tag{3.19}$$

Consider the case where $l=1$ and n is even. Then, if we take into account that $\varphi\in M(\sigma,\overline{\sigma})$, from (3.9_1), according to (3.2), (3.16) and (3.17) we obtain for sufficiently large k

$$\begin{aligned}u(\widetilde{\sigma}(t_k)) &\ge \frac{\widetilde{\sigma}(t_k)}{(n-2)!}\int_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-2}s^{-n}\int_{t_*}^{s}\xi^{n-1}\varphi(u)(\xi)d\xi ds\\
&\ge \frac{\widetilde{\sigma}(t_k)}{(n-2)!}\int_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-2}s^{-n}\int_{t_*}^{s}\xi^{n-1}\varphi(\widetilde{u}\Theta_{\lambda_*})(\xi)d\xi ds\\
&\ge \frac{\widetilde{\sigma}(t_k)}{(n-2)!}\int_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-2}s^{-n}\int_{t_*}^{s}\xi^{n-1}\widetilde{u}(\overline{\sigma}(\xi))\varphi(\theta_{\lambda^*})(\xi)d\xi ds\\
&\ge \frac{\widetilde{\sigma}(t_k)}{(n-2)!}\int_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-2}s^{-n}\int_{t_*}^{s}\xi^{n-1}\widetilde{u}(\widetilde{\sigma}(\xi))\varphi(\theta_{\lambda^*})(\xi)d\xi ds\\
&\ge \frac{\widetilde{\sigma}(t_k)}{(n-2)!}\int_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-2}s^{-n}\widetilde{u}(\widetilde{\sigma}(s))\int_{t_*}^{s}\xi^{n-1}\varphi(\theta_{\lambda^*})(\xi)d\xi ds,\end{aligned}$$

where $\Theta_{\lambda^*}(t) = t^{\lambda^*}$. From the latter inequality, for sufficiently large k, Taking into account (3.13_1), (3.15) and (3.18) we obtain

$$u(\widetilde{\sigma}(t_k)) \geq \frac{\gamma_1(\widetilde{\sigma}(t_k))^{1+\varepsilon_0}\prod_{i=1}^{n-1}|\lambda^*-i|\widetilde{u}(\widetilde{\sigma}(t_k))}{(n-2)!}$$
$$\times \int\limits_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-2}s^{-n-\varepsilon_0+\lambda^*}\Big(\frac{s}{\widetilde{\sigma}(s)}\Big)^{\varepsilon_0}ds$$
$$\geq \frac{(\frac{c}{2})^{\varepsilon_0}\gamma_1(\widetilde{\sigma}(t_k))^{1+\varepsilon_0}\prod_{i=1}^{n-1}|\lambda^*-i|\widetilde{u}(\widetilde{\sigma}(t_k))}{(n-2)!}$$
$$\times \int\limits_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-2}s^{-n+\lambda^*-\varepsilon_0}ds$$
$$= \frac{(\frac{c}{2})^{\varepsilon_0}\gamma_1(\widetilde{\sigma}(t_k))^{1+\varepsilon_0}\prod_{i=1}^{n-1}|\lambda^*-i|\widetilde{u}(\widetilde{\sigma}(t_k))(\widetilde{\sigma}(t_k))^{\lambda^*-\varepsilon_0-1}}{\prod_{i=1}^{n-1}|\lambda^*-i-\varepsilon_0|}.$$

According to (3.14) and (3.19), we finally obtain

$$u(\widetilde{\sigma}(t_k)) \geq \frac{\gamma_1(\frac{c}{2})^{\varepsilon_0}\prod_{i=1}^{n-1}|\lambda^*-i|}{\prod_{i=1}^{n-1}|\lambda^*-i-\varepsilon_0|}u(\widetilde{\sigma}(t_k)) > u(\widetilde{\sigma}(t_k)),$$

where k is sufficiently large. The obtained contradiction proves that $l \neq 1$.

Now assume $l > 1$. From (3.9_l), if we take into account (3.2), that the function $\widetilde{u}$ is non-increasing and the fact that $\varphi \in M(\sigma,\overline{\sigma})$, we obtain

$$u(\widetilde{\sigma}(t_k)) \geq \frac{(\widetilde{\sigma}(t_k))^l}{(l-1)!(n-l-1)!}\int\limits_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-l-1}s^{-n}\widetilde{u}(\widetilde{\sigma}(s))$$
$$\times \int\limits_{t_*}^{s}(s-\xi)^{l-1}\xi^{n-l}\varphi(\Theta_{\lambda^*})(\xi)d\xi ds, \tag{3.20}$$

where the function $\widetilde{\sigma}(t)$ is defined by the third equality of (3.15). If we carry out integration by parts on the right hand side of the inequality, we obtain

$$u(\widetilde{\sigma}(t_k)) \geq \frac{(\widetilde{\sigma}(t_k))^l}{(l-1)!(n-l-1)!}\int\limits_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-l-1}s^{-n}\widetilde{u}(\widetilde{\sigma}(s))$$
$$\times \int\limits_{t_*}^{s}(s-\xi)^{l-1}d\int\limits_{t_*}^{\xi}\xi_1^{n-l}\varphi(\Theta_{\lambda^*})(\xi_1)d\xi_1 ds$$
$$= \frac{(\widetilde{\sigma}(t_k))^l}{(l-1)!(n-l-1)!}\int\limits_{\widetilde{\sigma}(t_k)}^{+\infty}(s-\widetilde{\sigma}(t_k))^{n-l-1}s^{-n}\widetilde{u}(\widetilde{\sigma}(s))$$

$$\times \int_{t^*}^{s} (s-\xi)^{l-2} \int_{t_*}^{\xi} \xi_1^{n-l} \varphi(\Theta_{\lambda^*})(\xi_1) d\xi_1 d\xi ds.$$

Hence, if we take into account (1.13_l), (3.18) and (3.19), for sufficiently large k we obtain

$$u(\widetilde{\sigma}(t_k)) \geq \frac{\gamma_1 (\frac{c}{2})^{\varepsilon_0} \prod_{i=0;i\neq l-1}^{n-1} |\lambda^* - i| u(\widetilde{\sigma}(t_k))(\widetilde{\sigma}(t_k))^{\varepsilon_0 + l - \lambda^*}}{(l-2)!(n-l-1)!}$$
$$\times \int_{\widetilde{\sigma}(t_k)}^{+\infty} s^{\varepsilon_0 - n} (s - \widetilde{\sigma}(t_k))^{n-l-1} \int_{t^*}^{s} (s-\xi)^{l-2} \xi^{\lambda^* + 1 - l} d\xi ds.$$

