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BADORA'S EQUATION ON NON-ABELIAN LOCALLY COMPACT GROUPS

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Abstract. This paper is mainly concerned with the following functional equation

$$\int_G \left\{ \int_K f(xtk \cdot y)dk \right\} d\mu(t) = f(x)f(y), \quad x, y \in G,$$

where G is a locally compact group, K a compact subgroup of its morphisms, and μ is a generalized Gelfand measure. It is shown that continuous and bounded solutions of this equation can be expressed in terms of μ-spherical functions. This extends the previous results obtained by Badora (1992) on locally compact abelian groups. In the case where G is a connected Lie group, we characterize solutions of the equation in question as joint eigenfunctions of certain operators associated to the left invariant differential operators.

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1. Introduction

Let G be a locally compact group, K a compact subgroup of morphisms of G. The action of $k \in K$ on $x \in G$ is denoted by $k \cdot x$, and the normalized Haar measure on K by dk. Furthermore for a complex bounded measure μ on G, i.e., $\mu \in M(G)$, the topological dual of $C_0(G)$, the Banach space of continuous functions vanishing at infinity (cf. [6], 13.1.2 and 13.20.1), $\check{\mu}$ (resp. μ_x) denotes the measure defined by $\langle \check{\mu}, f \rangle = \int_G f(t^{-1})d\mu(t)$ (resp. $\langle \mu_x, f \rangle = \int_G f(tx^{-1})d\mu(t)$) for all continuous and bounded functions f on G.

In the paper [4] Badora considered the following functional equation

$$\int_G \left\{ \int_K f(xtk \cdot y)dk \right\} d\check{\mu}(t) = f(x)f(y), \quad x, y \in G, \tag{1.1}$$

when G is abelian and μ is K-invariant, i.e., $\int_G f(k \cdot t)d\mu(t) = \int_K f(t)d\mu(t)$ for all continuous functions with compact support on G ($f \in \mathcal{K}(G)$) and for all $k \in K$. The essentially bounded non-zero solutions of equation (1.1) are completely defined as

$$f(x) = \int_K (\chi * \mu_{k \cdot x})(e)\, dk, \quad x \in G, \tag{1.2}$$

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where χ is a character of G and e is the identity element of the abelian group G (see [4]).

In the paper we are going to study the functional equation

$$\int_G \left\{ \int_K f(xtk \cdot y)dk \right\} d\mu(t) = f(x)f(y), \quad x, y \in G. \tag{1.3}$$

This category contains not only equations of the from

$$\int_K f(xk \cdot y)\, dk = f(x)f(y), \quad x, y \in G \tag{1.4}$$

(see [4], [5], [15], [16] and [17]), but also the generalized d'Alembert's functional equation

$$\int_G f(xty)\, d\mu(t) + \int_G f(xty^{-1})\, d\mu(t) = 2f(x)f(y) \quad x, y \in G, \tag{1.5}$$

which was studied in [9].

Concerning continuous solutions of (1.4), one of the main results is due to Shin'ya [14]: any non-zero continuous solution of equation (1.4) has the form

$$f(x) = \int_K \chi(k \cdot x)dk \quad \text{for all } x \in G,$$

where $\chi :\ G \longrightarrow \mathbb{C}\backslash\{0\}$ is a continuous homomorphism of the abelian group G (cf. [14], Corollary 3.12).

In Section 2 (Theorem 2.1), we give necessary and sufficient conditions for a (measurable) essentially bounded function f to satisfy equation (1.3). One of these conditions is

$$M_K(\check{\mu} * h * f) = \langle h, \check{f} \rangle f \tag{1.6}$$

for all $h \in L_1(G, dx)$, where $M_K(h)(x) = \int_K h(k \cdot x)dk$ for all $h \in L^1_{\text{loc}}(G, dx)$ and for all $x \in G$. This explains why we restrict ourselves to solutions $f \in C_b(G)$.

In Theorem 2.2, we prove that the map

$$f \longmapsto \langle f, \omega \rangle = \int_G f(x)\omega(x)\, dx$$

is a continuous character of the Banach algebra $\mu * M_K(L_1(G, dx)) * \mu = M_K(\mu * L_1(G, dx) * \mu)$ if and only if ω is a non zero solution of the functional equation (1.3).

In Section 3, μ is a generalized Gelfand measure which is K-invariant. We extend the above-mentioned results due to Badora [4]. More precisely, under an additionally condition that every closed ideal of the commutative Banach algebra $\mu * L_1(G, dx) * \mu$ is contained in some maximal regular ideal, we find explicit formulas for solutions expressing them in terms of μ-spherical functions (Theorem 3.1).

In the final part of our paper when G is a connected Lie group and μ is an idempotent measure which is K-invariant, we characterize solutions of (1.3) as joint eigenfunctions of certain operators associated to the left invariant differential operators (Theorem 4.1).

Our notation is described in the following set-up and we will stick to it in the rest of the paper.

General set-up and notation. Let G be a locally compact separable Hausdorff group, e its identity element, $C(G)$ (resp. $C_b(G)$) the complex algebra of all continuous (resp. continuous and bounded) complex-valued functions on G.

Let K be a subgroup of the group $\mathrm{Mor}(G)$ of all mappings k of G onto itself that are either automorphisms and homeomorphisms ($k \in K^+$) or antiautomorphisms and homeomorphisms ($k \in K^-$).

We assume that K has a topology making it a compact Hausdorff group with the property that the canonical map $K \times G \longrightarrow G$ sending each pair (k, x) onto $k \cdot x$ is continuous.

For any $k \in K$, and for any function f on G, we put $(k \cdot f)(x) = f(k^{-1} \cdot x)$ and say that f is K-invariant if $k \cdot f = f$ for all $k \in K$.

The algebra of all regular and complex bounded measures on G is denoted by $M(G)$. We recall that the convolution of $M(G)$ is given by

$$\langle \mu * \nu, f \rangle = \int_G \int_G f(ts)\, d\mu(t)\, d\nu(s), \quad f \in C_b(G).$$

For any $\mu \in M(G)$ and any $k \in K$, we put $\langle k \cdot \mu, f \rangle = \langle \mu, k \cdot f \rangle$, $f \in C_b(G)$, and say that μ is K-invariant if $k \cdot \mu = \mu$ for all $k \in K$.

