Georgian Mathematical Journal
Volume 11 (2004), Number 3, 539–548

ASYMPTOTIC BEHAVIOR OF GENERALIZED NONEXPANSIVE SEQUENCES AND MEAN POINTS

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Abstract. Let E be a real Banach space with norm $\|\cdot\|$ and let $\{x_n\}_{n\geq 0}$ be a generalized nonexpansive sequence in E (i.e., $\| x_{i+1} - x_{j+1} \|^2 \leq \|x_i - x_j\|^2 + (\varepsilon(i+1, j+1) - \varepsilon(i, j))^2$ for all $i, j \geq 0$, where the series of nonnegative terms $\sum\limits_{i,j} \varepsilon(i, j)$ is convergent). Let $K = \bigcap\limits_{n=1}^{\infty} \overline{\text{co}} \left\{ \{x_i - x_{i-1}\}_{i \geq n} \right\}$. We deal with the mean point of $\{\frac{x_n}{n}\}$ concerning a Banach limit μ. If E is reflexive and $d = d(0, K)$, then we show that $d = d\left(0, \ \overline{\text{co}}\left\{\frac{x_n - x_0}{n}\right\}\right)$ and there exists a point z_0 with $\| z_0 \| = d$ such that $z_0 \in \overline{\text{co}}\{\frac{x_n - x_0}{n}\}$. In the sequel, this result is applied to obtain the weak and strong convergence of $\{\frac{x_n}{n}\}$.

2000 Mathematics Subject Classification: 47H09.
Key words and phrases: Asymptotic behavior, Banach limit, mean point, nonexpansive sequence, generalized nonexpansive sequence.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$; we denote weak convergence and strong convergence in E respectively by $\rightharpoonup$ and $\longrightarrow$ and let $\{x_n\}_{n\geq 0}$ be a generalized nonexpansive sequence in E (see Definition 2.1 below). Let $K = \bigcap\limits_{n=1}^{\infty} \overline{\text{co}} \left\{ \{x_i - x_{i-1}\}_{i \geq n} \right\}$. Djafari Rouhani [3] considered nonexpansive sequences and obtained an interesting result on the weak convergence of $\{\frac{x_n}{n}\}$ under the assumption that E is reflexive and strictly convex. Recently, Jung and Park [7] dropped the strict convexity requirement in the result of Djafari Rouhani, that is, instead of the weak limit of $\{\frac{x_n}{n}\}$, they dealt with the mean point of $\{\frac{x_n}{n}\}$ concerning a Banach limit under the assumption that E is reflexive. The present paper is motivated in part by Jung and Park's application [8] of a Banach limit technique due to Takahashi [16]. We consider a generalized nonexpansive sequence and we use the mean point to obtain the weak convergence of $\{\frac{x_n}{n}\}$, in the case when E is reflexive and strictly convex. In addition we obtain the strong convergence of $\{\frac{x_n}{n}\}$, in the case when E^* has a Fréchet differentiable norm. Our result extend and improve the corresponding results in [3], [7]–[11].

2. Preliminaries

Let E be a real Banach space; the norms of both E and its dual E^* will be denoted by $\|.\|$. The duality pairing between E and E^* will be denoted by $(\cdot,\cdot)$. The duality mapping J from E into the family of nonempty closed convex

ISSN 1072-947X / $8.00 /

subsets of E^* is denoted by

$$J(x) = \left\{x^* \in E^* : (x, x^*) = \| x\|^2 = \|x^*\|^2\right\}.$$

It may be observed that for $x, y \in E$ and $j \in J(x)$,

$$(x - y, j) = \|x\|^2 - (y, j) \geq \|x\|^2 - \frac{1}{2}\left(\|y\|^2 + \|j\|^2\right) = \frac{1}{2}\left(\|x\|^2 - \|y\|^2\right).$$

We observe that if E is reflexive and strictly convex and K is a nonempty closed convex subset of E, then the nearest point projection mapping P_K of E onto K is well defined, i.e., K is a Chebyshev set (see [1], [6]).