On the other hand, since

$$\int_{t^*}^{s} (s-\xi)^{l-2} \xi^{\lambda^*+1-l} d\xi = \frac{(l-2)!}{\prod_{i=0}^{l-2} |\lambda_* - i|} s^{\lambda^*} (1 + o(1)),$$

from the latter inequality we obtain

$$1 \geq \frac{\gamma_1 (\frac{c}{2})^{\varepsilon_0} \prod_{i=0;i\neq l-1}^{n-1} |\lambda^* - i| (\widetilde{\sigma}(t_k))^{\varepsilon_0 + l - \lambda^*} (1 + o(1))}{(n-l-1)! \prod_{i=0}^{l-2} |\lambda^* - i|}$$
$$\times \int_{\widetilde{\sigma}(t_k)}^{+\infty} s^{-n-\varepsilon_0+\lambda^*} (s - \widetilde{\sigma}(t_k))^{n-l-1} ds$$
$$= \frac{\gamma_1 (\frac{c}{2})^{\varepsilon_0} \prod_{i=0;i\neq l-1}^{n-1} |\lambda^* - i| (\widetilde{\sigma}(t_k))^{\varepsilon_0 + l - \lambda^*} (\widetilde{\sigma}(t_k))^{\lambda^* - l - \varepsilon_0} (1 + o(1))}{\prod_{i=0}^{l-2} |\lambda^* - i| \prod_{i=l}^{n-1} |\lambda^* - i - \varepsilon_0|}.$$

Hence for sufficiently large k, according to (3.14), we obtain $1 > (1 + o(1))$. The obtained contradiction proves that $l \notin \{1, \dots, n-1\}$, where $l + n$ is odd. This proves the validity of the theorem.

Theorem 3.1′. *Let condition* (1.2) *be fulfilled,* $l \in \{1, \dots, n-1\}$ *with* $l + n$ *odd and there exist* $\varphi \in M(\sigma, \overline{\sigma})$ *such that conditions* (3.3)–(3.5_l) *are fulfilled. Let, moreover, there exist* $\gamma_1 > 1$ *such that for any* $\lambda \in (l-1, l]$

$$\liminf_{t\to+\infty} t^{-1} \int_{0}^{t} s^{n-\lambda} \varphi(\Theta_\lambda)(s) ds > \gamma_1 \prod_{i=0}^{n-1} |\lambda - i|, \tag{3.21_l}$$

where $\Theta_\lambda(t) = t^\lambda$*. Then equation* (1.1) *has no proper solution satisfying condition* (2.1_l)*.*

Proof. To prove Theorem 3.1′, it is sufficient to show the that condition (3.6_l) follows from condition (3.21_l). Indeed, by (3.21_l) there exist $t_* \in [1, +\infty)$ and

$\gamma \in (1, \gamma_1]$ such that

$$t^{-1}\int_0^t s^{n-\lambda}\varphi(\Theta_\lambda)(s)ds > \gamma \prod_{i=0}^{n-1}|\lambda - i| \quad \text{for} \quad t \geq t_*.$$

Therefore for any $\lambda \in (l-1, l]$ we have

$$\begin{aligned}
t^{l-1-\lambda}\int_0^t s^{n-l}\varphi(\Theta_\lambda)(s)ds &= t^{l-1-\lambda}\int_0^t s^{\lambda-l}d\int_0^s \xi^{n-\lambda}\varphi(\Theta_\lambda)(\xi)d\xi \\
&= t^{-1}\int_0^t s^{n-l}\varphi(\Theta_\lambda)(s)ds + (l-\lambda)t^{l-1-\lambda}\int_{t_*}^t s^{\lambda-l-1}\int_0^s \xi^{n-\lambda}\varphi(\Theta_\lambda)(\xi)d\xi ds \\
&\geq \gamma \prod_{i=0}^{n-1}|\lambda - i|\Big(1 + (l-\lambda)t^{l-1-\lambda}\int_{t_*}^t s^{\lambda-l}ds\Big) \\
&= \gamma \prod_{i=0}^{n-1}|\lambda - i|\Big(1 + \frac{l-\lambda}{\lambda+1-l} + o(1)\Big) \\
&= \gamma \prod_{i=0, i\neq l-1}^{n-1}|\lambda - i| + o(1) \quad \text{for} \quad t \geq t_*.
\end{aligned}$$

Hence condition (3.6_l) is fulfilled, which proves the validity of the theorem. □

4. Functional Differential Equations with a Non-Linear Minorant

Using the results of in Section 3, for differential equations of sufficiently general type (1.1) it is possible to obtain sufficient conditions for a given equation to have Property **A**. Moreover, the obtained results are optimal.

Theorem 4.1. *Let $F \in V(\tau)$, and conditions* (1.2), (3.3), (3.4) *and* (3.5_{n-1}) *be fulfilled with $\varphi \in M(\sigma, \overline{\sigma})$. If, moreover, conditions (3.6_l) are fulfilled for any $l \in \{1, \dots, n-1\}$ with $l+n$ odd, then equation* (1.1) *has Property* **A**.

Proof. First of all show that according to (3.1), (3.4) and (3.5_{n-1}) condition (3.5_l) is fulfilled for any $l \in \{0, \dots, n-1\}$. Indeed, by virtue of (3.4) there exist $c_0 > 0$ and $t_1 \in R_+$ such that $(\overline{\sigma}(t))^{1+l-n} \geq c_0 t^{1+l-n}$ for $t \geq t_1$ and for any $l \in \{0, \dots, n-1\}$. Therefore for sufficiently large t, by virtue of (3.1), for any $c > 0$ we have

$$\varphi(c\Theta_l)(t) \geq (\overline{\sigma}(t))^{1+l-n}\varphi(c\Theta_{n-1})(t) \geq c_0 t^{1+l-n}\varphi(c\Theta_{n-1})(t),$$

where $\Theta_i(t) = t^i$ $(i = l, n-1)$. Therefore, (3.5_{n-1}) implies that

$$\int^{\infty} t^{n-l-1}\varphi(c\Theta_l)(t)dt \geq c_0 \int^{+\infty} \varphi(c\Theta_{n-1})(t)dt = +\infty.$$

Hence (3.5_l) is fulfilled for any $l \in \{0, \dots, n-1\}$ and any $c > 0$.

Now assume that $u : [t_0, +\infty) \to R$ is a non-oscillatory solution of equation (1.1). Without loss of generality, we assume that $u(t) > 0$ for $t \ge t_0$. Then using Lemma 2.1 there exists $l \in \{0, \dots, n-1\}$, where $l+n$ is odd, such that conditions (2.1_l) are fulfilled. According to (3.6_l) and Theorem 3.1, $l \notin \{1, \dots, n-1\}$. Hence n is odd and $l = 0$. Then by (3.5_0) we can easily show that the solution u satisfies conditions (1.3). This proves the validity of the theorem. □

Theorem 4.1′. *Let $F \in V(\tau)$, and conditions* $(1,2)$, (3.3), (3.4) *and* (3.5_{n-1}) *be fulfilled with $\varphi \in M(\sigma, \overline{\sigma})$. If, moreover, conditions* (3.21_l) *are fulfilled for any $l \in \{0, \dots, n-1\}$, where $l+n$ is odd, then equation* (1.1) *has Property* **A**.

To prove the theorem it is sufficient to note that the validity of conditions (3.6_l) follows from conditions (3.21_l), where $l \in \{0, \dots, n-1\}$ and $l+n$ is odd.

Remark 4.1. We cannot ignore any of the conditions in Theorems 4.1 and 4.1′ the theorems will not be valid in general.

