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WITT'S THEOREM FOR GROUPS WITH ACTION AND FREE LEIBNIZ ALGEBRAS

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Abstract. A subcategory of the category of groups with action is determined and it is proved that the functor defined in [2] takes free objects from this category to free Leibniz algebras. This result gives a solution to a problem stated by J.-L. Loday [6], [7] and is an analogue of Witt's theorem for groups and Lie algebras [10].

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Introduction

In [6], [7] J.-L. Loday stated the problem to define algebraic objects called "coquecigrues", which would have an analogous role for Leibniz algebras as groups have for Lie algebras. In particular, the author writes: "... it is reasonable to expect that a coquecigrue is equipped with a lower central series whose graded associated object is a Leibniz algebra. Moreover, a free coquecigrue should give rise to a free Leibniz algebra (analogue of the Witt theorem which says that the Lie algebra associated to a free group is free)" [7].

In [2] we defined the notion of central series for groups with action on itself and gave an analogue of Witt's construction [10] for such groups. We introduced a condition for the action of a group (see Section 1 below, Condition 1); we defined the category of groups (abelian groups) with action on itself $\mathbb{G}\mathrm{r}^c$ ($\mathbb{A}\mathrm{b}^c$) satisfying Condition 1 and the category of Lie–Leibniz algebras $\mathbb{LL}$. It is proved in [2] that the analogue of Witt's construction defines a functor $LL : \mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{LL}$. This functor leads us to Leibniz algebras (defined in [5]) over the ring of integers $\mathbb{Z}$ by taking the compositions $\mathbb{G}\mathrm{r}^c \xrightarrow{A} \mathbb{A}\mathrm{b}^c \xrightarrow{L} \mathbb{L}\mathrm{eibniz}$, $\mathbb{G}\mathrm{r}^c \xrightarrow{LL} \mathbb{LL} \xrightarrow{S_2} \mathbb{L}\mathrm{eibniz}$, where A is the abelianization functor, $L = LL\big|_{\mathbb{A}\mathrm{b}^c}$ and S_2 is the functor which makes the round bracket operation trivial (see Section 1 for details).

In this paper we introduce two conditions (Conditions 2 and 3) between round and square brackets for the objects of $\mathbb{G}\mathrm{r}^c$ and according to these conditions define subcategories $\overline{\mathbb{G}\mathrm{r}}$ and $\overline{\mathbb{LL}}$ of $\mathbb{G}\mathrm{r}^c$ and $\mathbb{LL}$, respectively. We prove that the functor LL takes free objects from $\overline{\mathbb{G}\mathrm{r}}$ to free objects in $\overline{\mathbb{LL}}$ (Theorem 3.2). The composition $S_2 \circ LL\big|_{\overline{\mathbb{G}\mathrm{r}}} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{L}\mathrm{eibniz}$ gives free Leibniz algebras for free objects from $\overline{\mathbb{G}\mathrm{r}}$ (Corollary 3.11). In particular, the functor $L : \mathbb{A}\mathrm{b}^c \longrightarrow \mathbb{L}\mathrm{eibniz}$

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takes free objects to free Leibniz algebras (Theorem 3.3). The results obtained in [2] (Theorem 3.6) and in this paper (Section 3) give an analogue of Witt's theorem for Leibniz algebras and a solution to the problems stated by J.-L. Loday in [6], [7].

In Section 1 we recall some definitions and main results from [2]. We introduce Conditions 2 and 3 for groups with action on itself and denote the corresponding subcategory of $\mathbb{G}\mathrm{r}^c$ by $\overline{\mathbb{G}\mathrm{r}}$. It is proved that if A and B are ideals of G in $\overline{\mathbb{G}\mathrm{r}}$, then the commutator $[A, B]$ is also an ideal of G (Proposition 1.5). For ideals A, B, C of G in $\overline{\mathbb{G}\mathrm{r}}$ it is proved that

$$\big[A,[B,C]\big] \subset \big[[A,B],C\big] + \big[[A,C],B\big]$$

(Proposition 1.6). These two statements are well-known for the case of groups. Applying these results we prove that for the objects G_n, $n > 1$, in the definition of central series of groups with action from $\overline{\mathbb{G}\mathrm{r}}$ we have $G_n = [G_{n-1}, G]$. From this fact we deduce that for the objects $\overline{G}_n = G_n/G_{n+1}$, $G \in \mathbb{A}\mathrm{b}^c$, we have only those identities which are inherited from the identities of G by identifying the elements $\overline{x^{\,y}} = \overline{x}$, where $x \in G_n$, $y \in G$, x^y denotes an action, and $\overline{x}$ denotes the corresponding class in $\overline{G}_n$.

In Section 2 we construct free objects in the categories $\mathbb{G}\mathrm{r}^\bullet$ (resp. in $\mathbb{G}\mathrm{r}^c$, $\mathbb{A}\mathrm{b}^c$, $\overline{\mathbb{G}\mathrm{r}}$) and $\mathbb{L}$eibniz.

In Section 3 we discuss questions concerning identities in $\mathbb{G}\mathrm{r}^\bullet$ between round and square brackets. We consider a certain set of possible identities in $\mathbb{G}\mathrm{r}^\bullet$; easy computations show that none of them is true in $\mathbb{G}\mathrm{r}^\bullet$ (even in $\mathbb{G}\mathrm{r}^c$). Nevertheless we cannot claim that there are no more identities between round and square brackets in $\mathbb{G}\mathrm{r}^\bullet$ or in $\overline{\mathbb{G}\mathrm{r}}$. We denote the possible set of identities in $\overline{\mathbb{G}\mathrm{r}}$ by E and the corresponding set of identities in $\mathbb{LL}$, inherited from E due to the functor $\overline{LL} = LL\big|_{\overline{\mathbb{G}\mathrm{r}}} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{LL}$, by $\overline{E}$; we define the full subcategory $\overline{\mathbb{LL}} \subset \mathbb{LL}$ of all those Lie–Leibniz algebras over $\mathbb{Z}$ which satisfy identities from $\overline{E}$. We prove that if G is a free object in $\overline{\mathbb{G}\mathrm{r}}$, then $\overline{LL}(G)$ is a free object in $\overline{\mathbb{LL}}$ (Theorem 3.2). In the same way, applying Proposition 1.13 it is proved that the functor $L : \mathbb{A}\mathrm{b}^c \longrightarrow \mathbb{L}$eibniz preserves the freeness of objects (Theorem 3.3). As a corollary we also obtain that the composition $S_2 \cdot \overline{LL} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{L}$eibniz corresponds to free objects in $\overline{\mathbb{G}\mathrm{r}}$ free Leibniz algebras over $\mathbb{Z}$. Of course, it would be simpler to prove the commutator properties and Proposition 1.11 for $\mathbb{A}\mathrm{b}^c$, then to show that the functor L preserves freeness, and since the abelianization functor $A : \mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{A}\mathrm{b}^c$ has the same property, the composition $LA : \mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{L}$eibniz would preserve freeness, too. Nevertheless we think that the general Lie–Leibniz case is interesting and that under Conditions 2, 3 we can show the properties of commutators in $\overline{\mathbb{G}\mathrm{r}}$, prove Proposition 1.11 and that the functor $\overline{LL} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \overline{\mathbb{LL}}$ takes free objects to free objects, from which fact we easily deduce the corresponding result for Leibniz algebras.

1. The Categories $\mathbb{G}\mathrm{r}^c$ and $\overline{\mathbb{G}\mathrm{r}}$ and Some Properties of Commutators for Groups with Action

We recall from [2] the definitions of the categories $\mathbb{G}\mathrm{r}^\bullet$, $\mathbb{G}\mathrm{r}^c$, the notions of an ideal, commutator and central series of objects of $\mathbb{G}\mathrm{r}^\bullet$, the construction of the functor $LL : \mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{LL}$. Further we introduce the category $\overline{\mathbb{G}\mathrm{r}}$ and prove certain properties of commutators of objects of $\overline{\mathbb{G}\mathrm{r}}$, which are known for the case of groups.

Let $\mathbb{G}\mathrm{r}^\bullet$ be the category of groups with action on itself from the right side [2]. Thus the objects of $\mathbb{G}\mathrm{r}^\bullet$ are groups with the additional binary operation $\varepsilon : G \times G \to G$ with

$$\begin{aligned}
&\varepsilon(g, g' + g'') = \varepsilon(\varepsilon(g, g'), g''),\\
&\varepsilon(g, 0) = g,\\
&\varepsilon(g' + g'', g) = \varepsilon(g', g) + \varepsilon(g'', g),\\
&\varepsilon(0, g) = 0
\end{aligned}$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, $g, h \in G$. We denote the group operation additively, nevertheless the group is not commutative in general. A morphism $(G, \varepsilon) \to (G', \varepsilon')$ is a group homomorphism $\varphi : G \to G'$ with $\varphi(g^h) = \varphi(g)^{\varphi(h)}$. Let $\mathbb{G}\mathrm{r}^c$ be the full subcategory of $\mathbb{G}\mathrm{r}^\bullet$ of those objects which satisfy

Condition 1. For each $x, y, z \in G$, $G \in \mathbb{G}\mathrm{r}^\bullet$

$$x - x^{(z^x)} + x^{y+z^x} - x + x^z - x^{z+y^z} = 0.$$

Define $[g, h] = -g + g^h$, for $g, h \in G$, $g \in \mathbb{G}\mathrm{r}^\bullet$.