Definition 2.1. A sequence $\{x_n\}_{n\geq 0} \subset E$ is said to be a generalized nonexpansive sequence if it satisfies

$$\| x_{i+1} - x_{j+1} \|^2 \leq \| x_i - x_j \|^2 + (\varepsilon(i+1, j+1) - \varepsilon(i, j))^2 \qquad (2.1)$$

for all $i, j \geq 0$, where the series of nonnegative terms $\sum_{i,j} \varepsilon(i, j)$is convergent.

Let μ be a mean on the integers N, i.e., a linear functional μ defined on ℓ^∞ such that

(a) $\mu(a) \geq 0$ if $a_n \geq 0 \ \forall \ n$,

(b) $\mu(a) = \mu(\sigma a)$ where σ denotes the right shift

$$\sigma a = \sigma(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4, \dots),$$

(c) $\mu(a) = 1$ if $a = (1, 1, 1, \dots)$.

Then we know that μ is a mean on N if and only if

$$\inf \{a_n : n \in N\} \leq \mu(a) \leq \sup \{a_n : n \in N\}$$

for every $a = (a_1, a_2, \dots) \in \ell^\infty$. For convenience we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a *Banach limit* (see [14]) if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \dots) \in \ell^\infty$. The Hahn Banach theorem guarantees the existence of a Banach limit [15]. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} \ a_n \ \leq \ \mu_n(a_n) \leq \ \limsup_{n\to\infty} \ a_n$$

for every $a = (a_1, a_2, \dots) \in \ell^\infty$. Let E be a reflexive Banach space and let $\{x_n\}$ be a bounded sequence in E. We now show for a Banach limit μ , there exists a point x_0 in E such that

$$\mu_n(x_n, x^*) = (x_0, x^*) \quad \forall \ x^* \in E^*.$$

In fact, the function $\mu_n(x_n, x^*)$ is linear in x^*. Also since

$$|\mu_n\left(x_n, x^*\right)| \leq \left(\sup_n \|x_n\|\right) \cdot \|x^*\|,$$

it follows that the function $\mu_n(x_n, x^*)$ is also bounded in x^*. Thus there is a $x_0^{**} \in E^{**}$ such that $\mu_n(x_n, x^*) = (x_0^{**}, x^*)$ for every $x^* \in E^*$. Since E is reflexive, we can find a point x_0 in E such that $\mu_n(x_n, x^*) = (x_0, x^*)$ for every $x^* \in E^*$.

This point x_0 is called a mean point of $\{x_n\}$ concerning μ. Furthermore [7], we also know that this mean point $x_0 \in \bigcap_{n\geq 1}^{\infty} \overline{\text{co}}\,\{x_n\}$.

Let $S = \{x \in E :\| x \|= 1\}$. Then the norm of E is called Fréchet differentiable if for each $x \in S$, the limit

$$\lim_{t\to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists uniformly for each $y \in S$.

Lemma 2.1. *Let $\{a_n\}_{n\geq 0}$ be a sequence of nonnegative real numbers with $a_0 = 0$, the series of nonnegative terms $\sum_{i,j} \varepsilon(i,j)$ be convergent, and satisfying $a_{n+p} \leq a_n + a_p + \varepsilon(n+p, n),\ \forall\, n \geq 0, \forall\, p \geq 1$. Then the sequence $\{\frac{a_n}{n}\}$ converges as $n \to \infty$ and $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}$.*

Proof. Let $p \geq 1$ be fixed. Then by the division algorithm, for all $n \geq p$, there exists $k \geq 1$ such that $n = kp + i; 0 \leq i < p$.

Since the series of nonnegative terms $\sum_{i,j} \varepsilon(i,j)$ converges, there exists $\eta > 0$ such that $\sum_{i,j} \varepsilon(i,j) \leq \eta$. Now, for any $p \geq 1$ (for notational purposes $\sum_2^1 = 0$) we have

$$a_{kp} \leq k \cdot a_p + \sum_{j=2}^{k} \varepsilon(jp, (j-1)p) \leq k \cdot a_p + \sum_{l,m} \varepsilon(l,m) \leq k \cdot a_p + \eta.$$

Thus, we have

$$\frac{a_{kp}}{kp+i} \leq \frac{k \cdot a_p + \eta}{kp+i} \leq \frac{a_p}{p} + \frac{\eta}{k} \quad \forall\, p \geq 1.$$