5. Equations with a Linear Minorant

Theorem 5.1. *Let $F \in V(\tau)$, condition* (1.2) *be fulfilled and let for any $t_0 \in R_+$*

$$|F(u)(t)| \ge \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)| d_s r_i(s,t) \quad \text{for} \quad t \ge t_0, \quad u \in H_{t_0,\tau}, \tag{5.1}$$

where τ_i, $\sigma_i \in C(R_+; R_+)$, $\tau_i(t) \le \sigma_i(t)$ for $t \in R_+$, $\lim_{t\to+\infty} \tau_i(t) = +\infty$, while the functions $r_i : R_+ \times R_+ \to R$ are nondecreasing in the first argument and Lebesgue integrable in the second argument on any finite subsegment of $[0, +\infty)$ $(i = 1, \dots, m)$*, and*

$$\liminf_{t\to+\infty} \frac{t}{\sigma_i(t)} > 0 \quad (i = 1, \dots, m). \tag{5.2}$$

If, moreover, for any $l \in \{1, \dots, n-1\}$ and $\lambda \in (l-1, l]$, where $l+n$ is odd, there exists $\gamma > 1$ such that

$$\liminf_{t\to+\infty} t^{l-1-\lambda} \int_0^t \xi^{n-l} \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{\lambda} d_s r_i(s,\xi) d\xi > \gamma \prod_{i=0; i\ne l-1}^{n-1} |\lambda - i|, \tag{5.3_l}$$

then equation (1.1) *has Property* **A**.

Proof. First of all show that from (5.3_l) it follows that

$$\int^{+\infty} \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{n-1} d_s r_i(s,\xi) d\xi = +\infty. \tag{5.4}$$

Indeed, when $\lambda = n-1$ from (5.3_{n-1}) we have

$$\liminf_{t\to+\infty} t^{-1}\int_0^t \xi\sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{n-1} d_s r_i(s,\xi)d\xi > 0. \tag{5.5}$$

Now assume that condition (5.4) is not fulfilled. Then for any $\varepsilon > 0$ there exists $t_0 \in R_+$, such that

$$\int_{t_0}^{+\infty} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{n-1} d_s r_i(s,\xi)d\xi < \varepsilon.$$

Therefore

$$\limsup_{t\to+\infty} t^{-1}\int_0^t \xi\sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{n-1} d_s r_i(s,\xi)d\xi =$$

$$= \limsup_{t\to+\infty}\bigg\{t^{-1}\int_0^{t_0} \xi\sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{n-1} d_s r_i(s,\xi)d\xi + t^{-1}\int_{t_0}^{t} \xi\sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{n-1} d_s r_i(s,\xi)d\xi\bigg\}$$

$$\le \limsup_{t\to+\infty}\int_{t_0}^{t} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{n-1} d_s r_i(s,\xi)d\xi < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this contradicts condition (5.5), i.e., (5.4) is fulfilled. Therefore according to (5.1)–(5.4) it is obvious that the conditions of Theorem 4.1 are fulfilled, where

$$\begin{gathered}\varphi(x)(t) = \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} x(s)d_s r_i(s,t),\\ \overline{\sigma}(t) = \max\{\sigma_i(t) : i = 1,\dots,m\},\end{gathered} \tag{5.6}$$

which proves the validity of Theorem 5.1. □

Theorem 5.1′. *Let $F \in V(\tau)$, and conditions* (1.2), (5.1), (5.2) *be fulfilled. Besides, let for any $l \in \{1,\dots,n-1\}$ with $l+n$ is odd and $\lambda \in (l-1,l]$*

$$\liminf_{t\to+\infty} t^{-1}\int_0^t \xi^{n-\lambda}\sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^{\lambda} d_s r_i(s,\xi)d\xi > \gamma\prod_{i=01}^{n-1}|\lambda - i|. \tag{5.7_l}$$

Then equation (1.1) *has Property* **A**.

Proof. It suffices to note that the conditions of Theorem 5.1 are fulfilled with φ and $\overline{\sigma}$ defined by (5.6). □

Theorem 5.2. *Let $F \in V(\tau)$, condition* (1.2) *be fulfilled and for any $t_0 \in R_+$*

$$|F(u)(t)| \geq \sum_{i=1}^{m} p_i(t)|u(\delta_i(t))| \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau}, \tag{5.8}$$

$$\begin{gathered} p_i \in L_{\mathrm{loc}}(R_+;R_+), \quad \delta_i \in C(R_+;R_+), \quad \lim_{t\to+\infty} \delta_i(t) = +\infty, \\ \liminf_{t\to+\infty} \frac{t}{\delta_i(t)} > 0 \quad (i = 1,\dots,m). \end{gathered} \tag{5.9}$$

Besides, let for any $l \in \{1,\dots,n-1\}$ with $l+n$ odd and $\lambda \in (l-1,l]$, there exist $\gamma > 1$ such that

$$\liminf_{t\to+\infty} t^{l-1-\lambda} \int_0^t \xi^{n-\lambda} \times \sum_{i=1}^{m} p_i(\xi)\delta_i^{\lambda}(\xi)d\xi > \gamma \prod_{i=0;i\neq l-1}^{n-1} |\lambda - i|. \tag{5.10$_l$}$$

Then equation (1.1) *has Property* **A**.

Proof. According to (5.8) and (5.10$_l$), conditions (5.1) and (5.3$_l$) are fulfilled with

$$\begin{gathered} \tau_i(t) = \delta_i(t) - 1, \quad \sigma_i(t) = \delta_i(t), \\ r_i(s,t) = p_i(t)e(s - \delta_i(t)) \quad (i = 1,\dots,n), \end{gathered} \tag{5.11}$$

where the function $e(t)$ is defined by the equality

$$e(t) = \begin{cases} 0 & \text{for} \quad t \in (-\infty, 0), \\ 1 & \text{for} \quad t \in [0, +\infty). \end{cases} \tag{5.12}$$

The theorem is proved. □

Theorem 5.2′. *Let $F \in V(\tau)$, and conditions* (1.2), (5.8) *and* (5.9) *be fulfilled. Besides, if for any $l \in \{1,\dots,n-1\}$ with $l+n$ odd and $\lambda \in (l-1,l]$ there exists $\gamma > 1$ such that*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t \xi^{n-\lambda} \sum_{i=1}^{m} p_i(\xi)\delta_i^{\lambda}(\xi)d\xi > \gamma \prod_{i=0}^{n-1} |\lambda - i|, \tag{5.13$_l$}$$

then equation (1.1) *has Property* **A**.