A function $f \in C_b(G)$ is μ-biinvariant if $f_\mu = f$, where f_μ is the continuous and bounded function defined by $f_\mu(x) = \int_G \int_G f(sxt) d\mu(t) d\mu(s)$ for all $x \in G$.

Note that if $\mu * \mu = \mu$, then f is μ-biinvariant if and only if $\int_G f(tx) d\mu(t) = \int_G f(xt) d\mu(t) = f(x)$ for all $x \in G$.

Definition 1.1. A measure $\mu \in M(G)$ is a Gelfand measure if $\check{\overline{\mu}} = \mu * \mu = \mu$ and the Banach algebra $\mu * M(G) * \mu$ is commutative under the convolution. μ is a generalized Gelfand measure if $\mu * \mu = \mu$ and the Banach algebra $\mu * M(G) * \mu$ is commutative.

Any non-zero solution $\phi \in C_b(G)$ of the functional equation

$$\int_G \phi(xty)\, d\mu(t) = \phi(x)\phi(y) \quad \text{for all} \quad x, y \in G, \tag{1.7}$$

is a μ-spherical function.

μ-spherical functions have been introduced and studied by Akkouchi and Bakali (see [1] and [2]).

When H is a compact subgroup of G and dh is the normalized Haar measure of H, then dh is a generalized Gelfand measure of G if and only if (G, H) is a Gelfand pair (see [7]).

Finally, if $\mu \in M(G)$ is a generalized Gelfand measure, then, according to Definition 1.1,

$$L_1^\mu(G) = \mu * L_1(G, dx) * \mu = \left\{ f \in L_1(G, dx) \mid f = f * \mu = \mu * f = \mu * f * \mu = f^\mu \right\}$$

is a commutative Banach algebra, where

$$f^\mu(x) = \mu * f * \mu(x) := \int_G \int_G f(t^{-1} x s^{-1}) \Delta^{-1}(s)\, d\mu(t)\, d\mu(s),$$

Δ being the modular function of G.

Furthermore, note that if G is unimodular (i.e., $\Delta = 1$), $\mu \in M(G)$ is K-invariant and $f \in L_1(G, dx)$, then we have

$$\begin{aligned}(M_K(f))^\mu(x) &= \int_G \int_G M_K(f)(t^{-1} x s^{-1})\, d\mu(t)\, d\mu(s) \\ &= \int_{K^+} \int_G \int_G f(k \cdot t^{-1} k \cdot x k \cdot s^{-1})\, dk\, d\mu(t)\, d\mu(s) \\ &\quad + \int_{K^-} \int_G \int_G f(k \cdot s^{-1} k \cdot x k \cdot t^{-1})\, dk\, d\mu(t)\, d\mu(s).\end{aligned}$$

Since $\check{\mu}$ is also K-invariant, we have

$$\begin{aligned}&\int_{K^+} \int_G \int_G f(k \cdot t^{-1} k \cdot x k \cdot s^{-1})\, dk\, d\mu(t)\, d\mu(s) \\ &\quad = \int_{K^+} \int_G \int_G f(t^{-1} k \cdot x s^{-1})\, dk\, d\mu(t)\, d\mu(s)\end{aligned}$$

and

$$\begin{aligned}&\int_{K^-} \int_G \int_G f(k \cdot s^{-1} k \cdot x k \cdot t^{-1})\, dk\, d\mu(t)\, d\mu(s) \\ &\quad = \int_{K^-} \int_G \int_G f(t^{-1} k \cdot x s^{-1})\, dk\, d\mu(t)\, d\mu(s),\end{aligned}$$

which implies that

$$(M_K(f))^\mu(x) = \int_K \int_G \int_G f(s^{-1} k \cdot x t^{-1})\, dk\, d\mu(t)\, d\mu(s) = M_K(f^\mu)(x) \quad \text{for all } x \in G.$$

2. General Properties

We fix a measure μ in $M(G)$. The following result gives necessary and sufficient conditions for a function to be a solution of equation (1.3) and explains why we restrict ourselves to continuous and bounded solutions.

Theorem 2.1. *Let $\mu \in M(G)$. Let f be a measurable and essentially bounded function on G. Then the following conditions are equivalent*

$$\int_G \left\{ \int_K f(xtk\cdot y)dk \right\} d\mu(t) = f(x)f(y) \text{ for almost all } x,y \in G, \tag{2.1}$$

$$M_K(\check{\mu} * h * f) = \langle h, \check{f}\rangle f \text{ for all } h \in L_1(G,dx), \tag{2.2}$$

$$M_K(\check{\mu} * \nu * f) = \langle \nu, \check{f}\rangle f \text{ for all } \nu \in M(G), \tag{2.3}$$

$$M_K(\check{\mu} * \delta_x * f) = \check{f}(x) f \text{ for all } x \in G. \tag{2.4}$$

Consequently, if f satisfies one of these conditions, then it is continuous.

Proof. (2.1) $\Rightarrow$ (2.2). For all $g, h \in \mathcal{K}(G)$, we have

$$\begin{aligned}
\langle h, \check{f}\rangle\langle f, g\rangle &= \int_G\int_G f(x^{-1})f(y)h(x)g(y)\,dx\,dy \\
&= \int_G\int_G\int_K\int_G f(x^{-1}tk\cdot y)h(x)g(y)\,dx\,dy\,dk\,d\mu(t) \\
&= \int_G\int_K\int_G (h*f)(t^{-1}k\cdot y)g(y)\,dk\,dy\,d\check{\mu}(t) \\
&= \int_G\int_K (\check{\mu}*h*f)(k\cdot y)g(y)\,dk\,dy = \big\langle M_K(\check{\mu}*h*f), g\big\rangle.
\end{aligned}$$

Consequently, $M_K(\check{\mu} * h * f) = \langle h, \check{f}\rangle f$, almost everywhere for all $h \in L_1(G,dx)$.

Now by choosing $h \in \mathcal{K}(G)$ such that $\langle h, \check{f}\rangle \neq 0$ and by using ([6], 14.9.2), we deduce that f is a continuous function and we get (2.2).