Hence, we have

$$\frac{a_n}{n} = \frac{a_{kp+i}}{kp+i} \leq \frac{a_{kp} + a_i + \epsilon(kp+i, kp)}{kp+i} \leq \frac{a_p}{p} + \frac{2\cdot\eta}{k} + \frac{a_i}{k}$$
$$\leq \frac{a_p}{p} + \frac{2\cdot\eta}{k} + \frac{\max_{0\leq i<p} a_i}{k}.$$

Now letting $n \to \infty$, we have $k \to \infty$ and so for all $p \geq 1$, we have $\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_p}{p}$. Therefore

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_{p\geq 1} \frac{a_p}{p} \leq \liminf_{n\to\infty} \frac{a_n}{n}.$$

Hence

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

□

The following well known lemma will be useful later (cf. [3]).

Lemma 2.2. *E^* has a Fréchet differentiable norm if and only if E is reflexive and strictly convex, and has the following property: if $x_n \rightharpoonup x$ and $\| x_n \| \to \| x \|$ for a sequence $\{x_n\}$ in E, then $\{x_n\}$ converges strongly to x.*

Let D be a subset of E. Then we denote the closure of D by $\overline{D}$ and the closed convex hull of D by $\overline{\text{co}}D$, respectively. For a point x in E, we denote its distance from D by $d(x, D) = \inf_{y \in D} \|x - y\|$

3. Main Result

In this section, we deal with a generalized nonexpansive sequence $\{x_n\}$ in E and study the mean point of $\{\frac{x_n}{n}\}$ concerning a Banach limit. We begin with the following lemmas which will play crucial roles in the proof of our main result. We shall also use the following basic inequality

$$(a+b)^q \leq a^q + b^q \tag{3.1}$$

for $0 < q \leq 1$ and $a,\ b \geq 0$.

Lemma 3.1. *Let E be a Banach space and let $\{x_n\}$ be a generalized nonexpansive sequence in E. Then $\lim_{n\to\infty} \| \frac{x_n}{n} \|$ exists and*

$$\lim_{n\to\infty} \left\| \frac{x_n}{n} \right\| = \inf_{n \geq 1} \left\| \frac{x_n - x_0}{n} \right\|.$$

Proof. Let $a_n = \| x_n - x_0 \|\ \forall\ n \geq 1$. Now applying (3.1) to the generalized nonexpansive sequence $\{x_n\}$ successively, we obtain for all $p \geq 1$ that

$$\begin{aligned} a_{n+p} &= \| x_{n+p} - x_0 \| \leq \| x_{n+p} - x_n \| + \| x_n - x_0 \| \\ &\leq \| x_p - x_0 \| + \sum_{j=1}^{n} [\varepsilon(j+p, j) - \varepsilon(j-1+p, j-1)] + \| x_n - x_0 \| \\ &= \| x_p - x_0 \| + \varepsilon(n+p, n) - \varepsilon(p, 0) + \| x_n - x_0 \| \\ &\leq a_n + a_p + \varepsilon(n+p, n). \end{aligned}$$

Hence the result follows from Lemma 2.1. □

Lemma 3.2. *Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers (i.e., $a_n > 0$ for each n) and $b_n = \sum_{i=1}^{n} a_i$. Assume that $b_n \uparrow \sum_{i=1}^{\infty} a_i = \infty$. If $\{x_n\}$ is a sequence of real numbers such that $x_n \to x$, then we have*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i x_i = x.$$

Proof. Let $\varepsilon > 0$. Choose some k such that $|x_n - x| < \frac{\varepsilon}{2}$ for each $n \geq k$. Put $M = \max\{|x_i - x| : i = 1, \ldots, k\}$, and then select $l > k$ such that $\frac{Mb_k}{b_n} < \frac{\varepsilon}{2}$ for all $n \geq l$. Now notice that if $n \geq l$, then

$$\begin{aligned}\left|\frac{1}{b_n}\sum_{i=1}^{n} a_i x_i - x\right| &= \left|\frac{1}{b_n}\sum_{i=1}^{n} a_i x_i - \frac{1}{b_n}\sum_{i=1}^{n} a_i x\right| \\ &\leq \frac{1}{b_n}\sum_{i=1}^{k} a_i|x_i - x| + \frac{1}{b_n}\sum_{i=k+1}^{n} a_i|x_i - x| \\ &\leq \frac{Mb_k}{b_n} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and the conclusion follows. □

Lemma 3.3. *Let E be a reflexive Banach space and let $\{x_n\}$ be a generalized nonexpansive sequence in E. Let*

$$K = \bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\left\{\{x_i - x_{i-1}\}_{i\geq n}\right\}.$$

Then $\lim_{n\to\infty} \| \frac{x_n}{n} \| = d(0, K) = \inf_{n\geq 1} \| \frac{x_n - x_0}{n} \|$.