Condition 1 is equivalent to

Condition 1′.

$$\big[x^y, [y, z]\big] = \big[[x, y], z^x\big] + \big[-[x, z], y^z\big], \quad x, y, z \in G.$$

In [2] we introduced the notion of an ideal in $\mathbb{G}\mathrm{r}^\bullet$:

A nonempty set A of $G \in \mathbb{G}\mathrm{r}^\bullet$ is called an ideal of G if it satisfies the following conditions:

1. A is a normal subgroup of G as a group;
2. $a^g \in A$, for $a \in A$, $g \in G$;
3. $-g + g^a \in A$, for $a \in A$ and $g \in G$.

Let A and B be subobjects of G, $G \in \mathbb{G}\mathrm{r}^\bullet$. Denote by $\{A, B\}$ the subobject of G generated by A and B. By definition [2], the commutator $[A, B]$ is an ideal of $\{A, B\}$ generated by the elements

$$\big\{[a, b], [b, a], (a, b) \,|\, a \in A,\ b \in B\big\},$$

where $(a, b) = -a - b + a + b$.

Recall that an Ω-group is a group with a system of n-any algebraic operations Ω $(n \geq 1)$ which satisfies the condition

$$\underbrace{0\,0\,\ldots\,0}_{n}\,\omega = 0,$$

where 0 is the identity element of G and $\omega \in \Omega$ is an n-any operation [4]. For the original theory of Ω-groups see [3].

It is proved in [2] that for an object $G \in \mathbb{G}\mathrm{r}^{\bullet}$ considered as an Ω-group, where Ω consists of one binary operation of action, the definitions of a commutator and an ideal are equivalent to the corresponding definitions for Ω-groups.

For the square bracket we have [2]

$$\begin{aligned} &[g, h_1 + h_2] = [g, h_1] + [g^{h_1}, h_2] = [g, h_2] + [g, h_1]^{h_2}; \\ &[g + g', h] = [g, h]^{\underline{g'}} + [g', h]; \quad [g, 0] = [0, g] = 0, \end{aligned} \tag{1.1}$$

where $x^{\underline{g}} = -g + x + g, \ \ x, g \in G$.

From (1.1) it follows that

$$[g, -h] = -[g^{-h}, h] = -[g, h]^{-h}; \quad [-g, h] = -[g, h]^{\underline{-g}}. \tag{1.2}$$

For the round bracket we have $(g, h) = -(h, g)$ and the analogous to (1.1) and (1.2) identities

$$\begin{aligned} &(g, h_1 + h_2) = (g, h_1) + (g^{\underline{h_1}}, h_2) = (g, h_2) + (g, h_1)^{\underline{h_2}}; \\ &(g + g', h) = (g, h)^{\underline{g'}} + (g', h); \\ &(g, 0) = (0, g) = 0, \end{aligned} \tag{1.3}$$

$$(g, -h) = -(g, h)^{\underline{-h}} \Leftrightarrow (-g, h) = -(g, h)^{\underline{-g}}. \tag{1.4}$$

These identities are well-known for groups (see, e.g., [9]) and are the special cases of (1.1) and (1.2).

Recall from [2] that the (lower) central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$$

of an object G of $\mathbb{G}\mathrm{r}^{\bullet}$ is defined inductively by

$$G_n = [G_1, G_{n-1}] + [G_2, G_{n-2}] + \cdots + [G_{n-1}, G_1],$$

where for the subobjects $A, B \subset G$, $[A, B]$ denotes the commutator and $A + B$ the subset of G defined by $A + B = \{a + b, \ a \in A, \ b \in B\}$ [2]. We have $[G_n, G_m] \subset G_{n+m}$ and it is proved (Proposition 3.2 [2]) that for each $n \geq 1$, G_{n+1} is an ideal of G_n.

Let k be a commutative ring with the unit. Recall [5] that a Leibniz algebra A over k is a k-module A equipped with a k-module homomorphism called a square bracket

$$[\ ,\ ] : A \otimes_k A \to A$$

satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for $x, y, z \in A$.

A Lie–Leibniz algebra over k is a k-module A together with two k-module homomorphisms

$$(\ ,\ ), [\ ,\ ] : A \otimes_k A \to A$$

called a round and a square bracket, respectively, such that $(x, x) = 0$ for $x \in A$ and both Jacobi and Leibniz identities hold [2].

Let $\mathbb{L}$ie, $\mathbb{L}$eibniz and $\mathbb{LL}$ denote the categories of Lie, Leibniz and Lie–Leibniz algebras over the ring of integers $\mathbb{Z}$, respectively. The functor $LL : \mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{LL}$ is defined on objects as

$$LL(G) = G_1/G_2 \oplus G_2/G_3 \oplus \cdots \oplus G_n/G_{n+1} \oplus \cdots .$$

Here each G_i/G_{i+1} is an abelian group and $\oplus$ denotes the direct sum of abelian groups. The round and square bracket operations

$$(\ ,\ ),\ \ [\ ,\ ] : LL(G) \otimes L(G) \to LL(G)$$

on $LL(G)$ are induced by the maps

$$(\ ,\ )_{mn},\ \ [\ ,\ ]_{mn} : G_m \times G_n \to G_{m+n}$$

defined by the round and square brackets in G, respectively:

$$\begin{aligned} x, y &\mapsto (x, y), \\ x, y &\mapsto [x, y] \end{aligned}$$

(see Theorem 3.6 of [2]).

We have the following diagram

$$\begin{tikzcd}
\mathbb{A}\mathrm{b}^c \arrow[r, shift left, "E"] \arrow[d, "L"'] & \mathbb{G}\mathrm{r}^c \arrow[l, shift left, "A"] \arrow[r, shift left=3, "Q_1"] \arrow[r, shift right=1, "Q_2"] \arrow[d, "LL"] & \mathbb{G}\mathrm{r} \arrow[l, shift right=1, "T"'] \arrow[l, shift left=3, "C"] \arrow[d, "W"] \\
\mathbb{L}\mathrm{eibniz} \arrow[r, shift left, "E_2"] & \mathbb{LL} \arrow[l, shift left, "S_1"] \arrow[r, shift right, "S_2"'] & \mathbb{L}\mathrm{ie}\,, \arrow[l, shift right, "E_1"']
\end{tikzcd} \tag{1.5}$$

where $\mathbb{A}\mathrm{b}^c$ is the category of abelian groups with action on itself satisfying Condition 1, A is the abelianization functor, which is left adjoint to the full embedding functor E. The functors S_1 and S_2 make square and round brackets trivial, respectively, and are left adjoints to the full embedding functors E_1 and E_2, respectively. The functor L is constructed analogously to the functor LL, we can write $L = LL\big|_{\mathbb{A}\mathrm{b}^c}$ and W is the functor defined by Witt's theorem [9], [10], the functor Q_1 makes the action of the group trivial and is left adjoint to the functor T which considers a group as a group with trivial action on itself; the functor Q_2 makes the action of the group as an action by conjugation (i.e., $Q_2(G)$ is the quotient of G by the equivalence relation generated by the relation $g^h \sim -h + g + h$, $g, h \in G$, $G \in \mathbb{G}\mathrm{r}^c$), and C is the functor which considers a group as a group with the action by conjugation on itself. The functors Q_1 and Q_2 are left adjoints to T and C, respectively. We have $LL \circ T = E_1 \circ W$, $E_2 \circ L = LL \circ E$ [2]. We will return to this diagram in Section 3.

For the case of groups it is proved that if A and B are normal subgroups of G, then the commutator (A, B) is also a normal subgroup of G. Below we will show that the analogous statement is true for a certain type of groups with action on itself.

Condition 2. $\big[x^y,(y,z)\big]=\big[(x,y),z^x\big]+\big[-(x,z),y^z\big]$.

Condition 3. $\big(x^y,[y,z]\big)=\big([x,y],z^x\big)+\big(-[x,z],y^z\big)$.

In Section 3 we will see that the objects of $\mathbb{G}\mathrm{r}^c$ do not generally satisfy these conditions. Note that for groups with trivial action on itself, or with action by conjugation, Conditions 1′, 2 and 3 are always satisfied. The same is true for the example $\mathbb{Z}^\bullet$ from [2]; recall that $\mathbb{Z}^\bullet$ is the abelian group of integers with the action on itself $x^y=(-1)^y x$. Thus the round bracket is zero in this case. For any set X, consider a free object F_X on the set X in the category $\mathbb{G}\mathrm{r}^\bullet$ (see Section 2 for the construction of free objects in this category). Let $F_X/\sim$ be the quotient object, where $\sim$ is the minimal equivalence relation generated by the relations expressed in Conditions 1′, 2 and 3. Then $F_X/\sim$ is an object of $\mathbb{G}\mathrm{r}^c$ which satisfies the above two conditions.

Denote by $\overline{\mathbb{G}\mathrm{r}}$ the full subcategory of $\mathbb{G}\mathrm{r}^\bullet$ of those objects which satisfy Conditions 1′, 2 and 3. Thus $\overline{\mathbb{G}\mathrm{r}}$ is the full subcategory of $\mathbb{G}\mathrm{r}^c$.

Since groups with action are Ω-groups, $[A,B]$ is an ideal of G if and only if $[[A,B],G]\subseteq[A,B]$ (see [4] or Proposition 2.12 of [2]).

Now we are going to prove the statements concerning some properties of elements of $[A,B]$, $\{A,B\}$ and G, where A and B are ideals of G. These statements readily imply that $[A,B]$ is an ideal of G if A and B are ideals of G and $G\in\overline{\mathbb{G}\mathrm{r}}$. Note that in this case $\{A,B\}=A+B$ and this object is also an ideal of G (Proposition 2.5 of [2]).

Below for $g,h\in G$, $g^{\underline{h}}=-h+g+h$.

Lemma 1.1. *Let $a,b,g\in\mathbb{G}\mathrm{r}^\bullet$. Then we have*

(i) $\big(a^{\underline{g}}\big)^b=\big(a^b\big)^{\underline{(g^b)}}$;

(ii) $\big(a^b\big)^{\underline{g}}=\big(a^{\underline{g^{(-b)}}}\big)^b$.