Proof. To prove the theorem it suffices to note that from inequality (5.13$_l$) follows inequality (5.10$_l$) (see the proof of Theorem 3.1′). □

Corollary 5.1. *Let $F \in V(\tau)$, conditions* (1.2) *and* (5.8) *be fulfilled, and*

$$\alpha_i \in (0,+\infty), \quad \delta_i(t) \geq \alpha_i t \quad \text{for} \quad t \in R_+ \quad (i = 1,\dots,m). \tag{5.14}$$

Besides, let there exist $\widetilde{p} \in L_{\mathrm{loc}}(R_+;R_+)$ such that

$$\liminf_{t\to+\infty} t^{-1} \int_0^t \xi^n \sum_{i=1}^{m} (p_i(\xi) - \widetilde{p}(\xi))\alpha_i^{\lambda} d\xi \geq 0 \quad \text{for any} \quad l \in \{1,\dots,n-1\}$$
$$\text{where} \quad l+n \quad \text{odd and} \quad \lambda \in (l-1,l]. \tag{5.15$_l$}$$

Then the condition

$$\liminf_{t\to+\infty} t^{-1}\int\limits_0^t \xi^n\widetilde{p}(\xi)d\xi$$
$$> \max\bigg\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\bigg(\sum_{i=1}^m \alpha_i^\lambda\bigg)^{-1} : \lambda\in[0,n-1]\bigg\} \quad (5.16)$$

is sufficient for equation (1.1) *to have Property* **A**.

Proof. To prove the corollary it suffices to note that by (5.14), (5.15_l) and (5.16) conditions (5.13_l) are fulfilled for any $l\in\{1,\dots,n-1\}$ with $l+n$ odd. □

Corollary 5.1′. *Let $F\in V(\tau)$, and conditions* (1.2), (5.8) *and* (5.14) *be fulfilled. Besides, let there exist $\widetilde{p}\in L_{\rm loc}(R_+;R_+)$ such that*

$$\widetilde{p}(t)=p_i(t)+o(t^n) \quad (i=1,\dots,m). \quad (5.17)$$

Then condition (5.16) *is sufficient for equation* (1.1) *to have Property* **A**.

Proof. To prove the corollary, it suffices to note that (5.17) imply the validity of conditions (5.15_l) for any $l\in\{1,\dots,n-1\}$ with $n+l$ odd and $\lambda\in(l-1,l]$. □

Corollary 5.2. *Let $F\in V(\tau)$, condition* (1.2) *be fulfilled and for any $t_0\geq 1$*

$$|F(u)(t)|\geq \frac{c}{t^{n+1}}\int\limits_{\alpha t}^{\overline{\alpha}t} u(s)ds \quad for \quad t\geq t_0, \quad u\in H_{t_0,\tau}, \quad (5.18)$$

where $0<\alpha<\overline{\alpha}$ and

$$c>\max\{-(\lambda+1)\lambda\dots(\lambda-n+1)(\overline{\alpha}^{\,\lambda+1}-\alpha^{\lambda+1})^{-1} : \lambda\in[0,n-1]\}. \quad (5.19)$$

Then equation (1.1) *has Property* **A**.

Proof. By (5.18) and (5.19) the conditions of Theorem 4.1′ are fulfilled, where

$$\varphi(x)(t)=\frac{c}{t^{n+1}}\int\limits_{\alpha t}^{\overline{\alpha}t} x(s)ds, \quad \overline{\sigma}(t)=\overline{\alpha}t.$$

The corollary is proved. □

Corollary 5.3. *Let $c>0$, $0<\alpha<\overline{\alpha}$. Then for the equation*

$$u^{(n)}(t)+\frac{c}{t^{n+1}}\int\limits_{\alpha t}^{\overline{\alpha}t} u(s)ds=0, \quad t\geq 1, \quad (5.20)$$

to have Property **A** *it is necessary and sufficient that* (5.19) *be fulfilled.*

Proof. The necessity is obvious. The sufficiency follows from Corollary 5.2. □

Corollary 5.4. *Let $F \in V(\tau)$, condition* (1.2) *be fulfilled and for any $t_0 \geq 1$*

$$|F(x)(t)| \geq \sum_{i=1}^{m} \frac{c_i}{t^n} |u(\alpha_i t)| \quad \textit{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau}. \tag{5.21}$$

If, moreover, the inequality

$$\max\{-\lambda(\lambda-1)\dots(\lambda-n+1)\bigg(\sum_{i=1}^{m} c_i \alpha_i^{\lambda}\bigg)^{-1} : \lambda \in [0, n-1]\} < 1 \tag{5.22}$$

holds, then equation (1.1) *has Property* **A**.

Proof. According to (5.21) and (5.22), it is obvious that conditions (5.13_l) are fulfilled for any $l \in \{1, \dots, n-1\}$ with $l + n$ odd, $\delta_i(t) = \alpha_i t$, $p_i(t) = \frac{c_i}{t^n}$ $(i = 1, \dots, m)$. Therefore the validity of the corollary follows from Theorem 5.2′. □

Corollary 5.5. *Let c_i, $\alpha_i \in (0, +\infty)$ $(i = 1, \dots, m)$. Then for the equation*

$$u^{(n)}(t) + \sum_{i=1}^{m} \frac{c_i}{t^n} u(\alpha_i t) = 0, \quad t \geq 1, \tag{5.23}$$

to have Property **A**, *it is necessary and sufficient that* (5.22) *be fulfilled.*

Proof. The necessity is obvious. The sufficiency follows from Corollary 5.4. □

6. Some Auxiliary Lemmas for Volterra Type Differential Inequalities

As was mentioned at the beginning of Section 3, we denote by $M(\sigma, \overline{\sigma})$ the set of continuous mappings φ satisfying conditions (3.1) and (3.2). Moreover, we assume below that one of the conditions

$$\overline{\sigma}(t) \leq t \quad \text{for} \quad t \in R_+ \tag{6.1}$$

and

$$\sigma(t) \geq t \quad \text{for} \quad t \in R_+ \tag{6.2}$$

is fulfilled.

Consider a differential inequality of the type

$$u^{(n)}(t)\operatorname{sign} u(t) + p(t)|u(\delta(t))| \leq 0, \tag{6.3}$$

with $n \geq 2$, $p \in L_{\text{loc}}(R_+; R_+)$, $\delta \in C(R_+; R_+)$, $\lim_{t \to +\infty} \delta(t) = +\infty$,

$$\int^{+\infty} \delta_0^{n-1}(t) p(t) dt = +\infty, \tag{6.4}$$

where $\delta_0(t) = \min\{t, \delta(t)\}$.

Lemma 6.1. *Let $\delta(t) \leq t$ for $t \in R_+$. For the differential inequality* (6.3) *to have Property* **A** *it is necessary and sufficient that it have no solution satisfying* (2.1_{n-1}).

Lemma 6.2. *Let $\delta(t) \geq t$ for $t \in R_+$ and n be even (n be odd). Then for the differential inequality* (6.3) *to have Property* **A** *it is necessary and sufficient that it have no solution satisfying* (2.1_1) $((2.1_2)$ *and* $(2.1_{n-1}))$.

The proof of Lemmas 6.1 and 6.2 see in [9, Lemmas 5.2 5.4].

Lemma 6.3. *Let $\varphi \in M(\sigma, \overline{\sigma})$, condition* (6.1) *be fulfilled and for sufficiently large t*

$$\varphi(xy)(t) \leq y(\overline{\sigma}(t))\varphi(x)(t) \quad \textit{when} \quad x(t) \uparrow +\infty \quad \textit{and} \quad y(t) \uparrow +\infty. \tag{6.5}$$

Then for the differential inequality

$$u^{(n)}(t)\mathrm{sign}u(t) + \varphi(|u|)(t) \leq 0 \tag{6.6}$$

to have Property **A** *it is necessary and sufficient that the differential inequality* (6.6) *have no solution of type* (2.1_{n-1}).