Proof. Let $k \geq 1$ be fixed and $j_n \in J(x_n - x_{k-1})$ for $n \geq k$. Now the generalized sequence $\{x_n\}$ yields for $n \geq k$ that

$$(x_k - x_{k-1}, j_n) \geq \frac{1}{2} \| x_n - x_{k-1} \|^2 - \frac{1}{2} \| x_n - x_k \|^2$$
$$\geq \frac{1}{2} \| x_n - x_{k-1} \|^2 - \frac{1}{2} \| x_{n-1} - x_{k-1} \|^2 - \frac{1}{2} \cdot (\varepsilon(n,k) - \varepsilon(n-1,k-1))^2.$$

Hence we obtain

$$\begin{aligned}\frac{2}{n^2}\left(x_k - x_{k-1}, \sum_{i=k}^{n} j_i\right) &\geq \left\|\frac{x_n - x_{k-1}}{n}\right\|^2 - \frac{1}{n^2}\sum_{i=k}^{n}(\varepsilon(i,k) - \varepsilon(i-1,k-1))^2 \\ &\geq \left\|\frac{x_n - x_{k-1}}{n}\right\|^2 - \frac{1}{n^2} \cdot 2\sum_{i,j} \varepsilon^2(i,j) \ \forall\ k \geq 1. \end{aligned} \tag{3.2}$$

Let $S_n = \frac{2}{n^2} \sum_{i=k}^{n} j_i$ for $n \geq k$. Then we have

$$\| S_n \| \leq \frac{2}{n^2}\sum_{i=k}^{n} \| x_i - x_{k-1} \| = \frac{2}{n^2}\sum_{i=k}^{n} i\left\|\frac{x_i - x_{k-1}}{i}\right\|.$$

Since $\{\frac{x_n}{n}\}$ is bounded by Lemma 3.1, it then follows that $\{S_n\}$ is bounded. Hence by the weak* compactness of the closed unit ball of E^* the sequence $\{S_n\}$ has a weak* cluster point $j \in E^*$ (obviously independent of $k \geq 1$). Since $\sum_{i,j} \varepsilon(i,j)$ is bounded $\forall\ k \geq 1$, we obtain from Lemma 3.1 and (3.2) that

$$(x_k - x_{k-1}, j) \geq \lim_{n\to\infty} \left\|\frac{x_n}{n}\right\|^2 \ \forall\ k \geq 1.$$

Hence for any $n \geq 1$, we have

$$\left(\frac{x_n - x_0}{n}, j\right) \geq \lim_{n\to\infty} \left\|\frac{x_n}{n}\right\|^2. \tag{3.3}$$

From Lemma 3.2, replacing a_i by i, b_n by $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and note that $\lim_{n\to\infty} b_n = \infty$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{2}{n^2}\sum_{i=k}^{n} i\left\|\frac{x_i - x_{k-1}}{i}\right\| &= \limsup_{n\to\infty}\left[\frac{n(n+1)}{n^2}\cdot\frac{2}{n(n+1)}\sum_{i=k}^{n} i\left\|\frac{x_i - x_{k-1}}{i}\right\|\right]\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot \limsup_{n\to\infty}\frac{2}{n(n+1)}\sum_{i=1}^{n} i\left\|\frac{x_i - x_{k-1}}{i}\right\|\\ &= \lim_{n\to\infty}\left\|\frac{x_n - x_{k-1}}{n}\right\|.\end{aligned}$$