The proof is an easy computation of both sides.

Lemma 1.2. *Let A and B be ideals of $G\in\overline{\mathbb{G}\mathrm{r}}$. Then for any $a\in A$, $b\in B$, $g\in G$ the elements*

$$[a,b]^{\underline{g}},[b,a]^{\underline{g}},(a,b)^{\underline{g}},[a,b]^g,[b,a]^g,$$
$$(a,b)^g,\big[g,[a,b]\big],\big[g,[b,a]\big],\big[g,(a,b)\big]$$

belong to $[A,B]$.

Proof. We have

$$\begin{aligned}[a,b]^{\underline{g}}&=-g+[a,b]+g=-g-a+a^b+g=-g-a+g+(a^b)^{\underline{g}}\\&=-g-a+g+\big(a^{\underline{g^{(-b)}}}\big)^b=-g-a+g+a^{\underline{g^{(-b)}}}-a^{\underline{g^{(-b)}}}\\&\quad+\big(a^{\underline{g^{(-b)}}}\big)^b=-g-a+g-g^{(-b)}+a+g^{(-b)}+\big[a^{\underline{g^{(-b)}}},b\big]\end{aligned}$$

$$
\begin{aligned}
&= - g - a + g - g^{(-b)} + a + g^{(-b)} - g + g + \big[a^{\underline{g^{(-b)}}}, b\big] \\
&= (a, b')^{\underline{g}} + \big[a^{\underline{g^{(-b)}}}, b\big],
\end{aligned}
$$

where $b' = g - g^{(-b)} \in B$, since B is an ideal, which proves that $[a, b]^{\underline{g}} \in [A, B]$.

We have $[b, a]^{\underline{g}} \in [A, B]$, since $[b, a]^{\underline{g}} \in [B, A]$ by the above-given proof and $[B, A] = [A, B]$ [2],

$$(a, b)^{\underline{g}} \in [A, B], \quad \text{since} \quad (a, b)^{\underline{g}} = (a^{\underline{g}}, b^{\underline{g}}).$$

For the next element we have

$$[a, b]^g = -a^g + a^{b+g} = -a^g + a^{g+b'} = [a^g, b'] \in [A, B],$$

where $b' = -g + b + g \in B$, here we apply the fact that B is an ideal of G.

From the previous result and from $[B, A] = [A, B]$ it follows that $[b, a]^g \in [A, B]$.

It is easy to see that

$$(a, b)^g = (a^g, b^g) \in [A, B].$$

For the element $[g, [a, b]]$ we apply Condition $1'$:

$$\big[g, [a, b]\big] = \big[(g^{-a})^a, [a, b]\big] = \big[[g^{-a}, a], b^{(g^{-a})}\big] + \big[-[g^{-a}, b], a^b\big].$$

This element is from $[A, B]$, since A and B are ideals of G and $[A, B] = [B, A]$.

From the previous result it follows that $[g, [b, a]] \in [A, B]$. In the same way applying Condition 2, we prove that $[g, [a, b]] \in [A, B]$. □

Remark. We do not need to check that elements of the type (g, t) belong to $[A, B]$, where t is a generator of $[A, B]$, since

$$(g, t) \in [A, B] \Leftrightarrow (t, g) \in [A, B] \Leftrightarrow t^{\underline{g}} \in [A, B].$$

The latter inclusion has been considered in Lemma 1.2.

Lemma 1.3. *Let A, B be ideals of G, $G \in \mathbb{Gr}^c$. For $g \in G$, $t, t_i \in [A, B]$, $i = 1, 2$*

(a) *If* $[g, t_i] \in [A, B]$, $i = 1, 2$, *then* $[g, t_1 + t_2] \in [A, B]$.
(b) *If* $[t_i, g] \in [A, B]$, $i = 1, 2$, *then* $[t_1 + t_2, g] \in [A, B]$.
(c) *If* $[g, t] \in [A, B]$, *then* $[g, -t] \in [A, B]$.

The proof follows from (1.1) and (1.2).

Lemma 1.4. *Let A and B be ideals of G, $G \in \overline{\mathbb{Gr}}$. If for $t \in [A, B]$ and any $g \in G$ we have $t^g, t^{\underline{g}}, [g, t] \in [A, B]$, then for any $g_1 \in \{A, B\}$ the following elements*

$$
\begin{gathered}
(t^{g_1})^g, \quad (t^{\underline{g_1}})^g, \quad [g_1, t]^g, \\
(t^{g_1})^{\underline{g}}, \quad (t^{\underline{g_1}})^{\underline{g}}, \quad [g_1, t]^{\underline{g}}, \\
[g, t^{g_1}], \quad [g, t^{\underline{g_1}}], \quad \big[g, [g_1, t]\big]
\end{gathered}
$$

belong to $[A, B]$.

Proof. It is obvious that $(t^{g_1})^g$, $(t^{\underline{g_1}})^{\underline{g}} \in [A,B]$. By Lemma 1.1, for the elements $(t^{g_1})^{\underline{g}}$, $(t^{\underline{g_1}})^g$ we have $(t^{g_1})^{\underline{g}} = (t^{\underline{g^{(-g_1)}}})^{g_1} \in [A,B]$. Since $\{A,B\} = A+B$ is an ideal, $(t^{\underline{g_1}})^g = (t^g)^{\underline{(g_1^g)}} \in [A,B]$, and therefore $g_1^g \in \{A,B\}$.

For the element $[g_1,t]^g$ we have

$$[g_1,t]^g = -g_1^g + g_1^{t+g} = -g_1^g + g_1^{g+t'} = [g_1^g, t'] \in [A,B],$$

where $t' = -g+t+g \in [A,B]$, and $g_1^g \in \{A,B\}$.

For the element $[g_1,t]^{\underline{g}}$ we will show that $([g_1,t],g) \in [A,B]$, from which it follows that $[g_1,t]^{\underline{g}} \in [A,B]$. Applying Condition 3, we obtain

$$\big([g_1,t],(g^{-g_1})^{g_1}\big) = \big(g_1^t,[t,g^{-g_1}]\big) - \big(-[g_1,g^{-g_1}],t^{g^{(-g_1)}}\big) \in [A,B].$$

For the element $[g,t^{g_1}]$ we show that $[g,[t,g_1]] \in [A,B]$, from which, by (1.1), it follows that $[g,t^{g_1}] + [g,-t]^{g_1} \in [A,B]$. Since $[g,t] \in [A,B] \Rightarrow [g,-t] \in [A,B] \Rightarrow [g,-t]^{t^{g_1}} \in [A,B]$, which implies that $[g,t^{g_1}] \in [A,B]$.

By Condition 1$'$ we have

$$\begin{aligned}\big[g,[t,g_1]\big] &= \big[[g^{-t},t],g_1^{(g^{-t})}\big] + \big[-[g^{-t},g_1],t^{g_1}\big]\\ &\in \big[[A,B,\{A,B\}\big] + \big[\{A,B\},[A,B]\big] \subset [A,B].\end{aligned}$$

For $[g,t^{\underline{g_1}}] \in [A,B]$ we show that $[g,(t,g_1)] \in [A,B]$, which can be done analogously to the previous proof by applying Condition 2.

For the element $[g,[g_1,t]]$ we have

$$\begin{aligned}\big[g,[g_1,t]\big] &= \big[(g^{-g_1})^{g_1},[g_1,t]\big] = \big[[g^{-g_1},g_1],t^{(g^{-g_1})}\big] + \big[-[g^{-g_1},t],g_1\big]\\ &\in \big[\{A,B\},[A,B]\big] + \big[[A,B],\{A,B\}\big] \subset [A,B].\end{aligned}$$

□

Proposition 1.5. *Let A and B be ideals of $G \in \overline{\mathbb{G}\mathrm{r}}$. Then the commutator $[A,B]$ is also an ideal of G.*

Proof. By Lemmas 1.1–1.4 we have proved that the generators of $[A,B]$ (as an ideal of $\{A,B\}$) satisfy the conditions: t^g, $t^{\underline{g}}$, $[g,t] \in [A,B]$ for any $g \in G$, where t is any generator of $[A,B]$ (Lemma 1.2), and from Lemmas 1.3, 1.4 it follows that if the generators satisfy these conditions, then any element of $[A,B]$ satisfies the same conditions, which is a necessary and sufficient condition for $[A,B]$ to be an ideal of G, which proves the proposition. □

Remark. From the above-proven lemmas we obtain $[[A,B],C] \subset [A,B]$ which is a necessary and sufficient condition for $[A,B]$ to be an ideal of G [2], [4], and this is another similar way to prove Proposition 1.5 by applying the same lemmas.

If A,B,C are normal subgroups of a group G, we have

$$\big(A,(B,C)\big) \subset \big(B,(C,A)\big) + \big(C,(A,B)\big), \tag{1.6}$$

where (A,B) denotes the commutator subgroup of G (see, e.g., [9]).

For groups with action on itself the analogous inclusion for square brackets does not hold in general for the ideals A,B,C of G, when $G \in \mathbb{G}\mathrm{r}^\bullet$, nor in the case when G satisfies the Condition 1$'$ (i.e., $G \in \mathbb{G}\mathrm{r}^c$).