(2.2) $\Rightarrow$ (2.3). By ([6] 14.11.1), there exists a sequence of regularizing functions $f_n \in L_1(G,dx)$.

In view of (2.2), for all $h \in \mathcal{K}(G)$ we have

$$\big\langle M_K(\check{\mu}*\nu*f_n*f), h\big\rangle = \big\langle \nu*f_n, \check{f}\big\rangle\langle f, h\rangle \text{ for all } \nu \in M(G) \text{ and } n \in \mathbb{N}.$$

Since

$$\big\langle M_K(\check{\mu}*\nu*f_n*f), h\big\rangle = \big\langle \check{\mu}*\nu*f_n*f, L_K(h)\big\rangle = \big\langle \check{\nu}*\mu*L_K(h), f_n*f\big\rangle,$$

where $L_K(h) = \int_K (k\cdot h)\text{mod}(k)dk$ and $\text{mod}(k)$ is defined by the formula $\int_G g(k\cdot x)dx = \text{mod}(k)\int_G g(x)dx$ for all $g \in L_1(G,dx)$ (cf. [6], 14.3.6.1.), by letting $n \longrightarrow +\infty$ and by using ([6], 14.11.1), we get

$$\big\langle \check{\nu}*\mu*L_K(h), f\big\rangle = \langle \nu, \check{f}\rangle\langle f, h\rangle = \big\langle M_K(\check{\mu}*\nu*f), h\big\rangle.$$

In view of ([6], 14.9.2)) and the fact that f is continuous, we obtain (2.3).

(2.4) $\Rightarrow$ (2.1). Since $(\check{\mu}*\delta_x*f)(y) = \int_G f(x^{-1}ty)d\mu(t)$, we get for all $z \in G$

$$M_K(\check{\mu}*\delta_x*f)(z) = \int_K (\check{\mu}*\delta_x*f)(k\cdot z)dk$$

$$= \int_K \int_G f(x^{-1}tk \cdot z)\, dk\, d\mu(t) = f(x^{-1})f(z),$$

which implies (2.1). □

The following theorem explains some relations existing between solutions of equation (1.3) and continuous characters of the commutative Banach algebra $\mu * M_K(L_1(G, dx)) * \mu$, where μ is a generalized Gelfand measure which is K-invariant.

Theorem 2.2. *Let G be unimodular, let μ be a K-invariant generalized Gelfand measure and let $\omega \in C_b(G)$. Then the map*

$$f \longmapsto \langle f, \omega \rangle := \int_G f(x)\omega(x)\, dx$$

is a character of the commutative Banach algebra $\mu * M_K(L_1(G, dx)) * \mu$ if and only if ω is a nonzero solution of equation (1.3).

Proof. To prove Theorem 2.2, we need the following result. □

Proposition 2.1. *Let μ be a K-invariant generalized Gelfand measure. If $\omega \in C_b(G)$ is a solution of equation* (1.3), *then ω is a μ-biinvariant function on G and ω is K-invariant.*

Consequently,

$$\int_G \omega(xty)\, d\mu(t) = \int_G \omega(ytx)\, d\mu(t) \quad \text{for all } x, y \in G.$$

Proof. According to equation (1.3), we have

$$\int_K \int_G \int_G \omega(xtsk \cdot y)\, dk\, d\mu(t)\, d\mu(s) = \int_G \omega(xt)\, d\mu(t)\omega(y).$$

Since $\mu * \mu = \mu$, we get

$$\omega(y) \int_G \omega(xt)\, d\mu(t) = \omega(x)\omega(y) \quad \text{for all} \quad x, y \in G.$$

which implies that

$$\int_G \omega(xt)\, d\mu(t) = \omega(x), \quad x \in G.$$

On the other hand,

$$\int_K \int_G \int_G \omega(xtk \cdot (sy))\, dk\, d\mu(t)\, d\mu(s) = \omega(x) \int_G \omega(sy)\, d\mu(s).$$

Since $\mu * \mu = \mu$, μ is K-invariant and $\int_G \omega(xs)\,d\mu(s) = \omega(x)$, we get from

$$\int_G \int_G \int_K \omega(xtk \cdot (sy))\,dk\,d\mu(t)\,d\mu(s)$$

$$= \int_G \int_{K^+} \int_G \omega(xtk \cdot sk \cdot y)\,dk\,d\mu(t)\,d\mu(s) + \int_G \int_{K^-} \int_G \omega(xtk \cdot yk \cdot s)\,dk\,d\mu(t)\,d\mu(s)$$

and

$$\int_G \int_{K^+} \int_G \omega(xtk \cdot sk \cdot y)\,dk\,d\mu(t)\,d\mu(s) = \int_{K^+} \int_G \omega(xtk \cdot y)\,dk\,d\mu(t),$$

$$\int_G \int_{K^-} \int_G \omega(xtk \cdot yk \cdot s)\,dk\,d\mu(t)\,d\mu(s) = \int_G \int_{K^-} \omega(xtk \cdot y)\,dk\,d\mu(t)$$

that

$$\int_G \int_K \omega(xtk \cdot y)\,dk\,d\mu(t) = \omega(x) \int_G \omega(sy)\,d\mu(s) \text{ for all } x, y \in G.$$

Thus

$$\int_G \omega(sy)\,d\mu(s) = \omega(y) \quad \text{for all } \ y \in G.$$

Now, since μ is a generalized Gelfand measure, by using ([9], Proposition 2.1) we obtain

$$\int_G \omega(xsy)\,d\mu(s) = \int_G \omega(ysx)\,d\mu(s), \quad x, y \in G.$$

If we fix $k' \in K$, then using equation (1.3), we get

$$\int_K \int_G \omega(xtk \cdot (k' \cdot y))\,dk\,d\mu(t) = \omega(x)\omega(k' \cdot y). \tag{2.5}$$