Now using above inequality, we also have

$$\begin{aligned}\| j \| \leq \liminf_{n\to\infty} \| S_n \| &\leq \liminf_{n\to\infty} \frac{2}{n^2}\sum_{i=k}^{n} i\left\|\frac{x_i - x_{k-1}}{i}\right\|\\ &\leq \limsup_{n\to\infty} \frac{2}{n^2}\sum_{i=k}^{n} i\left\|\frac{x_i - x_{k-1}}{i}\right\| = \lim_{n\to\infty}\left\|\frac{x_n - x_{k-1}}{n}\right\| = \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\|\end{aligned}$$

and so it follows that

$$(x_k - x_{k-1}, j) \geq \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\|^2 \geq \| j\|^2 \ \forall\ k \geq 1.$$

Hence for any $z \in \overline{\text{co}}\left\{\{x_{i+1} - x_i\}_{i\geq 0}\right\}$

$$\begin{aligned}\frac{1}{2}\lim_{n\to\infty}\left\|\frac{x_n}{n}\right\|^2 + \frac{1}{2}\| z\|^2 &\geq \frac{1}{2}\| j\|^2 + \frac{1}{2}\| z\|^2\\ &\geq (z, j) \geq \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\|^2 \geq \| j\|^2.\end{aligned} \tag{3.4}$$

Since $K \subset \overline{\text{co}}\left\{\{x_{i+1} - x_i\}_{i\geq 0}\right\}$, it follows from (3.4) that

$$\| j \| \leq \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\| \leq \inf_{z\in K} \| z \| = d(0, K).$$

Since $\{\frac{x_n}{n}\}$ is bounded, it follows that $\{\frac{x_n - x_0}{n}\}$ is bounded and E is reflexive, therefore, by Eberlein–Smulian theorem the sequence $\{\frac{x_n - x_0}{n}\}$ contains a weakly convergent subsequence $\{\frac{x_{n_i} - x_0}{n_i}\}$. Suppose $\frac{x_{n_i} - x_0}{n_i} \rightharpoonup q$ for some $q \in K$. Then we have

$$\|q\| \leq \liminf_{i\to\infty}\left\|\frac{x_{n_i} - x_0}{n_i}\right\| = \lim_{n\to\infty}\left\|\frac{x_n}{n}\right\|.$$

Hence

$$\lim_{n\to\infty}\left\|\frac{x_n}{n}\right\| = d(0, K).$$

This completes the proof. □

Now using Lemmas 3.1 and 3.3, we obtain the main result.

Theorem 3.4. *Let E be a reflexive Banach space and let $\{x_n\}$ be a generalized nonexpansive sequence in E. Let*

$$K = \bigcap_{n=1}^{\infty} \overline{\text{co}} \left\{ \{x_i - x_{i-1}\}_{i \geq n} \right\}$$

and $d = d(0, K)$. Then $d = d\left(0, \overline{\text{co}} \left\{ \frac{x_n - x_0}{n} \right\}\right)$ and there exists a point z_0 with $\| z_0 \| = d$ such that $z_0 \in \overline{\text{co}}\{\frac{x_n - x_0}{n}\}$.

Proof. In view of Lemma 3.3, we may assume that $\{\frac{x_n - x_0}{n}\}_{n \geq 1}$ is bounded. Now, it follows from the reflexiveness of E that for a Banach limit μ, there exists $z_0 \in \overline{\text{co}}\{\frac{x_n - x_0}{n}\}$ such that

$$\mu_n \left(\frac{x_n - x_0}{n}, x^* \right) = (z_0, x^*) \quad \forall\, x^* \in E^*. \tag{3.5}$$

Now, for $j_0 \in J(z_0)$ we have

$$\begin{aligned} \| z_0 \|^2 &= (z_0, j_0) = \mu_n \left(\frac{x_n - x_0}{n}, j_0 \right) \\ &\leq \mu_n \left(\left\| \frac{x_n - x_0}{n} \right\| \right) \cdot \| j_0 \| = d \cdot \| j_0 \| = d \cdot \| z_0 \|, \end{aligned}$$

and hence $\| z_0 \| \leq d$. From the proof of Lemma 3.3 and (3.3), there exists a functional $j \in E^*$ with $\| z_0 \| \leq d$ such that

$$\left(\frac{x_n - x_0}{n}, j \right) \geq d^2 \quad \forall\, n \geq 1. \tag{3.6}$$