Proposition 1.6. *Let $G \in \overline{\mathbb{G}\mathrm{r}}$ and A, B, C be ideals of G. Then we have*

$$\big[A, [B, C]\big] \subset \big[[A, B], C\big] + \big[[A, C], B\big].$$

For the case of groups this result gives (1.6). We have formulated the right side of the inclusion in this form, since it is more convenient for the proof using Conditions 1′, 2, 3. We need several lemmas. For simplicity, denote

$$D_{A,B,C} = \big[[A, B], C\big] + \big[[A, C], B\big].$$

By Proposition 1.5, $[A, [B, C]]$ and D are ideals of G, therefore it is sufficient to prove that the generators of $[A, [B, C]]$ (as an ideal of $\{A, [B, C]\}$) belong to D. By the definition of a commutator, $[A, [B, C]]$ is an ideal of $\{A, [B, C]\}$ generated by the elements

$$\big\{[a, t], [t, a], (a, t) \,|\, a \in A,\ t \in [B, C]\big\}.$$

The commutator $[B, C]$ itself is an ideal of $\{B, C\}$ generated by the elements

$$\big\{[b, c], [c, b], (b, c) \,|\, b \in B,\ c \in C\big\},$$

and we have $\{B, C\} = B + C$, since B and C are ideals of G.

Lemma 1.7. *Let A, B and C be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$. For $a \in A$, $b \in B$, $c \in C$ the elements*

$$\big[a, [b, c]\big],\ \big[a, [c, b]\big],\ \big[a, (b, c)\big],\ \big[[b, c], a\big],$$
$$\big[[c, b], a\big],\ \big[(b, c), a\big],\ \big(a, [b, c]\big),\ \big(a, [c, b]\big),\ \big(a, (b, c)\big)$$

belong to D_{ABC}.

Proof. For the first element we apply Condition 1′. We have

$$\big[a, [b, c]\big] = \big[(a^{-b})^b, [b, c]\big] = \big[(a^{-b}, b), c^{a^{(-b)}}\big] + \big[-[(a, c), b^c\big] \in D_{ABC}.$$

For the next element we apply the first result and we have $[a, [c, b]] \in D_{ACB} = D_{ABC}$.

In the same way, applying Conditions 2, 3 and also the corresponding Witt–Hall identity for commutators in groups, we prove that all elements given in the lemma belong to D. □

Lemma 1.8. *Let A, B and C be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$, and $t_i \in [B, C]$, $i = 1, 2$.*

If $(a, t_i) \in D_{ABC}$, $i = 1, 2$ for any $a \in A$, then

$$(a, t_1 + t_2) \in D_{ABC}.$$

If $[a, t_i] \in D_{ABC}$, $i = 1, 2$ for any $a \in A$, then

$$[a, t_1 + t_2] \in D_{ABC}.$$

If $[t_i, a] \in D_{ABC}$, $i = 1, 2$ for any $a \in A$, then

$$[t_1 + t_2, a] \in D_{ABC}.$$

The proof follows from (1.1) and (1.3) and the fact that D is an ideal of G.

Lemma 1.9. *For any ideal I of G, $G \in \mathbb{G}\mathrm{r}^{\bullet}$ and elements $g, h \in G$,*
If $[g,h] \in I$, then $[-g,h]$, $[g,-h] \in I$.
If $(g,h) \in I$, then $(-g,h)$, $(g,-h) \in I$.

The proof follows from (1.2) and (1.4) and the fact that I is an ideal of G.

Lemma 1.10. *Let A, B and C be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$. For any $t \in [B,C]$, any $a \in A$ and any $x \in \{B,C\}$ we have:*
(a) $[a,t] \in D_{ABC}$, *then* $[a,t^x] \in D_{ABC}$.
(b) $[a,t] \in D_{ABC}$, *then* $[a,t^{\underline{x}}] \in D_{ABC}$.
(c) $[a,[x,t]] \in D_{ABC}$.
(a′) *If* $(a,t) \in D_{ABC}$, *then* $(a,t^x) \in D_{ABC}$.
(b′) *If* $(a,t) \in D_{ABC}$, *then* $(a,t^{\underline{x}}) \in D_{ABC}$.
(c′) $(a,[x,t]) \in D_{ABC}$.
(a″) *If* $[t,a] \in D_{ABC}$, *then* $[t^x,a] \in D_{ABC}$.
(b″) *If* $[t,a] \in D_{ABC}$, *then* $[t^{\underline{x}},a] \in D_{ABC}$.
(c″) $[[x,t],a] \in D_{ABC}$.

Proof. We will show that $[a,[t,x]] \in D_{ABC}$, from which it follows that $[a,t^x] \in D_{ABC}$. Since B and C are ideals of G, $\{B,C\} = B+C$, any element $x \in \{B,C\}$ has the form $x = b+c$, $b \in B$, $c \in C$. We have

$$\big[a,[t,b+c]\big] = \big[a,[t,b]+[t^b,c]\big] = \big[a,[t,b]\big] + \big[a^{[t,b]},[t^b,c]\big].$$

By Proposition 1.5, $[B,C]$ is an ideal of G. By Lemma 1.7 applied for $A, [B,C], B$ and A, $[B,C]$, C we obtain

$$\big[a,[t,b+c]\big] \subset D_{A,[B,C],B} + D_{A,[B,C],C} \subset D_{A,C,B} + D_{A,B,C} = D_{A,B,C},$$

since $[B,C] \subset C$, $[B,C] \subset B$ (since B and C are ideals of G) and $D_{ACB} = D_{ABC}$. We have

$$\big[a,[t,x]\big] = [a,-t+t^x] = [a,t^x] + [a,-t]^{(t^x)}.$$

Since $[a,t] \in D$, by Lemma 1.9 $[a,-t] \in D$, and since D is an ideal of G, $[a,-t]^{(t^x)} \in G$. This proves that $[a,t^x] \in D_{ABC}$.

(b) is proved in an analogous way; we prove first that $[a,(t,x)] \in D_{ABC}$ for any $a \in A$, $t \in [B,C]$, $x \in \{B,C\}$, from which it follows that $[a,t^{\underline{x}}] \in D_{ABC}$.

(c) Since $x = b+c$, for $b \in B$, $c \in C$, we have

$$\big[a,[x,t]\big] = \big[a,[b+c,t]\big] = \big[a,[b,t]^{\underline{c}}+[c,t]\big] = \big[a,[b,t]^{\underline{c}}\big] + \big[a^{[b,t]^{\underline{c}}},[c,t]\big].$$

In the same way as in (a), applying Lemma 1.7 we can prove that $[a,[b,t]] \in D_{AB[B,C]} \subset D_{ABC}$ and $[a^{[b,t]^{\underline{c}}},[c,t]] \subset D_{AC[B,C]} \subset D_{ACB} \subset D_{ABC}$. By (b) we have $[a,[b,t]^{\underline{c}}] \subset D_{ABC}$, since $[b,t] \in [B,[B,C]] \subset [B,C]$ and $c \in \{B,C\}$.

(a′), (b′), (c′) are proved in a similar way.

For (a″) we first show that $[[t,x],a] \in D_{ABC}$. We have

$$\big[[t,x],a\big] = \big[[t,b+c],a\big] = \big[[t,b]+[t^b,c],a\big] = \big[[t,b],a\big]^{\underline{[t^b,c]}} + \big[[t^b,c],a\big].$$

Applying Lemma 1.7, we show that $[[t,b],a] \in D_{AB[B,C]} \subset D_{ABC}$ and since D_{ABC} is an ideal of G, we have $[[t,b],a]^{\underline{[t^b,c]}} \in D_{ABC}$.

Next, we show by Lemma 1.7 applied for $t^b \in [B,C]$, $c \in C$, $a \in A$, that the element $[[t^b, c], a]$ from $[A, [[B,C], C]]$ is included in $D_{A[B,C]C}$ and hence in D_{ABC}, since B is an ideal of G and $[B,C] \subset B$.

Applying Lemma 1.9, from $[[t,x],a] \in D_{ABC}$ it follows that $[t^x, a] \in D_{ABC}$.

(b″) We begin with proving that $[(t,x),a] \in D_{ABC}$. We have

$$\big[(t, b+c), a\big] = \big[(t,c) + (t,b)^{\underline{c}}, a\big] = \big[(t,c),a\big]^{\underline{(t,b)^{\underline{c}}}} + \big[(t,b)^{\underline{c}}, a\big].$$

Again by Lemma 1.7 $[(t,c),a] \in D_{A[B,C]C} \subset D_{ABC}$, from which $[(t,c),a]^{\underline{(t,b)^c}} \in D_{ABC}$.

For the second summand we have

$$\big[(t,b)^{\underline{c}}, a\big] = \big[(t^c, b^c), a\big] \in D_{A[B,C]B} \subset D_{ACB} = D_{ABC};$$

hence $[(t,x),a] \in D_{ABC}$.

We have

$$\big[(t,x),a\big] = \big[-t + t^{\underline{x}}, a\big] = \big[-t, a\big]^{\underline{t^{\underline{x}}}} + \big[t^{\underline{x}}, a\big].$$

By Lemma 1.9, $[-t,a] \in D_{ABC}$ and therefore $[-t,a]^{\underline{t^{\underline{x}}}} \in D_{ABC}$, from which $[t^{\underline{x}}, a] \in D_{ABC}$.

(c″) We have $\big[[x,t],a\big] = \big[[c+b,t],a\big] = \big[[c,t]^{\underline{b}} + [b,t], a\big] = \big[[c,t]^{\underline{b}}, a\big]^{\underline{[b,t]}} + \big[[b,t],a\big]$.

By Lemma 1.7,

$$\big[[b,t],a\big] \subset D_{A[B,C]B} \subset D_{ABC}.$$

For the first summand we have $[c,t] \in \big[B,[B,C]\big] \subset [B,C]$; $\big[[c,t],a\big] \in D_{A[B,C]C} \subset D_{ABC}$ by Lemma 1.7. Thus for $t' = [c,t]$ we have $[t', a] \in D_{ABC}$. From (b″) we obtain

$$\big[(t')^{\underline{b}}, a\big] \in D_{ABC} \quad \text{since} \quad b \in \{B, C\},$$

and therefore

$$\big[[c,t]^{\underline{b}}, a\big]^{\underline{[b,t]}} \in D_{ABC},$$

since D_{ABC} is an ideal of G. This ends the proof of the lemma. ☐

The proof of Proposition 1.6 follows from Lemmas 1.7–1.10.