Proof. The necessity is obvious. Show the sufficiency. Let the inequality have no solution of type (2.1_{n-1}) and show that it has Property **A**. First of all note that since inequality (6.6) has no solution of type (2.1_{n-1}), according to Lemma 4.1 in [9] we have

$$\int^{+\infty} \varphi(c\Theta_{n-1})(t)dt = +\infty \quad \text{for any} \quad c > 0, \tag{6.7}$$

where $\Theta_{n-1}(t) = t^{n-1}$. On the other hand, if we take into account (3.2), (6.1) and (6.7), we obtain

$$\int^{+\infty} t^{n-i-1}\varphi(c\Theta_i)(t)dt = +\infty \quad \text{for any} \quad c > 0, \tag{6.8_i}$$

where $\Theta_i(t) = t^i$ $(i = 0, \dots, n-1)$. □

Now suppose that the differential inequality (6.6) has no Property **A** and $u : [t_0, +\infty) \to R$ is a proper nonoscillatory solution of the differential inequality (6.6). Then if we take into account Lemma 2.1 and (6.6), without loss of generality we can assume that the function u satisfies conditions (2.1_l), where $l \in \{0, \dots, n-3\}$, $l+n$ is odd. Show that if n is odd and $l = 0$, then conditions (1.3) are fulfilled. Indeed, otherwise there exist $c_0 > 0$ and $t_* \geq t_0$ such that $u(t) \geq c_0$ for $t \geq t_*$. Therefore, if we take into account (2.1_0) and (3.1), from inequality (6.6) we obtain

$$\int_{t^*}^{+\infty} t^{n-1}\varphi(c_0)(t)dt \leq (n-1)! \sum_{i=1}^{n} |u^{(i-1)}(t^*)|,$$

where $t^* > t_*$ is a sufficiently large number. The latter inequality contradicts condition (5.8_0). The obtained contradiction proves that condition (1.3) is fulfilled. Therefore, since by the assumption the differential inequality has no Property **A**, inequality (6.6) has a solution of type (2.1_l), where $l \in \{1, \dots, n-3\}$, $l+n$ is odd. By (2.1_l), there exists $c_0 > 0$ such that $u(t) \geq c_0 t^{l-1}$ for $t \geq t_1$, where t_1 is a sufficiently large number. Therefore taking into account (3.1), (3.2) and (6.1), from (6.6) for sufficiently large t we have

$$|u^{(n)}(t)| \geq \varphi(|u|)(t) \geq \varphi(c_0\Theta_{l-1})(t) \geq (\overline{\sigma}(t))^{-1}\varphi(c_0\Theta_l)(t) \geq \frac{c_1}{2}t^{-1}\varphi(c_0\Theta_l)(t),$$

where $\Theta_i(t) = t^i$ $(i = l-1, l)$, $c_1 = \liminf_{t\to+\infty}\frac{t}{\overline{\sigma}(t)}$. From the latter inequality by (6.8_l) it is obvious that condition (2.6) is fulfilled. Hence the conditions of Lemma 2.2 are fulfilled. Therefore, by the first condition of (3.2) and (2.9_0), the function u for sufficiently large t satisfies the differential inequality

$$u^{(n)}(t) + \varphi(\Theta_l)(t)\frac{u(\overline{\sigma}(t))}{(\overline{\sigma}(t))^l} \leq 0, \tag{6.9}$$

where $\Theta_l(t) = t^l$. On the other hand, according to (6.1), (6.5) and (6.7) condition (6.4) is fulfilled, where $\delta_0(t) = \overline{\sigma}(t)$ and $p(t) = \varphi(\Theta_l)(t)\ (\overline{\sigma}(t))^{-l}$. Therefore by Lemma 6.1 inequality (6.9) has a solution u_1 of type (2.1_{n-1}). Since $l \leq n-3$, it is obvious that $\frac{u_1(t)}{t^l} \uparrow +\infty$ as $t \uparrow +\infty$. Therefore (6.5) gives that for sufficiently large t u_1 is a solution of type (2.1_{n-1}) of the differential inequality (6.6). This contradicts the condition of the lemma. The obtained contradiction proves the validity of the lemma.

Lemma 6.4. *Let $\varphi \in M(\sigma, \overline{\sigma})$, condition* (6.2) *be fulfilled,*

$$\liminf_{t\to+\infty}\frac{t}{\sigma(t)} > 0, \tag{6.10}$$

$$\varphi(xy)(t) \geq x(\sigma(t))\varphi(y)(t) \quad \text{for } x(t)\uparrow+\infty \text{ and } y(t)\uparrow+\infty \text{ for } t\uparrow+\infty, \tag{6.11}$$

and

$$\int^{+\infty} t^{n-1}\varphi(c)(t)dt = +\infty \quad \text{for any} \quad c > 0. \tag{6.12}$$

Then for the differential inequality (6.6) *to have Property* **A**, *it is necessary and sufficient in the case of even n (odd n) that the differential inequality* (6.6) *have no solutions of type* (2.1_1) $((2.1_2)$ *and* $(2.1_{n-1}))$.

Proof. The necessity is obvious. Show the sufficiency. Consider the case where n is even and show that the differential inequality (6.6) has Property **A**. First of all note that since inequality (6.6) has no solution of type (2.1_1), due to Lemma 4.1 in [9] we have

$$\int^{+\infty} t^{n-2}\varphi(c\Theta_1)(t)dt = +\infty \quad \text{for any} \quad c > 0,$$

where $\Theta_1(t) = t$. Therefore, if we take into account (5.2) and (5.11), we obtain

$$\int^{+\infty} t^{n-i-1}\varphi(c\Theta_i)(t)dt = +\infty \quad \text{for any} \quad c > 0 \quad (i = 1, \dots, n-1), \tag{6.13_i}$$

where $\Theta_i(t) = t^i$.

Now suppose that the differential inequality (6.6) has no Property **A** and $u : [t_0, +\infty) \to R$ is a proper nonoscillatory solution of the differential inequality (6.6). Then using to (6.6) and Lemma 2.1, without loss of generality we can assume that conditions (2.1_l) are fulfilled, where $l \in \{3, \dots, n-1\}$ and $l+n$ is odd. According to (2.1_l), (6.13_{l-1}), (6.2), (3.1) and (6.6) it is obvious that condition (2.6) is fulfilled. Therefore according to the second condition of (2.9_0) and (6.11) from (6.6) it follows that for sufficiently large t u is a proper solution of type (2.1_l) of the differential inequality

$$u^{(n)}(t) + \varphi(\Theta_{l-1})(t)\frac{u(\sigma(t))}{(\sigma(t))^{l-1}} \le 0. \tag{6.14}$$