So from the fact that K is compact and hence unimodular it follows that the first term of equation (2.5) becomes

$$\int_K \int_G \omega(xtk \cdot (k' \cdot y))\,dk\,d\mu(t) = \int_K \int_G \omega(xtk \cdot y)\,dk\,d\mu(t), \tag{2.6}$$

whence we deduce that $\omega(k' \cdot x) = \omega(x)$ for all $x \in G$. The proof of the proposition is finished. □

Proof of Theorem 2.2. Let $\omega \in C_b(G)$ be a solution of equation (1.3), then for all $f, g \in L_1(G, dx)$, we have

$$\left\langle \int_K (k \cdot f)^\mu\,dk * \int_K (k' \cdot g)^\mu\,dk', \omega \right\rangle = \int_K \int_K \big\langle (k \cdot f)^\mu * (k' \cdot g)^\mu, \omega \big\rangle dk\,dk'$$

$$= \int_K \int_K \int_G \int_G (k \cdot f)^\mu(y)(k' \cdot g)^\mu(y^{-1}x)\omega(x)\, dk\, dk'\, dx\, dy$$
$$= \int_K \int_K \int_G \cdots \int_G (k \cdot f)(t^{-1}ys^{-1})(k' \cdot g)(l^{-1}y^{-1}xr^{-1}) \times \omega(x)\, dk\, dk'\, dx\, dy\, d\mu(t)\, d\mu(s)\, d\mu(l)\, d\mu(r)$$
$$= \int_K \int_K \int_G \cdots \int_G \int_G (k \cdot f)(y)(k' \cdot g)(x) \times \omega(tylsxr)\, dx\, dy\, dk\, dk'\, d\mu(l)\, d\mu(t)\, d\mu(s)\, d\mu(r).$$

Hence by Proposition 2.1 and $\mu * \mu = \mu$,

$$\int_K \int_K \int_G \cdots \int_G \int_G (k \cdot f)(y)(k' \cdot g)(x) \times \omega(tylsxr)\, dk\, dk'\, dx\, dy\, d\mu(l)\, d\mu(t)\, d\mu(s)\, d\mu(r)$$
$$= \int_K \int_K \int_G \int_G \int_G (k \cdot f)(y)(k' \cdot g)(x)\omega(ysx)\, dx\, dy\, dk\, dk'\, d\mu(s)$$
$$= \int_K \int_K \int_G \int_G \int_G (k \cdot f)(y)(k'.g)(x)k^{-1} \cdot \omega(k^{-1} \cdot ysk^{-1} \cdot x)\, dx\, dy\, dk\, dk'\, d\mu(s)$$
$$= \int_K \int_K \int_G \int_G \int_G f(k \cdot y)(k' \cdot g)(x)\omega(k \cdot ysk \cdot x)\, dx\, dy\, dk\, dk'\, d\mu(s)$$
$$= \int_K \int_K \int_G \int_G \int_G f(y)(k' \cdot g)(x)\omega(ysk \cdot x)\, dx\, dy\, dk\, dk'\, d\mu(s),$$

which follows from $\int_G k \cdot f(x)dx = \int_G f(x)dx$ for all $f \in L_1(G, dx)$ and $k \in K$. Consequently, we have

$$\left\langle \int_K (k \cdot f)^\mu\, dk * \int_K (k' \cdot g)^\mu\, dk', \omega \right\rangle$$
$$= \int_K \int_K \int_G \int_G \int_G f(y)(k' \cdot g)(x)\omega(ysk \cdot x)\, dx\, dy\, dk\, dk'\, d\mu(s)$$
$$= \int_K \int_G \int_G (k' \cdot g)(x) f(y)\omega(y)\omega(x)\, dx\, dy\, dk'$$
$$= \int_K \int_K \int_G \int_G (k \cdot f)^\mu(y)(k' \cdot g)^\mu(x)\omega(y)\omega(x)\, dx\, dy\, dk\, dk'$$
$$= \left\langle \int_K (k \cdot f)^\mu\, dk, \omega \right\rangle \left\langle \int_K (k' \cdot g)^\mu\, dk', \omega \right\rangle,$$

which means that $f \longmapsto \langle f, \omega \rangle$ is a character of $\mu * (M_K(L_1(G, dx))) * \mu$.

Let conversely χ be a continuous character of the commutative Banach algebra $\mu * (M_K(L_1(G, dx))) * \mu$. Then $f \longmapsto \chi((M_K(f))^\mu)$ is a continuous linear mapping of the Banach algebra $L_1(G, dx)$ into $\mathbb{C}$. Consequently, there exists $\omega \in L_\infty(G)$ such that

$$\chi((M_K(f))^\mu) = \langle f, \omega \rangle \ \text{ for all } \ f \in L_1(G, dx).$$

In addition, ω may be chosen continuous: Let $f_0 \in \mathcal{K}(G)$ be a K-invariant function such that $\chi((M_K(f_0))^\mu) = 1$. Then for all $f \in \mathcal{K}(G)$ we have

$$\begin{aligned}\langle f, \omega \rangle =& \chi\big((M_K(f_0))^\mu\big)\chi\big((M_K(f))^\mu\big) = \chi\big((M_K(f))^\mu\big) * (M_K(f_0))^\mu\big)\\ =& \chi\big((M_K(M_K(f) * f_0^\mu)^\mu\big) = \big\langle M_K(f) * f_0^\mu, \omega\big\rangle = \big\langle M_K(f), \omega * (f_0^\mu)^\vee)\big\rangle\\ =& \big\langle f, M_K\big(\omega * (f_0^\mu)^\vee\big)\big\rangle.\end{aligned}$$

Consequently, $\omega = M_K(\omega * (f_0^\mu)^\vee)$ locally almost everywhere.

By using $L_\infty(G, dx) * L_1(G, dx) \subseteq C_b(G)$, we conclude that ω is a continuous function on G.

On the other hand, in view of

$$\chi\big((M_K(f))^\mu\big) = \chi\big((M_K(f^\mu))^\mu\big) = \langle f^\mu, \omega \rangle = \langle f, \omega_\mu \rangle,$$

we deduce that $\langle f, \omega \rangle = \langle f, \omega_\mu \rangle$ for all $f \in L_1(G, dx)$, which implies that $\omega_\mu = \omega$, i.e., ω is μ-biinvariant.