Lemma 1.11. *If $G \in \overline{\mathbb{G}\mathrm{r}}$, then for*

$$G_n = [G_1, G_{n-1}] + [G_2, G_{n-2}] + \cdots + G_{n-1}, G_1]$$

we have

$$G_n = [G_{n-1}, G], \tag{1.7}$$

for $n > 1$, where $G_1 = G$.

Proof. For $n = 2, 3$ (1.7) is trivial. For $n = 4$ we have

$$G_4 = [G_1, G_3] + [G_2, G_2] + [G_3, G_1].$$

Thus $[G_3, G_1] \subset G_4$, and for $G_4 \subset [G_3, G_1]$ we will show that $[G_2, G_2] \subset [G_3, G_1]$. We have

$$[G_2, G_2] = \big[[G_1, G_1], G_2\big] \subset \big[G_1, [G_1, G_2]\big] + \big[G_1, [G_1, G_2]\big] \subset [G_3, G_1],$$

since $[G_1, G_2] \subset G_3$.

Assume that (1.7) is true for any G_l, where $l < n$. For $l = n$ we have $[G_{n-1}, G_1] \subseteq G_n$. We have to show that

$$[G_k, G_{n-k}] \subseteq [G_{n-1}, G] \quad \text{for} \quad 1 \leq k < n. \tag{1.8}$$

For $k = 1$, $[G_1, G_{n-1}] = [G_{n-1}, G]$.

For $k = 2$, by Proposition 1.6, $[G_2, G_{n-2}] = [[G_1, G_1], G_{n-2}] \subset [G_1, [G_1, G_{n-2}]] + [G_1, [G_1, G_{n-2}]] = [G_{n-1}, G]$, since $[G_1, G_{n-2}] = G_{n-1}$ by our assumption.

Suppose that (1.8) is true for $1 \leq k \leq t-1$, where $t < n$. We will show (1.8) for $k = t$.

By our assumption, $G_t = [G_{t-1}, G]$; therefore

$$\begin{aligned}[G_t, G_{n-t}] &= \big[[G, G_{t-1}], G_{n-t}\big] \subset \big[G, [G_{t-1}, G_{n-t}\big] + \big[G_{t-1}, [G, G_{n-t}\big] \\ &\subset [G, G_{n-1}] + [G_{t-1}, G_{n-t+1}] \subset [G, G_{n-1}] + [G_{n-1}, G] = [G_{n-1}, G];\end{aligned}$$

here we have used the facts that $[G_{t-1}, G_{n-t}] \subset G_{n-1}$, $[G, G_{n-t}] \subset G_{n-t+1}$ and that, by our assumption, $[G_{t-1}, G_{n-t+1}] \subset [G_{n-1}, G]$,which proves the lemma. □

From this lemma the construction of the functor $\mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{LL}$ becomes simpler for the objects of $\overline{\mathbb{G}\mathrm{r}}$. Namely, if $G \in \overline{\mathbb{G}\mathrm{r}}$, then

$$LL(G) = \sum_{n=1}^{\infty} G_n \,/\, [G_n, G]. \tag{1.9}$$

Let G be a free object in $\overline{\mathbb{G}\mathrm{r}}$ (see Section 2 for the construction) and $G_n = [G_{n-1}, G]$, $n > 1$. Let E be the set of all defining identities between the brackets (both round and square) in G_n, $n \geq 1$, and $\overline{E}$ the set of all defining identities which satisfy the elements of the groups $\overline{G}_n = G_n/[G_n, G]$, $n \geq 1$. Under "defining identities" we mean that any identity in G follows from the identities from E.

Remark. We could define G_n from the beginning by (1.7), but we would need Propositions 1.5 and 1.6 for proving $[G_n, G_m] \subset G_{n+m}$, which we have applied in proving Theorem 3.6 [2].

If G is a free object in $\overline{\mathbb{G}\mathrm{r}}$, then we have Conditions $1'$,2,3 for the elements of G, but there can be more identities between the round, and round and square brackets. In the case of $\mathbb{A}\mathrm{b}^c$ we have another picture, the only identity we have in $\mathbb{A}\mathrm{b}^c$ is Condition $1'$ (and of course its consequences).

Let G be a free object of $\mathbb{A}\mathrm{b}^c$ and $g_1, \dots, g_k \in G$. Let $P(g_1, \dots, g_k)$ be any expression of the elements $g_i, i = 1, \dots, k$ and bracket operations in G.

We say that P is a pure n-bracket if after decomposing each g_i in terms of brackets it contains only n-brackets. Here we have in mind that $\mathbb{A}\mathrm{b}^\bullet \cong \mathbb{A}\mathrm{b}^{[\,]}$ and the corresponding isomorphism for $\mathbb{A}\mathrm{b}^c$. For example, for the basis elements x_1, x_2, x_3 of G, $[x_1, [x_2, x_3]]$ is a pure 3-bracket. If g is a pure m-bracket and h is a pure k-bracket, then $[g, h]$ is a pure $m + k$-bracket.

According to Condition $1'$, it may happen that a linear combination of pure n-brackets is an element of G_{n+1}.

Lemma 1.12. *Let G be a free object in $\mathbb{A}\mathrm{b}^c$. If $P(g_1,\dots,g_t) \in G_n$ is a linear combination of pure n-brackets in G and $P(g_1,\dots,g_t) \in G_{n+1}$, then $P(\overline{g_1},\dots,\overline{g_t}) = 0$ in $\overline{G}_n = G_n/G_{n+1}$ is either the Leibniz identity or its consequence.*

Proof. There exists an expression $Q() \in G_{n+1}$ with $n+1$ brackets such that $P() - Q() = 0$. Since G is free, $P() - Q() = 0$ is either equivalent to Condition $1'$ or to its consequence. Now the proof is a direct computation. Take x, y, z as pure k, l, m-brackets, respectively, in Condition $1'$, with $k + l + m = n$. Then from (1.1), (1,3) and the fact that $g^h = g + [g, h]$, for any $g, h \in G$ we obtain that the pure n-bracket combination part of Condition $1'$ has the form $[x,[y,z]]-[[x,y],z]+[[x,z],y]$. Note that in $\mathbb{A}\mathrm{b}^c$ we have $[-g,h] = -[g,h]$. The same result we have in the case $P() - Q() = 0$ is equivalent to a consequence of Condition $1'$, which ends the proof. $\square$

Proposition 1.13. *Let G be a free object in $\mathbb{A}\mathrm{b}^c$. Then the elements of the object $L(G)$ ($L : \mathbb{A}\mathrm{b}^c \longrightarrow \mathbb{L}\mathrm{eibniz}$) satisfy only the Leibniz algebra identities for square brackets i.e., the square bracket operation is bilinear and in $\overline{G}_n = G_n/G_{n+1}$, $n \geq 1$we have the Leibniz identity*

$$\big[\overline{x},[\overline{y},\overline{z}]\big] = \big[[\overline{x},\overline{y}],\overline{z}\big] - \big[[\overline{x},\overline{z}],\overline{y}\big],$$

where $x, y, z \in G$ and $\overline{x} \in \overline{G}_m$, $\overline{y} \in \overline{G}_l$, $\overline{z} \in \overline{G}_t$ denote the corresponding elements with $m + l + t = n$.

Proof. Suppose G is free in $\mathbb{A}\mathrm{b}^c$ and we have in $\overline{G}_n$ the identity or relation $P(\overline{x}_{ji}) = \sum_{j=1}^{l} P_j(\overline{x}_{j1},\dots,\overline{x}_{jt}) = 0$, where each P_j denotes a bracket element in P, Σ denotes the sum of these elements in $\overline{G}_n$, $\overline{x}_{ij} \in \overline{G}_{kji}$, $k_{j1} + \cdots + k_{jt} = n$,$j = 1,\dots,l$. We suppose that each $\overline{x}_{ji} \neq 0$ and P contains at most n brackets. For each inverse image x'_{ji} in G_{kji}, $j = 1,\dots,l$, $i = 1,\dots,t$ (i.e., $x'_{ji} \mapsto \overline{x}_{ji}$, by the natural homomorphism $G_{kji} \to \overline{G}_{kji}$) we have $P(x'_{ji}) = \sum_{j=1}^{l} P_J(x'_{j1},\dots,x'_{jt}) \in G_{n+1}$. Since each $\overline{x}_{ji} \neq 0$, we have $x_{ji} \notin G_{kji+1}$, thus each x_{ji} contains kji brackets as a summand. Hence each $\overline{x}_{ji}$ has an inverse image $\widetilde{x}_{ji} \in G_{kji}$, $\widetilde{x}_{ji} \mapsto \overline{x}_{ji}$, and $\widetilde{x}_{ji}$ is a pure kji-bracket. We have $P(\widetilde{x}_{ji}) = \sum_{j=1}^{l} P_j(\widetilde{x}_{j1},\dots,\widetilde{x}_{jt}) \in G_{n+1}$, and each $P_j(\widetilde{x}_{j1},\dots,\widetilde{x}_{jt})$ is a pure n-bracket. $P(\overline{\widetilde{x}}_{ji}) = P(\overline{x}_{ji})$ and, by Lemma 1.12, $P(\overline{x}_{ji}) = 0$ is either the Leibniz identity or its consequence. $\square$

Remark. In $\mathbb{A}\mathrm{b}^\bullet$, Condition 1 has the form

$$-x^{(z^x)} + x^{y+z^x} + x^z - x^{z+y^z} = 0,$$

which is, of course, equivalent to Condition $1'$.