On the other hand, by (6.10) and (6.13_{l-1}) condition (6.4) is fulfilled with $\delta_0(t) = t$, $p(t) = (\sigma(t))^{1-l}\varphi(\Theta_{l-1})(t)$. Therefore, by Lemma 6.2 the differential inequality (6.14) has a proper solution u_1 of type (2.1_1). Since for the function u_1 the conditions of Lemma 2.2 are fulfilled, we have $\frac{u_1(t)}{t^{l-1}} \downarrow 0$ for $t \uparrow +\infty$. Therefore, due to (6.11), for sufficiently large t we have

$$\varphi(\Theta_{l-1})(t) \ge \frac{(\sigma(t))^{l-1}}{u_1(\sigma(t))}\varphi(u_1)(t).$$

If we take into account the latter inequality, from (6.14) we obtain that for sufficiently large t the function u_1 is a proper solution of type (2.1_1) of the differential inequality (6.6), which contradicts the condition of the lemma. In case of even n the obtained contradiction proves that the differential inequality (6.6) has Property **A**. As for the case of odd n, by a reasoning analogous to the above we will show that the differential inequality (6.6) has no proper solution of type (2.1_l), where $l \in \{1, \dots, n-1\}$ and $l+n$ odd. On the other hand, if the differential inequality (6.6) has a proper nonoscillatory solution of type (2.1_0), then using (6.12) we easily show that the function u satisfies conditions (1.3). Hence, in the case of odd n too inequality (6.6) has Property **A**, which proves the validity of the lemma. □

Remark 6.1. It is obvious that if the operator φ satisfies the conditions of any of the theorems given in Section 3, then the differential inequality (6.6) has no solution of type (2.1_l), where $l \in \{1, \dots, n-1\}$ and $l+n$ is odd.

7. Functional Differential Equations with a Volterra Type Minorant

In the theorems given in Sections 4 and 5, in the case of existence of a Volterra type minorant $[\frac{n}{2}]$ the conditions (which are essential in the theorem) can be replaced by one or two conditions.

Theorem 7.1. *Let $F \in V(\tau)$, and conditions* (1.2), (3.3), (6.1), (6.5) *be fulfilled with $\varphi \in M(\sigma, \overline{\sigma})$. Then condition* ($3.6_{n-1}$) *is sufficient for equation* (1.1) *to have Property* **A**.

Proof. Let $u : [t_0, +\infty) \to R$ be a proper nonoscillatory solution of equation (1.1). Then according to Lemma 1.1 and (1.2), without loss of generality we can assume that u satisfies conditions (2.1_l) where $l + n$ is odd. According to (3.3) the function u for sufficiently large t satisfies the differential inequality (6.6). Let $l \in \{1, \dots, n-1\}$, where $l+n$ is odd. Since the differential inequality (6.6) has no Property **A**, according to Lemma 6.3 it has a proper solution of type (2.1_{n-1}). On the other hand, if we take into account Remark 6.1, since the operator φ satisfies the conditions of Theorem 3.1, by virtue of (3.6_{n-1}) the differential inequality (6.6) has no solution of type (2.1_{n-1}). The obtained contradiction proves that $l \notin \{1, \dots, n-1\}$. Let n be odd and $l = 0$. Taking into account (6.8_0), (2.1_0) and (3.3), since $\varphi \in M(\sigma, \overline{\sigma})$, from (1.1) we can easily show that conditions (1.3) are fulfilled. Hence equation (1.1) has Property **A**. The theorem is proved. □

Analogously, we can prove the following statement, taing into account Theorem 3.1'.

Theorem 7.1′. *Let* $F \in V(\tau)$*, and conditions* (1.2)*,* (3.3)*,* (6.1) *and* (6.5) *be fulfilled with* $\varphi \in M(\sigma, \overline{\sigma})$*. Then condition* (3.21_{n-1}) *is sufficient for equation* (1.1) *to have Property* **A**.

If we use Lemma 6.4 instead of Lemma 6.3, Theorems 7.2 and 7.2′ below can be proved analogously to Theorems 7.1 and 7.1′.

Theorem 7.2. *Let* $F \in V(\tau)$*, and conditions* (1.2)*,* (3.3)*,* (3.4)*,* (6.2)*,* (6.11) *and* (6.12) *be fulfilled, where* $\varphi \in M(\sigma, \overline{\sigma})$*. Then for equation* (1.1) *to have Property* **A** *it is sufficient in the case of an even* n *that condition* (3.6_1) *be fulfilled and in the case of an odd* n *conditions* (3.6_2) *and* (3.6_{n-1}) *be fulfilled.*

Theorem 7.2′. *Let* $F \in V(\tau)$*, and conditions* (1.2)*,* (3.3)*,* (3.4)*,* (6.2)*,* (6.11) *and* (6.12) *be fulfilled, where* $\varphi \in M(\sigma, \overline{\sigma})$*. Then for equation* (1.1) *to have Property* **A***, it is sufficient in the case of an even* n *that condition* (3.21_1) *be fulfilled and in the case of an odd* n *conditions* (3.21_2) *and* (3.21_{n-1}) *be fulfilled.*

Remark 7.1. In the case of an odd n in Theorem 7.2 (in Theorem 7.2′) if at least one of conditions (3.6_2) and (3.6_{n-1}) (conditions (3.21_2) and (3.21_{n-1})) is not fulfilled, then the theorem is not valid in general.

8. Functional Differential Equations with a Linear Volterra Minorant

Theorem 8.1. *Let* $F \in V(\tau)$*, conditions* (1.2) *and* (5.1) *be fulfilled and*

$$\tau_i(t) \le \sigma_i(t) \le t \quad for \quad t \in R_+ \quad (i = 1, \dots, m). \tag{8.1}$$

Then condition (5.3_{n-1}) *is sufficient for equation* (1.1) *to have Property* **A**.

Proof. To prove the Theorem it is sufficient to note that due to (1.2), (5.1), (5.3_{n-1}) and (8.1) the conditions of Theorem 7.1 are fulfilled where the operator φ is given by equality (5.6),

$$\sigma(t) = \min\{\tau_i : i = 1, \dots, m\}, \quad \overline{\sigma}(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}. \tag{8.2}$$

The theorem is proved. □

Theorem 8.1′. *Let* $F \in V(\tau)$, *and conditions* (1.2), (5.1) *and* (8.1) *be fulfilled. Then condition* (5.7_{n-1}) *is sufficient for equation* (1.1) *to have Property* **A**.

Proof. To prove the theorem it is sufficient to note that the conditions of Theorem 7.1′ are fulfilled, where the operator φ and the functions σ and $\overline{\sigma}$ are given respectively by equalities (5.6) and (8.2). □

Theorem 8.2. *Let* $F \in V(\tau)$, *conditions* (1.2) *and* (5.8) *be fulfilled and*

$$\delta_i(t) \le t \quad \text{for} \quad t \in R_+ \quad (i = 1, \dots, m). \tag{8.3}$$

Then condition (5.10_{n-1}) *is sufficient for equation* (1.1) *to have Property* **A**.

Proof. By virtue of the conditions of Theorem 8.2 it is obvious that the conditions of Theorem 7.1 are fulfilled, where the operator φ is given by (5.6), and the functions $\tau_i(t)$, $\sigma_i(t)$ and $r_i(s,t)$ $(i = 1, \dots, m)$ are given by equalities (5.11) and (5.12). □

Using Theorem 7.1′, we can analogously prove the following

Theorem 8.2′. *Let* $F \in V(\tau)$, *and conditions* (1.2), (5.8) *and* (8.3) *be fulfilled. Then condition* (5.13_{n-1}) *is sufficient for equation* (1.1) *to have Property* **A**.