To show that ω is a K-invariant function, we will use the formula $M_K(M_K(f)) = M_K(f)$; $f \in L_1(G, dx)$. For all $f \in L_1(G, dx)$ we have

$$\langle f, \omega \rangle = \chi\big((M_K(f))^\mu\big) = \chi\big((M_K(M_K(f)))^\mu\big) = \big\langle M_K(f), \omega\big\rangle = \big\langle f, M_K(\omega)\big\rangle,$$

from which we conclude that $\omega = M_K(\omega)$ and $k' \cdot \omega = \omega$ for all $k' \in K$.

Now we are going to show that ω is a solution of equation (1.3).

For all $f, g \in L_1(G, dx)$ we have

$$\begin{aligned}&\bigg\langle \Big(\int_K (k \cdot f)\, dk\Big)^\mu * \Big(\int_K (k' \cdot g)\, dk'\Big)^\mu, \omega \bigg\rangle\\ &= \int_G \int_G \int_K \int_G \int_K f(y) k' \cdot g(x) \omega(k \cdot ytx)\, dk\, dk'\, dx\, dy\, d\mu(t)\\ &= \int_G \int_G \int_K \int_K \int_G f(y) g(x) \omega(k' \cdot ytk \cdot x)\, dk\, dk'\, dx\, dy\, d\mu(t)\\ &= \int_G \int_G \int_K \int_{K^+} \int_G f(y) g(x) k'^{-1} \cdot \omega(yt(k'^{-1}k) \cdot x)\, d\mu(t)\, dk\, dk'\, dx\, dy\\ &\quad + \int_G \int_G \int_K \int_{K^-} \int_G f(y) g(x) k'^{-1} \cdot \omega((k'^{-1}k) \cdot xty)\, dk\, dk' dx\, dy\, d\mu(t)\\ &= \int_G \int_G \int_K \int_K \int_G f(y) g(x) \omega(yt(k'^{-1}k) \cdot x)\, dk\, dk'\, dx\, dy\, d\mu(t)\end{aligned}$$

$$= \int\limits_G \int\limits_G \int\limits_K \int\limits_G f(y)g(x)\omega(ytk \cdot x)\, dk\, dx\, dy\, d\mu(t)$$

and it follows that

$$\int\limits_G \int\limits_G \int\limits_K \int\limits_G f(y)g(x)\omega(ytk \cdot x)\, dk\, dx\, dy\, d\mu(t) = \int\limits_G \int\limits_G f(y)g(x)\omega(y)\omega(x)\, dx\, dy$$

for all $f, g \in L_1(G, dx)$. Consequently, ω is a solution of equation (1.3).

On the other hand, in view of $\chi(f) = \langle f, \omega\rangle$ for all $f \in \mu * M_K(L_1(G, dx)) * \mu$ we deduce that $\omega \neq 0$. This ends the proof of Theorem 2.2. □

3. Solutions of Equation (1.3)

In this section, μ is a generalized Gelfand measure which is K-invariant and we assume that every closed ideal of the commutative Banach algebra $L_1^{\mu}(G)$ is contained in some maximal regular ideal. The latter condition is satisfied for example in the following situations:

- G is a compact group and μ is a Gelfand measure (cf. [2]).
- G is unimodular with a growth being almost of polynomial type and μ is a Gelfand measure with compact support (cf. [10], Theorem 3.3.5, [13], Corollary on p. 227, and [8]).

In Theorem 3.1 below, we give a full description of continuous and bounded solutions of the equation

$$\int\limits_G \left\{ \int\limits_K f(xtk \,.\, y)\, dk \right\} d\mu(t) = f(x)f(y), \quad x, y \in G. \tag{3.1}$$

Theorem 3.1. *Let μ be a Gelfand measure which is K-invariant. Let f be a nonzero continuous and bounded solution of the functional equation* (3.1). *Then there exists a μ-spherical function ψ on G such that*

$$f(x) = \int\limits_K \psi(k \cdot x)\, dk, \quad x \in G. \tag{3.2}$$

Conversely, any function of this form is a solution of equation (3.1).

Proof. Let ψ be a μ-spherical function. Let f be a continuous and bounded function defined by the formula

$$f(x) = \int\limits_K \psi(k \cdot x)\, dk, \quad x \in G.$$

Hence $f(xtk' \cdot y) = \int_K \psi(k \cdot (xtk' \cdot y))dk$ and

$$\int\limits_G \int\limits_K f(xtk' \cdot y)\, dk'\, d\mu(t) = \int\limits_K \int\limits_K \int\limits_G \psi\big(k \cdot (xtk' \cdot y)\big)\, dk\, dk'\, d\mu(t)$$

$$= \int\limits_{K^+} \int\limits_K \int\limits_G \psi\big(k \cdot xt(kk') \cdot y\big)\, dk\, dk'\, d\mu(t) + \int\limits_{K^-} \int\limits_K \int\limits_G \psi\big((kk') \cdot ytk \cdot x\big)\, dk\, dk'\, d\mu(t)$$

$$= \int_K \int_K \int_G \psi\big((kk')\cdot ytk\cdot x\big)\, dk\, dk'\, d\mu(t)$$

$$= \int_K \int_K \psi\big((kk')\cdot y\big)\psi(k\cdot x)\, dk\, dk' = \int_K \psi(k'\cdot y)\, dk' \int_K \psi(k\cdot x)\, dk = f(x)f(y).$$

Conversely, if $f \in C_b(G)$ is a non-zero continuous and bounded solution of the functional equation (3.1), then it has the form (3.2). To prove this first we choose an arbitrary function $g \in L_1(G, dx)$.

Multiplying (3.1) by $g(x)$ and integrating the result over G, we get

$$\int_G g(x)\bigg(\int_K \int_G f(xtk\cdot y)\, dk\, d\mu(t)\bigg)\, dx = f(y)\int_G f(x)g(x)\, dx \ \text{ for all } \ y \in G.$$

Hence, in view of

$$\int_K \int_G f(xtk\cdot y)\, dk\, d\mu(t) = \int_K \int_G f(k\cdot ytx)\, dk\, d\mu(t) = \int_K (\check{\mu}_{k\cdot y} * f)(x)\, dk,$$

we get

$$\int_G f(x)\bigg[\int_K ((\check{\nu_{k\cdot y}}) * g)(x)\, dk - f(y)g(x)\bigg]\, dx = 0 \ \text{ for all } \ y \in G, \tag{3.3}$$

where $\nu = \check{\mu}$.