Direct computations show that in $\mathbb{A}\mathrm{b}^c$ we have the identities

$$[-g,h] = -[g,h],$$
$$[g,-h] = [-g,h]^{-h}, \qquad [g,h]^x = [g^x,h], \quad x,g,h \in G \in \mathbb{A}\mathrm{b}^c.$$

The first two identities could be obtained from the identities in $\mathbb{G}\mathrm{r}^c$

$$[-g,h]^{\underline{g}} = -[g,h],$$

$$[g,-h]^{-g}=[-g,h]^{-h}, \quad g,h\in G'\in \mathbb{G}\mathrm{r}^c,$$

applying the functor $A:\mathbb{G}\mathrm{r}^c\longrightarrow \mathbb{A}\mathrm{b}^c$. It is easy to see that these identities follow from (1.2) and all the above identities do not give new identities for $LL(G')$, or $L(G)$.

2. Free Objects in $\mathbb{G}\mathrm{r}^\bullet$, $\overline{\mathbb{G}\mathrm{r}}$, $\mathbb{A}\mathrm{b}^c$ and $\mathbb{L}$eibniz

In this section we give the construction of free objects in the categories of groups with action. We define free objects in Leibniz and Lie–Leibniz algebras and recall the free Lie algebra definition; we give the construction of free Leibniz algebras.

Let X be a set, M_X be the free magma generated by X. Recall (see, e.g., [1] or [9]) that a magma is a set M with a (generally nonassociative) binary operation

$$M\times M\longrightarrow M\,.$$

We write the elements of M_X in a "vertical way"; so the elements of M_X have the form

$$\begin{matrix} & \begin{pmatrix} & & x_{ti_t}\\ & \iddots & \\ \begin{pmatrix} \begin{pmatrix} & x_{t2}\\ x_{t1} & \end{pmatrix} & x_{t3} \end{pmatrix} & & \end{pmatrix} \\ \begin{pmatrix} & & \begin{pmatrix} & & x_{t-1,i_{t-1}}\\ & \iddots & \\ \begin{pmatrix} \begin{pmatrix} & x_{t-1,2}\\ x_{t-1,1} & \end{pmatrix} & x_{t-1,3} \end{pmatrix} & & \end{pmatrix} \\ & \iddots & \\ \begin{pmatrix} & \begin{pmatrix} & & x_{1i_1}\\ & \iddots & \\ \begin{pmatrix} \begin{pmatrix} & x_{12}\\ x_{11} & \end{pmatrix} & x_{13} \end{pmatrix} & & \end{pmatrix} \\ x & \end{pmatrix} & & \end{pmatrix} & \end{matrix} \tag{2.1}$$

where $x,\, x_{js}\in X$, $j=1,2,\dots,t$, $s=1,2,\dots i_j$.

We denote this kind of elements by x to indicate that the element (2.1) is presented by the element $x\in X$.

Let $\mathcal{F}(M_X)$ be a free group generated by M_X. The operation in $\mathcal{F}(M_X)$ we denote by "+", so the elements of $\mathcal{F}(M_X)$ have the form

$$\pm x_1 \pm x_2 \pm\cdots\pm x_n\,,$$

where x_i is an element of type (2.1) for each $i = 1, \dots, n$. The empty word (neutral element) of $\mathcal{F}(M_X)$ we denote by 0.

Define in $\mathcal{F}(M_X)$ the action of elements by

$$\begin{aligned}
&(x_1 + \dots + x_n)^{y_1 + \dots + y_m} \\
&\quad = \Bigg(\bigg(\Big((x_1)\Big)^{(y_1)}\bigg)^{(y_{m-1})}\Bigg)^{(y_m)} + \dots + \Bigg(\bigg(\Big((x_n)\Big)^{(y_1)}\bigg)^{(y_{m-1})}\Bigg)^{(y_m)}, \\
&(x_1 + \dots + x_n)^0 = x_1 + \dots + x_n, \quad 0^{(x_1 + \dots + x_n)} = 0.
\end{aligned}$$

Now it is easy to see that the following statement holds.

Proposition 2.1. *The object $\mathcal{F}(M_X)$ is a group with action on itself and it is the free object in $\mathbb{G}\mathrm{r}^\bullet$ generated by the set X.*

Let $\sim$ be a minimal equivalence relation on $\mathcal{F}(M_X)$ generated by the relation defined by Condition 1. Then we obtain

Proposition 2.2. *The object $\mathcal{F}(M_X)/\sim$ is the free object in $\mathbb{G}\mathrm{r}^c$ generated by the set X.*

In the same way we construct free objects in $\overline{\mathbb{G}\mathrm{r}}$ and $\mathbb{A}\mathrm{b}^c$.

On the other hand, in diagram (1.5) the functor A is left adjoint to the full embedding functor E and therefore we obtain

Proposition 2.3. *$A(\mathcal{F}(M_X)/\sim)$ is the free object in $\mathbb{A}\mathrm{b}^c$ generated by the set X.*

In the following definition all algebras are considered over a commutative ring k with the unit.

Definition 2.4. Let X be a set. A is a free Lie–Leibniz (respectively Lie, Leibniz) algebra with basis X if there is an injection $X \longrightarrow A$ and for any Lie–Leibniz (resp. Lie, Leibniz) algebra B and a map $\alpha : X \longrightarrow B$, there exists a unique homomorphism $\overline{\alpha} : A \longrightarrow B$ of Lie–Leibniz (resp. Lie, Leibniz) algebras such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\overline{\alpha}"] & B \\
X \arrow[u] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

is commutative.

Here we give a construction of free Leibniz algebras. Let k be a commutative ring with the unit and X be any set. Denote by $W(X)$ the set of all those formal combinations of square brackets and elements of X, which do not contain the words of the form $[a, [b, c]]$, where a, b, c are elements of X or combinations of elements of X and brackets. Let $F(W(X))$ be the free k-module generated by the set $W(X)$. Consider the map $\eta : W(X) \times W(X) \to F(W(X))$ defined by $\eta(w_1, w_2) = [w_1, w_2]$ if $[w_1, w_2] \in W(X)$; for $[w_1, w_2] \notin W(X)$ we decompose the word $[w_1, w_2]$ according to the Leibniz identity and express it as a sum of the words from $W(X)$ in $F(W(X))$. We define $\eta(w_1, w_2)$ as this final sum. Note that any two different decompositions give the same element of $F(W(X))$. We define the bracket operation on $F(W(X))$ as the k-linear extension of the map η to $F(W(X))$. It is easy to see that the obtained object is a free Leibniz algebra on the set X (cf. [8]).

3. Identities in $\mathbb{G}\mathrm{r}^c$ and the Main Results

In this section all algebras (Lie, Leibniz, Lie–Leibniz) are considered over the ring of integers $\mathbb{Z}$.

We investigate a question of the existence of identities between round and square brackets in $\mathbb{G}\mathrm{r}^\bullet$. If E is the set of identities for the category $\overline{\mathbb{G}\mathrm{r}}$, we define the full subcategory $\overline{\mathbb{LL}}$ of $\mathbb{LL}$ (Lie–Leibniz algebras) of those objects satisfying identities $\overline{E}$, where $\overline{E}$ denotes the set of all identities inherited in $\mathbb{LL}$ from E. We prove that if G is the free object in $\overline{\mathbb{G}\mathrm{r}}$ generated by the set X, then every element of $\overline{G}_n = G_n/G_{n+1}$ is represented as a combination of elements of the form

$$\Big[\Big(\cdots\big[\big(\cdots\big[\big(\overline{y}_k,\cdots[(\overline{y}_3,[(\overline{x},\overline{y}_1)],\overline{y}_2)]\ \big)\big],\dots\overline{y}_{n-1}\Big)\Big],$$

where two brackets mean that we have either a round or a square bracket for $x, y_1, \dots, y_{n-1} \in X$ and this representation is unique up to identities from $\overline{E}$. By this result we easily prove that the functor LL takes free objects from $\overline{\mathbb{G}\mathrm{r}}$ to free objects in $\overline{\mathbb{LL}}$ and $L(G)$ (resp. $LA(G')$) is a free Leibniz algebra if G (resp. G') is a free object in $\mathbb{A}\mathrm{b}^c$ (resp. in $\overline{\mathbb{G}\mathrm{r}}$). The category $\overline{\mathbb{G}\mathrm{r}}$ is defined in Section 1 as the full subcategory of those objects of $\mathbb{G}\mathrm{r}^\bullet$ which satisfy Conditions $1'$, 2, 3. We look for possible identities in $\overline{\mathbb{G}\mathrm{r}}$ between the round and square brackets. We have well-known Witt–Hall identities for round brackets in $\mathbb{G}\mathrm{r}$. By Witt's theorem [9], [10] the functor $W : \mathbb{G}\mathrm{r} \longrightarrow \mathbb{L}\mathrm{ie}$ in diagram (1.5) takes free objects to free objects. Taking into account the same kind of argument as we have at the end of Section 1 for the case of groups with action and Lie–Leibniz algebras, we conclude that in $\mathbb{G}\mathrm{r}$ we do not have such identities for the round brackets which "inherit" new identities for Lie algebras. Thus if new identities exist in $\mathbb{G}\mathrm{r}$ they give the same Jacobi identity, the identity $(x, x) = 0$ and bilinear property for the operation $(\ ,\ )$ in the corresponding Lie algebra. Below we consider in $\overline{\mathbb{G}\mathrm{r}}$ those "variations" of the well-known identities in $\mathbb{G}\mathrm{r}$ which by applying the usual functors (see diagram (1.5))