Corollary 8.1. *Let* $F \in V(\tau)$, *conditions* (1.2) *and* (5.8) *be fulfilled and*

$$\alpha_i t \le \delta_i(t) \le t \quad \text{for} \quad t \in R_+ \quad (i = 1, \dots, m), \tag{8.4}$$

where $\alpha_i \in (0, 1]$. *If, moreover, there exists* $\widetilde{p} \in L_{\rm loc}(R_+; R_+)$ *such that*

$$\liminf_{t \to +\infty} t^{-1} \int_0^t \xi^n \sum_{i=1}^m \alpha_i^\lambda (p_i(\xi) - \widetilde{p}(\xi)) d\xi \ge 0 \quad \text{for any} \quad \lambda \in (n-2, n-1], \tag{8.5}$$

then for equation (1.1) *to have Property* **A** *it is sufficient that*

$$\liminf_{t \to +\infty} t^{-1} \int_0^t \xi^n \widetilde{p}(\xi) d\xi$$
$$> \max\bigg\{ -\lambda(\lambda-1)\dots(\lambda-n+1)\Big(\sum_{i=1}^m \alpha_i^\lambda\Big)^{-1} : \lambda \in [n-2, n-1]\bigg\}. \tag{8.6}$$

Proof. By Theorem 8.2′, to prove the corollary it is sufficient to note that from conditions (8.4)–(8.6) imply condition (5.13_{n-1}). □

Corollary 8.1′. *Let* $F \in V(\tau)$, *and conditions* (1.2), (5.8) *and* (8.4) *be fulfilled. If, moreover, there exists* $\widetilde{p} \in L_{\rm loc}(R_+; R_+)$ *such that*

$$p_i(t) = \widetilde{p}(t) + o(t^n) \quad (i = 1, \dots, m), \tag{8.7}$$

then condition (8.6) *is sufficient for equation* (1.1) *to have Property* **A**.

Proof. The validity of Corollary 8.1′ follows from Corollary 8.1 since (8.5) follows from (8.7). □

Corollary 8.2. *Let* $p \in L_{\mathrm{loc}}(R_+; R_+)$ *and*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t \xi^n p(\xi)d\xi$$
$$> \max\{-\lambda(\lambda-1)\dots(\lambda-1+n+1) : \lambda \in [n-2, n-1]\}.$$

Then equation (1.4) *has Property* **A**.

Remark 8.1. Corollary 8.2 is an integral generalization of a well-known theorem by V. Kondrat'ev.

Corollary 8.3. *Let* $F \in V(\tau)$ *and conditions* (1.2) *and* (5.8) *be fulfilled, where*

$$m = 1 \quad \delta_1(t) \le t \quad \textit{for} \quad t \in R_+, \quad \liminf_{t\to+\infty} \frac{\delta_1(t)}{t^\alpha} > 0, \tag{8.8}$$

with $\alpha \in (0,1)$. *Then for equation* (1.1) *to have Property* **A** *it is sufficient that*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t \xi^{1+\alpha(n-1)} p_1(\xi)\, d\xi > 0. \tag{8.9}$$

Proof. To prove the corollary to Theorem 8.1′, it is sufficient to show that condition (5.13_{n-1}) is fulfilled with $m = 1$. In the case $\lambda = n-1$, it is obvious by (8.9) that condition (5.13_{n-1}) is fulfilled. Let $\lambda \in (n-2, n-1)$. According to the second condition of (8.8) there exist $t_0 > 0$ and $c_0 > 0$ such that $\delta_1(t) \ge c_0 t^\alpha$ when $t \ge t_0$. Therefore, since $(1-\alpha)(n-1-\lambda) > 0$ for any $T > t_0$, we have

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^{n-\lambda}\delta_1^\lambda(s)p_1(s)ds = \liminf_{t\to+\infty} t^{-1} \int_T^t s^{n-\lambda}\delta_1^\lambda(s)p_1(s)ds$$
$$\ge c_0^\lambda \liminf t^{-1} \int_T^t s^{n-\lambda+\alpha\lambda} p_1(s)ds$$
$$= c_0^\lambda \liminf_{t\to+\infty} t^{-1} \int_T^t s^{(1-\alpha)(n-1-\lambda)} s^{1+\alpha(n-1)} p_1(s)ds$$
$$\ge c_0^\lambda T^{(1-\alpha)(n-1-\lambda)} \liminf_{t\to+\infty} t^{-1} \int_T^t s^{1+\alpha(n-1)} p_1(s)ds.$$

Hence, taking into account (8.9) and the fact that T is arbitrary, we have

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^{n-\lambda}\delta_1^\lambda(s)p_1(s)ds = +\infty \quad \text{for} \quad \lambda \in (n-2, n-1).$$

Thus for any $\lambda \in (n-2, n-1]$ (5.13_{n-1}) is fulfilled with $m=1$, which proves the validity of the corollary. □

Corollary 8.4. *Let $F \in V(\tau)$ and conditions* (1.2) *and* (5.21) *be fulfilled, where $c_i \in (0,+\infty)$, $\alpha_i \in (0,1]$ $(i=1,\dots,m)$. Then for equation* (1.1) *to have Property* **A** *it is sufficient that*

$$\max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^{m} c_i\alpha_i^{\lambda}\Big)^{-1} : \lambda \in [n-2, n-1]\Big\} < 1. \quad (8.10)$$

The validity of the corollary follows from Theorem 8.1′.

Corollary 8.5. *Let $c_i \in (0,+\infty)$, $\alpha_i \in (0,1]$ $(i=1,\dots,m)$. Then condition* (8.10) *is necessary and sufficient for equation* (5.23) *to have Property* **A**.

The necessity is obvious. The sufficiency follows from Corollary 8.4.

Corollary 8.6. *If $F \in V(\tau)$, conditions* (1.2) *and* (5.18) *are fulfilled with $0 < \alpha < \overline{\alpha} \le 1$ and*

$$c > \max\big\{-(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\overline{\alpha}^{\lambda+1} - \alpha^{\lambda+1}\big)^{-1} : \lambda \in [n-2, n-1]\big\}, \quad (8.11)$$

then equation (1.1) *has Property* **A**.

Proof. According to (8.11), the validity of the corollary follows from Theorem 7.1′, where the operator φ and the functions σ and $\overline{\sigma}$ are given by the equalities

$$\varphi(x)(t) = \frac{c}{t^{n+1}} \int_{\alpha t}^{\overline{\alpha} t} x(s)\,ds, \quad \sigma(t) = \alpha t, \quad \overline{\sigma}(t) = \overline{\alpha} t.$$

The corollary is proved. □

Corollary 8.7. *Let $c > 0$, $0 < \alpha < \overline{\alpha} \le 1$. Then for equation* (5.20) *to have Property* **A** *it is necessary and sufficient that* (8.11) *be fulfilled.*

The necessity is obvious. The sufficiency follows from Corollary 8.6.