We shall now consider the sets $A = \big\{ \int_K ((\check{\nu_{k\cdot y}}) * g)dk - f(y)g \mid g \in L_1(G, dx), y \in G\big\}$ and $B = \{h^\mu,\ h \in A\}$.

To establish that $I = \mathrm{Lin}(B)$ is an ideal in the algebra $L_1^\mu(G, dx)$, suppose that $h^\mu \in B$, where $h = \int_K ((\check{\nu_{k\cdot y}}) * g)dk - f(y)g \mid y \in G,\ \ g \in L_1(G, dx)$.

Now for all $L \in L_1^\mu(G, dx)$ we have

$$L * h^\mu = (L * h)^\mu = \bigg(\int_K \big((\check{\nu_{k\,.\,y}}) * N\big)dk - f(y)N\bigg)^\mu,$$

where $N = L * g$, which implies that $L * h^\mu \in I$, since I is a linear subspace of $L_1^\mu(G, dx)$. Thus we conclude that I is an ideal of the Banach algebra $L_1^\mu(G, dx)$.

There are two cases:

Case (1). $I = L_1^\mu(G, dx)$. Hence for all $h \in L_1(G, dx)$,

$$\int_G f(x)h^\mu\, dx = \int_G f(x)h(x) = 0,$$

which implies that $f(x) = 0$ for all $x \in G$, which contradicts our assumption.

Case (2). I is a proper ideal of the Banach algebra $L_1^\mu(G, dx)$.

By the assumption of the theorem, there exists a regular maximal ideal $I_{\max}$ of $L_1^\mu(G, dx)$, which contains I.

Now by adapting the same method as used in ([13], Theorem 1, p. 412) we conclude that there exists a continuous character χ_0 of the Banach algebra

$L_1^\mu(G, dx)$ such that $\chi_0(\theta) = 0$ for all $\theta \in I_{\max}$. Hence by [1], there exists a μ-spherical function ψ such that

$$\int_G h^\mu(x)\psi(x)\,dx = 0 \quad \text{for all} \;\; h^\mu \in I.$$

Therefore

$$\int_G \int_K \big((\check{\nu_{k\cdot y}}) * g\big)(x)\psi(x)\,dx\,dk = f(y)\int_G \psi(x)g(x)\,dx$$

for all $y \in G$ and for all $g \in L_1(G, dx)$.

Since

$$\int_G \int_K \big((\check{\nu_{k\cdot y}}) * g\big)(x)\psi(x)\,dx\,dk = \int_K \psi(k\cdot y)\,dk \int_G g(x)\psi(x)\,dx,$$

then by choosing $g_0 \in L_1(G, dx)$ such that $\int_G g_0(x)\psi(x)dx \neq 0$ we get

$$f(y) = \int_K \psi(k\cdot y)\,dk \quad \text{for all} \;\; y \in G. \tag{3.4}$$

The proof of the theorem is thus completed. □

4. Badora's Equation in Lie Groups

In this section, we characterize solutions of the equation

$$\int_G \left\{ \int_K f(xtk\cdot y)\,dk \right\} d\mu(t) = f(x)f(y), \quad x, y \in G, \tag{4.1}$$

on a connected Lie group G as joint eigenfunctions of certain operators associated to the left invariant differential operators, where K is a compact subgroup of the group $Aut(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms.

This extends the previous results obtained by Stetkær in [17] for equation (1.4) and by the authors in [3] for the equation

$$\int_G f(xty)\,d\mu(t) + \int_G f(xt\sigma(y))\,d\mu(t) = 2f(x)f(y), \quad x, y \in G,$$

where σ is a continuous automorphism of G such that $\sigma \circ \sigma = I$.

In the sequel, we need the following notations.

Let G be a connected Lie group. $\mathbb{D}(G)$ denote the algebra of the left invariant differential operators on G, i.e., for all $D \in \mathbb{D}(G)$ and for all $a \in G$

$$L(a)Df = DL(a)f \quad \text{for all} \;\; f \in C^\infty(G),$$

where $L(a)f(x) = f(a^{-1}x)$ for all $x \in G$.

We recall (see [17], Proposition II.3) that K has a Lie group structure, the canonical map $K \times G \to G$ sending (k, x) onto $k \cdot x$ is C^∞ and if $f \in C^\infty(G)$, then so does $k \cdot f$ for any $k \in K$ because continuous homomorphisms between Lie groups automatically are C^∞.

Throughout this section, we assume that μ satisfies the following conditions:

1) μ is a K-invariant measure with compact support on G and

2) $\mu * \mu = \mu$.

The symbol $C_\mu^\infty(G) = \check{\mu} * C^\infty * \Delta\check{\mu}$ stands for all C^∞-functions which are μ-biinvariant on G. The subspace of $C_\mu^\infty(G)$ of functions which are K-invariant will be denoted by $C_{\mu,K}^\infty(G)$.

For any operator D on $C^\infty(G)$, we define the new operator by

$$D_\mu^K f(x) = D\big\{M_K(L(x^{-1})f)_\mu\big\}(e)$$

for all $f \in C^\infty(G)$ and $x \in G$.

We will next describe some properties of D_μ^K that will be used later.