$$\mathbb{A}\mathrm{b}^c \xleftarrow{A} \mathbb{G}\mathrm{r}^c \xrightarrow{Q_2} \mathbb{G}\mathrm{r}$$

give the known identities in $\mathbb{A}\mathrm{b}^c$ and $\mathbb{G}\mathrm{r}$. As above, for $x, y \in G$, $G \in \mathbb{G}\mathrm{r}^\bullet$ we denote $x^{\underline{y}} = -y + x + y$. Consider the following expressions:

$$a_1 = \big[x^y, (y, z)\big]; \qquad b_1 = \big[(x, y), z^x\big]; \qquad c_1 = \big[-(x, z), y^z\big];$$

$$a_2 = \big(x^y, [y, z]\big); \qquad b_2 = \big([x, y], z^x\big); \qquad c_2 = \big(-[x, z], y^z\big);$$

$$a_3 = -\big[(y, z), x^y\big]; \qquad b_3 = -\big[z^x, (x, y)\big]; \qquad c_3 = -\big[y^z, -(x, z)\big];$$

$$
\begin{aligned}
a_4 &= \big(x^y, -[z,y]\big); & b_4 &= \big(-[y,x], z^x\big); & c_4 &= \big([z,x], y^z\big);\\
a_5 &= \big[x^{\underline{y}}, (y,z)\big]; & b_5 &= \big[(x,y), z^{\underline{x}}\big]; & c_5 &= \big[-(x,z), y^{\underline{z}}\big];\\
a_6 &= \big(x^{\underline{y}}, [y,z]\big); & b_6 &= \big([x,y], z^{\underline{x}}\big); & c_6 &= \big(-[x,z], y^{\underline{z}}\big);\\
a_7 &= -\big[(y,z), x^{\underline{y}}\big]; & b_7 &= -\big[z^{\underline{x}}, (x,y)\big]; & c_7 &= -\big[y^{\underline{z}}, -(x,z)\big];\\
a_8 &= \big(x^{\underline{y}}, -[z,y]\big); & b_8 &= \big(-[y,x], z^{\underline{x}}\big); & c_8 &= \big([z,x], y^{\underline{z}}\big).
\end{aligned}
$$

Consider all kinds of identities

$$a_i = b_j + c_k, \quad i,j,k = \overline{1,8}. \tag{3.1}$$

Applying the functor A or Q_2 to (3.1), we obtain that the resulting equalities are true in $\mathbb{A}\mathrm{b}^c$ and $\mathbb{G}\mathrm{r}$ (i.e., when $(\ \ ) = 0$ or $[\ \ ] = (\ \ )$).

Direct computations give:

$$
\begin{aligned}
a_1 &= -x^y + x^{-z+y+z};\\
a_2 &= -x^y - y^z + y + x^y - y + y^z;\\
a_3 &= -z^{(x^y)} - y^{(x^y)} + z^{(x^y)} + y^{(x^y)} - y - z + y + z;\\
a_4 &= -x^y - z + z^y + x^y - z^y + z;\\
a_5 &= -y - x + y - y^{-y-z+y+z} + x^{-y-z+y+z} + y^{-y-z+y+z};\\
a_6 &= -y - x + y - y^z + x + y^z;\\
a_7 &= -z^{-y+x+y} - y^{-y+x+y} + z^{-y+x+y} + y^{-y+x+y} - y - z + y + z;\\
a_8 &= -y - x + y - z + z^y - y + x + y - z^y + z;\\
b_1 &= -y - x + y + x - x^{(z^x)} - y^{(z^x)} + x^{(z^x)} + y^{(z^x)};\\
b_2 &= -x^y + x - z^x - x + x^y + z^x;\\
b_3 &= -z^{-y+x+y} + z^x;\\
b_4 &= -y + y^x - z^x - y^x + y + z^x;\\
b_5 &= -y - x + y + x - x^{-x+z+x} - y^{-x+z+x} + x^{-x+z+x} + y^{-x+z+x};\\
b_6 &= -x^y - z + x^y - x + z + x;\\
b_7 &= -x^{-x-y+x+y} - z^{-x-y+x+y} + x^{-x-y+x+y} - x + z + x;\\
b_8 &= -y + y^x - x - z + x - y^x + y - x + z + x;\\
c_1 &= -x - z + x + y - z^{(y^z)} - x^{(y^z)} + z^{(y^z)} + x^{(y^z)};\\
c_2 &= - - x + x^z - y^z - x^z + x + y^z;\\
c_3 &= -y^{-x+z+x} + y^z;\\
c_4 &= -z^x + z - y^z - z + z^x + y^z;\\
c_5 &= -x - z + x + z - z^{-z+y+z} - x^{-z+y+z} + z^{-z+y+z} + x^{-z+y+z};\\
c_6 &= -x + x^z - z - y + z - x^z + x - z + y + z;\\
c_7 &= -z^{-z-x+z+x} - y^{-z-x+z+x} + z^{-z-x+z+x} - z + y + z;\\
c_8 &= -z^x - y + z^x - z + y + z.
\end{aligned}
$$

The check shows that none of identities (3.1) holds for free objects in $\mathbb{G}\mathrm{r}^\bullet$. The same is true for the category $\mathbb{G}\mathrm{r}^c$, since Condition 1 represents any element x by a combination of elements with base element x, and therefore Condition 1 does not help any of identities (3.1) to hold in $\mathbb{G}\mathrm{r}^c$. Nevertheless we cannot claim that we do not have identities between round and square brackets in $\mathbb{G}\mathrm{r}^\bullet$ or in $\mathbb{G}\mathrm{r}^c$. The same situation is observed for $\overline{\mathbb{G}\mathrm{r}}$; by definition, here we have two identities from (3.1), these are Condition 2 and Condition 3 (for $i = j = k = 1$ and $i = j = k = 2$). Note also that we may have identities in $\mathbb{G}\mathrm{r}$ which do not give new identities for $W(F)$ (where F is a free group and $W(F)$ is the corresponding Lie algebra) but "variations" (with square brackets) of these identities in $\mathbb{G}\mathrm{r}^c$ (or in $\overline{\mathbb{G}\mathrm{r}}$) may give new identities in $LL(G)$, for a free object $G \in \mathbb{G}\mathrm{r}^c$, since, e.g., in $W(G)$ we have $(\overline{x}, \overline{x}) = 0$, but in $LL(G)$, $[\overline{x}, \overline{x}] \neq 0$, $x \in G$.

Let G be a free object in $\overline{\mathbb{G}\mathrm{r}}$. Let E be the set of all defining identities between both kinds of brackets in $\overline{\mathbb{G}\mathrm{r}}$, and let $\overline{E}$ be the set of corresponding identities for $LL(G)$, inherited from E.

Denote by $\overline{\mathbb{LL}}$ the full subcategory of $\mathbb{LL}$ consisting of those objects of $\mathbb{LL}$ which satisfy the conditions from $\overline{E}$. Of course, among the identities in $\overline{E}$ we have bilinear properties of $[\ ,\ ]$ and $(\ ,\ )$, the identities $(x, x) = 0$, $(x, 0) = (0, x) = 0$, $[x, 0] = [0, x] = 0$, the Jacobi identity

$$\big(x, (y, z)\big) + \big(y, (z, x)\big) + \big(z, (x, y)\big) = 0,$$

the Leibniz identity

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - \big[[x, z], y\big], \tag{3.2}$$

and also the identities

$$\big[x, (y, z)\big] = \big[(x, y), z\big] - \big[(x, z), y\big],$$
$$\big(x, [y, z]\big) = \big([x, y], z\big) - \big([x, z], y\big)$$

which correspond to the known identities for round and square brackets in $\mathbb{G}\mathrm{r}$ and $\mathbb{G}\mathrm{r}^\bullet$, respectively, Conditions $1'$, 2 and 3 in $\mathbb{G}\mathrm{r}^\bullet$.

For a free object $G \in \mathbb{A}\mathrm{b}^c$, E contains the usual identities (1.1) and only one additional identity, Condition $1'$; by virtue of Proposition 1.13 the set of all defining identities E (which satisfy the elements of $L(G)$) consists of identity (3.2), bilinear properties of square bracket operation and $[x, 0] = [0, x] = 0$. See also the remark after the proof of Proposition 1.13.

Proposition 3.1. *Let $G \in \overline{\mathbb{G}\mathrm{r}}$, $G_n = [G_{n-1}, G]$ for $n > 1$, where $G_1 = G$, and $\overline{G}_r = G_n/G_{n+1}$. If G is the free object in $\overline{\mathbb{G}\mathrm{r}}$ generated by the set X, then $\overline{G}_1$ is the free abelian group generated by the same set X and every element of $\overline{G}_n$, $n > 1$ has a representation as a combination of elements of the form*

$$\Big[\Big(\cdots\big[\big(\cdots\big[\big(\overline{y}_k\cdots\big[\big(\overline{y}_3, [(\overline{x}, \overline{y}_1)], \overline{y}_2)\big]\big)\big]\cdots\big)\big]\cdots\overline{y}_m\big]\cdots\overline{y}_{n-1}\Big)\Big] \tag{3.3}$$

($n-1$ round or square brackets), where $x, y_1, \ldots, y_{n-1} \in X$, and this representation is unique up to identities from $\overline{E}$.