Theorem 8.3. *Let $F \in V(\tau)$, conditions* (1.2), (5.1) *and* (5.2) *be fulfilled, and*

$$t \le \tau_i(t) \le \sigma_i(t) \quad \text{for} \quad t \in R_+ \quad (i=1,\dots,m). \quad (8.12)$$

Then for equation (1.1) *to have Property* **A** *it is sufficient that condition* (5.3_1) *be fulfilled for even n and conditions* (5.3_2) *and* (5.3_{n-1}) *be fulfilled for odd n.*

Proof. The validity of the theorem follows from Theorem 7.2, where the operator φ is given by equality (5.6) and the functions σ and $\overline{\sigma}$ are given by equalities (8.2). □

Analogously, Theorem 7.2′ leads to

Theorem 8.3′. *Let $F \in V(\tau)$, conditions* (1.2), (5.1), (5.2) *and* (8.12) *be fulfilled. Then for equation* (1.1) *to have Property* **A** *it is sufficient that* *condition* (5.7_1) *be fulfilled for even* n *and conditions* (5.7_2) *and* (5.7_{n-1}) *be fulfilled for odd* n.

Theorems 8.3 and 8.3′, Corollaries 8.8–8.12 below are proved analogously to Corollaries 8.1–8.7.

Corollary 8.8. *Let* $F \in V(\tau)$, *and conditions* (1.2) *and* (5.8) *be fulfilled, where*

$$\delta_i(t) \geq \alpha_i t \quad \text{for} \quad t \in R_+ \quad \alpha_i \in [1, +\infty) \quad (i = 1, \dots, m). \tag{8.13}$$

Besides, let there exist $\widetilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ *such that in the case of even* n *(in the case of odd* n*) condition* (5.15_1) *(*(5.15_2) *and* (5.15_{n-1})*) is fulfilled.*

Then for equation (1.1) *to have Property* **A** *it is sufficient that in the case of even* n *(in the case of odd* n*)*

$$\liminf_{t \to +\infty} t^{-1} \int_0^t s^n \widetilde{p}(s) ds > \max\Big\{ -\lambda(\lambda - 1) \cdots (\lambda - n + 1) \Big(\sum_{i=1}^m \alpha_i^\lambda \Big)^{-1} : \lambda \in [0, 1] \Big\} \tag{8.14}$$

$$\Big(\liminf_{t \to +\infty} t^{-1} \int_0^t s^n \widetilde{p}(s) ds > \max\Big\{ -\lambda(\lambda - 1) \cdots (\lambda - n + 1) \times \Big(\sum_{i=1}^m \alpha_i^\lambda \Big)^{-1} : \lambda \in [1, 2] \cup [n - 2, n - 1] \Big\} \Big). \tag{8.15}$$

Corollary 8.8′. *Let* $F \in V(\tau)$ *and conditions* (1.2), (5.8) *and* (8.13) *be fulfilled. Besides, let there exist* $\widetilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ *such that conditions* (5.17) *are fulfilled. Then for equation* (1.1) *to have Property* **A** *it is sufficient that in the case of even* n *(in the case of odd* n*) condition* (8.14) *((8.15)) be fulfilled.*

Corollary 8.9. *If* $F \in V(\tau)$, *conditions* (1.2) *and* (5.18) *are fulfilled with* $1 \leq \alpha < \overline{\alpha}$ *and* n *even (*n *odd)*

$$c > \max\big\{ -(\lambda + 1)\lambda(\lambda - 1) \cdots (\lambda - n + 1) \times \big(\overline{\alpha}^{\lambda+1} - \alpha^{\lambda+1}\big)^{-1} : \lambda \in [0, 1] \big\}, \tag{8.16}$$

$$\big(c > \max\big\{ -(\lambda + 1)\lambda(\lambda - 1) \cdots (\lambda - n + 1) \times \big(\overline{\alpha}^{\lambda+1} - \alpha^{\lambda+1}\big)^{-1} : \lambda \in [1, 2] \cup [n - 2, n - 1] \big\}\big), \tag{8.17}$$

then equation (1.1) *has Property* **A**.

Corollary 8.10. *Let* $c > 0$, $1 \leq \alpha < \overline{\alpha}$. *Then for equation* (5.20) *to have Property* **A** *it is necessary and sufficient that* (8.16) *((8.17)) be fulfilled when* n *is even (when* n *is odd).*

Corollary 8.11. *Let* $F \in V(\tau)$ *and conditions* (1.2) *and* (5.21) *be fulfilled, where* $\alpha_i \in [1, +\infty)$, $c_i \in (0, +\infty)$ $(i = 1, \dots, m)$. *Then for equation* (1.1) *to* *have Property* **A** *it is sufficient that the condition*

$$\max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^{m}c_i\alpha_i^{\lambda}\Big)^{-1}:\lambda\in[0,1]\Big\}<1 \tag{8.18}$$

$$\Big(\max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1) \times\Big(\sum_{i=1}^{m}c_i\alpha_i^{\lambda}\Big)^{-1}:\lambda\in[1,2]\cup[n-2,n-1]\Big\}<1\Big) \tag{8.19}$$

be fulfilled for even n (for odd n) .

Corollary 8.12. *Let $c_i\in(0,+\infty)$, $\alpha_i\in[1,+\infty)$ $(i=1,\dots,m)$. Then for equation* (5.23) *to have Property* **A** *it is necessary and sufficient that condition* (8.18) ((8.19)) *be fulfilled for even n (for odd n).*

References

1. V. A. Kondrat'ev, On oscillation of solutions of the equation $y^{(n)}+p(x)y=0$. (Russian) *Trudy Moskov. Mat. Obshch.* **10**(1961), 419–436.
2. A. Kneser, Untersuchungen über die reelen Nullstellen der Integrale linear Differentialgleichungen. *Math. Ann.* **42**(1893), 409–435.
3. I. T. Kiguradze and T. A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. *Kluwer Academic Publishers, Dordrecht, Boston, London*, 1993.
4. R. G. Koplatadze and T. A. Chanturia, Oscillation properties of differential equations with deviating argument. (Russian) *Tbilisi State University Press, Tbilisi,* 1977.
5. R. G. Koplatadze, Differential equations with deviating argument that have the properties A and B. (Russian) *Differentsial'nye Uravneniya* **25**(1989), No. 11, 1897–1909; English transl.: *Differential Equations* **25**(1989), No. 11, 1332–1342.
6. R. Koplatadze, G. Kvinikadze, and I. P. Stavroulakis, Properties **A** and **B** of n-th order linear differential equations with devieting argument. *Georgian Math. J.* **6**(1999), 287–298.
7. R. G. Koplatadze, On asymptotic behaviour of solutions of functional-differential equations. *Equadiff* 8 (*Bratislava,* 1993). *Tatra Mt. Math. Publ.* **4**(1994), 143–146.
8. R. G. Koplatadze, Oscillation properties of solutions of functional-differential equations. (Russian) *Dokl. Akad. Nauk* **340**(1995), No. 4, 473–475.
9. R. Koplatadze, On oscillatory properties of solutions of functional differential equations. *Mem. Differential Equations Math. Phys.* **3**(1994), 1–179.

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