Proposition 4.1. *Our assumptions imply*

(1) D_μ^K *is a left invariant operator;*

(2) $k \cdot D_\mu^K f = D_\mu^K k \cdot f$ *for all* $k \in K$ *and for all* $f \in C^\infty(G)$*;*

(3) $(D_\mu^K f)(e) = D(M_K f_\mu)(e)$. *In particular, if* f *is a* μ*-biinvariant and* K*-invariant function on* G*, then we have* $(D_\mu^K f)(e) = (Df)(e)$*;*

(4) *If* f *is a solution of equation* (4.1) *which satisfies* $\int_G f(xt)d\mu(t) = f(x)$ *for all* $x \in G$*, then* f *is an eigenfunction of the operators* D_μ^K*. More precisely* $D_\mu^K f = (Df)(e)f$ *and, consequently,* f *is analytic on* G*.*

Proof. (1) First we choose an arbitrary function $f \in C^\infty(G)$. For each fixed $a \in G$ and for all $x \in G$, we have

$$\begin{aligned} L(a)(D_\mu^K f)(x) &= D_\mu^K f(a^{-1}x) = D\big\{M_K(L(x^{-1}a)f)_\mu\big\}(e) \\ &= D\big\{M_K(L(x^{-1})L(a)f)_\mu\big\}(e) = D_\mu^K(L(a)f)(x), \end{aligned}$$

which implies that D_μ^K is a left invariant operator on G.

(2) If we now take arbitrary $k' \in K$, then for all $x \in G$ we have

$$k'^{-1} \cdot (D_\mu^K f)(x) = (D_\mu^K f)(k' \cdot x) = D\big\{M_K(L(k' \cdot x^{-1})f)_\mu\big\}.$$

For all $y \in G$ we have

$$\begin{aligned} M_K\big\{L(k' \cdot x^{-1}f)_\mu\big\}(y) &= \int_K \big(L(k' \cdot x^{-1})f\big)_\mu(k \cdot y)\, dk \\ &= \int_G \int_K \int_G L(k' \cdot x^{-1})f(tk \cdot ys)\, d\mu(t)\, d\mu(s)\, dk \\ &= \int_G \int_G \int_K f(k' \cdot xtk \cdot ys)\, d\mu(t)\, d\mu(s)\, dk \end{aligned}$$

and

$$M_K\big\{\big(L(x^{-1})k'^{-1} \cdot f\big)_\mu\big\}(y) = \int_K \big(L(x^{-1})k'^{-1} \cdot f\big)_\mu(k \cdot y)\, dk$$

$$= \int_G \int_K \int_G L(x^{-1})k'^{-1} \cdot f(tk \cdot ys)\, dk\, d\mu(t)\, d\mu(s)$$
$$= \int_G \int_K \int_G k'^{-1} \cdot f(xtk \cdot ys)\, dk\, d\mu(t)\, d\mu(s)$$
$$= \int_G \int_K \int_G f\big(k' \cdot xk' \cdot t(k'k) \cdot yk' \cdot s\big)\, dk\, d\mu(t)\, d\mu(s).$$

Since μ is K-invariant and K is unimodular, we get

$$\int_G \int_K \int_G f\big(k' \cdot xk' \cdot t(k'k) \cdot yk' \cdot s\big)\, dk\, d\mu(t)\, d\mu(s)$$
$$= \int_G \int_K \int_G f(k' \cdot xtk \cdot ys)\, dk\, d\mu(t)\, d\mu(s).$$

Consequently, we have the equality

$$k \cdot D_\mu^K f = D_\mu^K k \cdot f \quad \text{for all} \quad k \in K.$$

(4) Since f is a solution of equation (4.1), then for all $x, y \in G$,

$$\begin{aligned} M_K(L(x^{-1})f)_\mu(y) &= \int_K (L(x^{-1})f)_\mu(k \cdot y)\, dk \\ &= \int_G \int_G \int_K (L(x^{-1})f)(tk \cdot ys)\, dk\, d\mu(t)\, d\mu(s) \\ &= \int_K \int_G \int_G f(xtk \cdot ys)\, dk\, d\mu(t)\, d\mu(s). \end{aligned}$$

In view of $\int_G f(xs)d\mu(s) = f(x)$ for all $x \in G$ we get

$$M_K(L(x^{-1})f)_\mu(y) = \int_K \int_G f(xtk \cdot y)\, dk\, d\mu(t) = f(x)f(y),$$

which implies that $M_K(L(x^{-1})f)_\mu = f(x)f$ and consequently $D_\mu^K f = (Df)(e)f$.

Now let us prove that any solution f of (4.1) is analytic on G. We first show that $f \in C^\infty(G)$. Formula (2.2) shows that

$$M_K(\check{\mu} * h * f) = \langle h, \check{f} \rangle f \quad \text{for all} \quad h \in L_1(G, dx).$$

Consequently, by choosing $h \in C^\infty_{\mu,K}(G)$ with compact support and such that $\langle h, \check{f} \rangle = 1$, we get $h * f = M_K(\check{\mu} * h * f) = \langle h, \check{f} \rangle f = f$, i.e., $h * f = f$. Since $h \in C^\infty(G)$, so is the convolution and we conclude that $f \in C^\infty(G)$.

On the other hand, if $D = \triangle$ denotes the Laplace–Beltrami operator on G, then in view of ([11], p. 400), $\triangle$ is elliptic and has analytic coefficients. Since $\triangle f = \triangle_\mu^K f = \triangle(f)(e)f$, by using Theorem of S. Bernstein [12] we get that f is analytic. This ends the proof of the proposition. $\square$

We now state and prove the following proposition.

Proposition 4.2. *With the assumptions above on G, K and μ and if $D \in \mathbb{D}(G)$, then for all $f \in C^\infty_{\mu,K}(G)$,*

$$D^K_\mu f = M_K(Df * \triangle\check{\mu}).$$

In particular the restriction of D^K_μ to $C^\infty_{\mu,K}(G)$ is an endomorphism.