As a result we have $(z_0, j) \geq d^2$. Since $\| j \| \leq d$, we obtain

$$d^2 \geq \| z_0 \| \cdot \| j \| \geq (z_0, j) \geq d^2$$

and hence $\| z_0 \| = \| j \| = d$. From (3.6), it follows that $(z_0, j) \geq d^2$ for every $z \in \overline{\text{co}}\{\frac{x_n - x_0}{n}\}$ and so

$$\| z \| \cdot d = \| z \| \cdot \| j \| \geq (z, j) \geq d^2.$$

Hence $\| z \| \geq d$ for every $z \in \overline{\text{co}}\{\frac{x_n - x_0}{n}\}$. As a result we obtain

$$d = d \left(0, \overline{\text{co}} \left\{ \frac{x_n - x_0}{n} \right\} \right).$$

Now suppose that there is another point w_0 satisfying (3.5). Then for $j \in J(z_0 - w_0)$, we have

$$\| z_0 - w_0 \|^2 = (z_0 - w_0, j) = \mu_n \left(\frac{x_n - x_0}{n} - \frac{x_n - x_0}{n}, j \right) = 0,$$

and hence $z_0 = w_0$. This completes the proof. □

Corollary 3.5. *Suppose* $E, \{x_n\}, K$ *and* d *are as in Theorem* 3.4. *Then we have the following:*

(i) *If* E *is strictly convex, then the weak* $\lim_{n\to\infty} \frac{x_n}{n}$ *exists and coincides with* $P_K 0$ *with* $\| P_K 0 \| = d$;

(ii) *If* E^* *has a Fréchet differentiable norm, the strong* $\lim_{n\to\infty} \frac{x_n}{n}$ *exists and coincides with* $P_K 0$.

Proof. (i). Since E is strictly convex, the set

$$\left\{ z \in \overline{\mathrm{co}} \left\{ \frac{x_n - x_0}{n} \right\} : \| z \| = d \right\}$$

consists of exactly one point and $d(0, K) = \| P_K 0 \|$. It may be observed that this point equals z_0 in Theorem 3.4. Let $\{\frac{x_{n_i}}{n_i}\}$ be a subsequence of $\{\frac{x_n}{n}\}$ such that $\{\frac{x_{n_i}}{n_i}\}$ converges weakly to $p \in K$. Then since

$$\|p\| \le \liminf_{i\to\infty} \left\| \frac{x_{n_i}}{n_i} \right\| = \lim_{n\to\infty} \left\| \frac{x_n}{n} \right\| = \| P_K 0 \|$$

we have $p = z_0 = P_K 0$. Then $\{\frac{x_n}{n}\}$ converges weakly to $P_K 0$. This completes the proof.

(ii) This is an immediate consequence of (i) and Lemma 2.2. □

Remark 3.1. (1) Let $\{x_n\}_{n\ge 0}$ be a nonexpansive sequence in E (i.e., $\| x_{i+1} - x_{j+1} \| \le \| x_i - x_j \|$ for all $i, j \ge 0$). Then $\lim_{n\to\infty} \left\| \frac{x_n}{n} \right\|$ exists [2, Theorem 3.1] and $\{x_n\}_{n\ge 0}$ also satisfies (2.1). Thus Theorem 3.4 is a partial generalization of Theorem 3.3 in [7].

(2) Since our study is equivalent to the study of the asymptotic behavior of the sequence $\left\{\frac{T^n x}{n}\right\}_{n\ge 1}$ in E, T is a nonexpansive mapping from an arbitrary set K of E into itself and $x \in K$, Theorem 3.4 is a partial improvement of Theorem 5 in [13].

(3) Our results extend and improve the corresponding results in [7]–[11].

(4) Our result may also be applied to the asymptotic behavior of curves in E, and thus to the asymptotic behavior of unbounded trajectories for the quasi-autonomous dissipative system $\frac{du}{dt} + Au \ni f$ where A is an accretive (possibly multivalued) operator in $E \times E$, see [2] for the case when E is a Hilbert space.

Acknowledgement

This work was partially supported by University Grant Commission, New Delhi, India (4/41-(2001) Computer).

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(Received 14.04.2004)

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