Proof. Let $D \in \mathbb{D}(G)$ and $f \in C^\infty_{\mu,K}(G)$, then for all $x, y \in G$ we have

$$M_K\big(L(x^{-1})f\big)_\mu(y) = \int_G \int_G \int_K f(xtk \cdot ys)\, dk\, d\mu(t)\, d\mu(s)$$

$$= \int_G \int_K f(xtk \cdot y)\, d\mu(t)\, dk = \int_K \int_G k^{-1} \cdot f(k^{-1} \cdot xk^{-1} \cdot ty)\, d\mu(t)\, dk.$$

Since μ and f are K-invariant, we get

$$\int_K \int_G k^{-1} \cdot f(k^{-1} \cdot xk^{-1} \cdot ty)\, d\mu(t)\, dk = \int_K \int_G f(k^{-1} \cdot xty)\, d\mu(t)\, dk$$

$$= \int_K \int_G f(k \cdot xty)\, d\mu(t)\, dk = \int_K \int_G L((k \cdot xt)^{-1}) f(y)\, d\mu(t)\, dk.$$

Consequently,

$$D\big\{M_K(L(x^{-1})f)_\mu\big\}(e) = \int_K \int_G L((k \cdot xt)^{-1}) Df(e)\, dk\, d\mu(t)$$

$$= \int_K \int_G Df(k \cdot xt)\, dk\, d\mu(t) = \int_K Df * \triangle\check{\mu}(k \cdot x)\, dk = M_K(Df * \triangle\check{\mu})(x),$$

which proves that

$$D^K_\mu f = M_K(Df * \triangle\check{\mu}).$$

From this it follows that $D^K_\mu f$ is a K-invariant function on G.

Now we are able to prove that $D^K_\mu f$ is a μ-biinvariant function.

First, we show that $Df * \triangle\check{\mu}$ is μ-biinvariant.

$$\check{\mu} * (Df * \triangle\check{\mu}) * \triangle\check{\mu} = \check{\mu} * Df * \triangle\check{\mu} = D(\check{\mu} * f) * \triangle\check{\mu} = Df * \triangle\check{\mu},$$

which proves that $Df * \triangle\check{\mu}$ is a μ-biinvariant function on G.

Now for all $x \in G$ we have

$$\big(\check{\mu} * M_K(Df * \triangle\check{\mu}) * \triangle\check{\mu}\big)(x) = \int_G \int_G M_K(Df * \triangle\check{\mu})(txs)\, d\mu(t)\, d\mu(s)$$

$$= \int_G \int_G \int_K \big(Df * \triangle\check{\mu})(k \cdot tk \cdot xk \cdot s)\big)\, dk\, d\mu(t)\, d\mu(s)$$

$$= \int_G \int_G \int_K \big((Df * \triangle\check{\mu})(tk \cdot xs) \big)\, dk\, d\mu(t)\, d\mu(s).$$

Since $Df * \triangle\check{\mu}$ is μ-biinvariant, we get

$$\int_G \int_G \int_K (Df * \triangle\check{\mu})(tk \cdot xs)\, d\mu(t)\, d\mu(s)\, dk = \int_K (Df * \triangle\check{\mu})(k \cdot x)\, dk$$
$$= M_K (Df * \triangle\check{\mu})(x),$$

from which we deduce that $D^K_\mu f$ is μ-biinvariant. □

In the next theorem, we prove the main result of this section.

Theorem 4.1. *Let $\mu \in M(G)$ be a K-invariant, idempotent measure with compact support. If $f \in C(G)$, then the following statements are equivalent:*

(1) *f is a solution of the equation*

$$\int_G \left\{ \int_K f(xtk \cdot y)\, dk \right\} d\mu(t) = f(x)f(y) \ \text{ for all } \ x, y \in G,$$

(2) (a) *f is μ-biinvariant,*
(b) *f is K-invariant,*
(c) *$f \in C^\infty(G)$,*
(d) *f is analytic,*
(e) *f is a joint eigenfunction of the operators D^K_μ for all $D \in \mathbb{D}(G)$.*

Proof. (1) ⇒ (2) follows directly from Proposition 4.1. Conversely, suppose that the assumptions (a), (b), (c), d and (e) hold and let us consider the function defined on G by

$$F(y) = \int_K \int_G f(k \cdot xty)\, dk\, d\mu(t),$$

where x is an arbitrary element of G.

It is easy to verify that F is a μ-biinvariant function on G. Concerning the K-invariance of F, for each fixed $k' \in K$ and for all $y \in G$ we have

$$F(k' \cdot y) = \int_G \int_K f(k \cdot xtk' \cdot y)\, dk\, d\mu(t) = \int_G \int_K k'^{-1}.f((k'^{-1}k) \cdot xk'^{-1} \cdot ty)\, dk\, d\mu(t).$$

Since μ and f are K-invariant, we obtain

$$\int_G \int_K k'^{-1} \cdot f\big((k'^{-1}k) \cdot xk'^{-1} \cdot ty\big)\, dk\, d\mu(t) = \int_G \int_K f\big((k'^{-1}k) \cdot xty\big)\, dk\, d\mu(t),$$

and from the left-invariance of dk follows the equality

$$\int_G \int_K f\big((k'^{-1}k) \cdot xty\big)\, dk\, d\mu(t) = \int_G \int_K f(k \cdot xty)\, dk\, d\mu(t) = F(y),$$

which proves that F is K-invariant.

Now $F(y)$ can be written in the form

$$F(y) = \int_G \int_K L\big((k \cdot xt)^{-1}\big) f(y)\, dk\, d\mu(t).$$

By Proposition 4.1 D_μ^K is left invariant so that we get

$$\begin{aligned} D_\mu^K F(y) &= \int_G \int_K D_\mu^K f(k \cdot xty)\, dk\, d\mu(t) \\ &= D(f)(e) \int_G \int_K f(k \cdot xty)\, dk\, d\mu(t) = D(f)(e)F(y), \end{aligned}$$

which implies that

$$D_\mu^K F(e) = D(f)(e) f(x). \tag{4.2}$$

In view of Proposition 4.1, $D_\mu^K F(e) = DF(e)$, and from (4.2) we get

$$D(F - F(e)f)(e) = 0 \quad \text{for all} \quad D \in \mathbb{D}(G).$$

Since $F - F(e)f$ is analytic on the connected Lie group G, by Helgason [11], we obtain $F = f(x)f$, which can be rewritten as

$$\int_G \left\{ \int_K f(k \cdot xty)\, dk \right\} d\mu(t) = f(x)f(y) \quad \text{for all} \quad x, y \in G.$$

By using the same methods as in the proof of Proposition 4.2 we get

$$\int_G \left\{ \int_K f(xtk \cdot y)\, dk \right\} d\mu(t) = \int_G \left\{ \int_K f(k \cdot xty)\, dk \right\} d\mu(t) = f(x)f(y)$$

for all $x, y \in G$. This ends the proof of the theorem. □

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