Proof. It is obvious that $\overline{G}_1 = G_1/G_2$ is the free abelian group generated by the set X. We have $G_2 = [G_1, G]$, and, by definition, G_2 is an ideal of G generated by elements of the form $[(g,h)]$ (here we mean elements of both forms $[g,h]$ and (g,h)), $g,h \in G$. Since G is a free object in $\overline{\mathbb{G}\mathrm{r}}$, we have

$$g = x_{1} + \cdots + x_{n}\,, \quad h = y_1 + \cdots + y_k\,,$$

where $x_i, y_i \in X$, $i = \overline{1,n}$, $j = \overline{1,k}$. Then by (1.2) and (1.3) we obtain that $[(g,h)]$ has the form

$$[(g,h)] = \sum_{i,j} \left[\left(x_{i}\,, y_{j}\,\right)\right]^{-}; \tag{3.4}$$

here for $a \in G$, a^{-} means that the action operations represented by $\square$ include also actions by conjugation. Now we have to show that if $t, t_1, t_2 \in G_2$ and have form (3.4), then t^g, $t^{\underline{g}}$, $[g,t]$, $t_1 + t_2$ have the same form for $g \in G$. It is obvious that t^g, $t^{\underline{g}}$ and $t_1 + t_2$ have form (3.4). For $[g,t]$ we have the representation

$$[g,t] = \sum_{l,i,j} \left[z_{l}\,, \left[\left(x_{i}\,, y_{j}\,\right)\right]^{-}\right)\Big]^{-}.$$

If we open one bracket (square or round, as it is in the representation) in each summand

$$\begin{aligned} \left[x_{i}\,, y_{j}\,\right] &= -x_{i} + x_{i}^{(y_{j}\,)}, \\ \left(x_{i}\,, y_{i}\,\right) &= -x_{i} + x_{\underline{i}}^{(y_{j}\,)}, \end{aligned} \tag{3.5}$$

and then apply (1.2) and (1.3), we will see that $[(g,t)]$ has a representation of form (3.4). We have $\overline{[(g,t)]} = 0$ in $\overline{G}_2$, since $[(g,t)] \in G_3$, and this is also obvious from (3.5) and the fact that $\overline{x_i} = \overline{x_i^{(y_j^0)}}$ in $\overline{G}_2$ for $x_i \in G_2$. In the same way we prove that the elements of $G_3 = [G_2, G]$ have representations of the form

$$\sum \left[\left(z_{2}\,, \overset{\curvearrowleft\curvearrowright}{\left[\left(x_{i}\,, y_{i}\,\right)\right]^{-}}\right)\right]^{-},$$

where for $a, b \in G$, $[(\overset{\frown}{a,b})]$ denotes elements either of form $[(a,b)]$ or of form $[(b,a)]$.

Suppose that the elements of G_{n-1} can be represented as $\mathbb{Z}$-combinations of the elements of the form

$$\left[\left(\cdots\left[\left(\cdots\left[\left(y_{k} \cdots \left[\left(y_{3}\,, \left[\left(x\,\,, y_{1}\,\right)\right]\,\right)\right]^{-} \ldots\right)\right]^{-} \cdots y_{m}\right)\right]^{-} \cdots\,, y_{n-2}\right)\right]^{-}.$$

Then we obtain the corresponding result for G_n. These representations are unique up to identities from E. From this it follows that the elements of $\overline{G}_n$ are combinations with coefficients from $\mathbb{Z}$ of elements of form (3.3). Since $\overline{E}$ is the set of all identities in $L(G) = \sum\limits_{n=1}^{\infty} \overline{G}_n$, these representations of elements of $\overline{G}_n$ are unique up to identities from $\overline{E}$. $\square$

From Proposition 3.1 follows the main result.

Theorem 3.2. *Let G be the free object in $\overline{\mathbb{G}\mathrm{r}}$ generated by the set X. Then the Lie–Leibniz algebra $LL(G)$ is the free object in the category $\overline{\mathbb{LL}}$ with basis X.*

In the same way, applying Proposition 1.13 we obtain

Theorem 3.3. *Let G be the free object in $\mathbb{A}\mathrm{b}^c$ generated by the set X. Then $L(G)$ is the free Leibniz algebra on the set X.*

Corollary 3.4. *Any free Leibniz algebra can be obtained up to an isomorphism by the functor L; i.e., for any free Leibniz algebra A there is an object $G \in \mathbb{A}\mathrm{b}^c$ such that $L(G) \approx A$.*

Proof. Let A be the free Leibniz algebra on the set X. Take the free object G in $\mathbb{A}\mathrm{b}^c$ on the set X. Now, by Theorem 3.3 $L(G)$ is the free Leibniz algebra generated by the set X and therefore $L(G) \approx A$. $\square$

Consider $LL\big|_{\overline{\mathbb{G}\mathrm{r}}}$. It is obvious that $LL\big|_{\overline{\mathbb{G}\mathrm{r}}}$ factors through $\overline{\mathbb{LL}}$. Thus we have the commutative diagram

$$\begin{tikzcd}
\mathbb{G}\mathrm{r}^c \arrow[ddrr, "LL"'] & \overline{\mathbb{G}\mathrm{r}} \arrow[l, hook'] \arrow[r, "\overline{LL}"] \arrow[ddr, "LL/\overline{\mathbb{G}\mathrm{r}}"'] & \overline{\mathbb{LL}} \arrow[dd, "\mathrm{I}"] \\
& & \\
& & \mathbb{LL}\ .
\end{tikzcd}$$

Corollary 3.5. *Any free Leibniz algebra A can be considered as an object of $\overline{\mathbb{LL}}$, i.e., $E_2(A) \in \overline{\mathbb{LL}}$.*

Proof. It follows from Corollary 3.4 and the fact that $\mathbb{A}\mathrm{b}^c \hookrightarrow \overline{\mathbb{G}\mathrm{r}}$ and $E_2 \cdot L = LL\big|_{\mathbb{A}\mathrm{b}^c} = \overline{LL}\big|_{\mathbb{A}\mathrm{b}^c}$. $\square$

Corollary 3.6. *There is a full embedding functor $\overline{E_2} : \mathbb{L}\mathrm{eibniz} \longrightarrow \overline{\mathbb{LL}}$ such that $I\overline{E}_2 = E_2$; the functor $\overline{S}_2 = S_2 I$ is a left adjoint to $\overline{E}_2$.*

Proof. Let A be any Leibniz algebra, choose a free Leibniz algebra F_A on the basis A and an epimorphism $F_A \longrightarrow A$. We have $E_2(F_A) \in \overline{\mathbb{LL}}$ by Corollary 3.5 and $E_2(A) \in \mathbb{LL}$; from this it follows that the elements of A also satisfy identities from $\overline{E}$, thus $E_2(A) \in \overline{\mathbb{LL}}$, which means that there is a full embedding functor $\overline{E_2} : \mathbb{L}\mathrm{eibniz} \longrightarrow \overline{\mathbb{LL}}$ with $I\overline{E}_2 = E_2$. It is easy to see that $\overline{S}_2$ is a left adjoint to $\overline{E}_2$. $\square$

Applying Witt's theorem stating that the functor W takes free objects from $\mathbb{G}\mathrm{r}$ to free objects in $\mathbb{L}\mathrm{ie}$, we obtain the following results.

Corollary 3.7. *Any free Lie algebra can be obtained by the functor W.*

Corollary 3.8. *Any free Lie algebra A can be considered as an object of $\overline{\mathbb{LL}}$, i.e., $E_1(A) \in \overline{\mathbb{LL}}$.*

Corollary 3.9. *There is a full embedding functor $\overline{E}_1 : Lie \longrightarrow \overline{\mathbb{LL}}$ such that $I\overline{E}_1 = E_1$; the functor $\overline{S}_1 = S_1 I$ is a left adjoint to $\overline{E}_1$.*

Thus we have the diagram

$$
\begin{tikzcd}[column sep=large, row sep=large]
 & & \mathbb{G}\mathrm{r}^c \arrow[dll, bend right, "A"'] \arrow[dddd, bend right, "LL"'] \arrow[drr, bend left=50, "Q_1"] \arrow[drr, bend left=20, "Q_2"] & & \\
\mathbb{A}\mathrm{b}^c \arrow[urr, bend left, "E"] \arrow[rr, shift left, "\overline{E}"] \arrow[d, "L"'] & & \overline{\mathbb{G}\mathrm{r}} \arrow[u] \arrow[ll, shift left, "\overline{A}"] \arrow[rr, shift left=3, "\overline{Q}_1"] \arrow[rr, shift right=1, "\overline{Q}_2"] \arrow[d, "\overline{LL}"] & & \mathbb{G}\mathrm{r} \arrow[ull, bend right=35, "T"'] \arrow[ull, bend right=10, "C"'] \arrow[ll, shift right=-1, "\overline{T}"'] \arrow[ll, shift right=3, "\overline{C}"] \arrow[d, "W"] \\
\mathbb{L}\mathrm{eibniz} \arrow[rr, shift left, "\overline{E}_2"] \arrow[drr, shift left, "E_2"] & & \overline{\mathbb{L}\mathbb{L}} \arrow[ll, shift left, "\overline{S}_2"] \arrow[rr, shift right, "\overline{S}_1"'] \arrow[d, "I"] & & \mathbb{L}\mathrm{ie} \arrow[ll, shift right, "\overline{E}_1"'] \arrow[dll, shift right, "E_1"'] \\
 & & \mathbb{L}\mathbb{L} \arrow[ull, shift left, "S_2"] \arrow[urr, shift right, "S_1"'] & &
\end{tikzcd}
\qquad ,
\tag{3.6}
$$

where $\overline{A} = A\big|_{\overline{\mathbb{G}\mathrm{r}}}$, $\overline{E}$ is the obvious full embedding, (i.e., it is clear that E factors through $\overline{\mathbb{G}\mathrm{r}}$), $\overline{Q}_i = Q_i\big|_{\overline{\mathbb{G}\mathrm{r}}}$, $i = 1, 2$. Since Conditions 2 and 3 are satisfied for groups with trivial action or action by conjugation, it follows that T and C factor through $\overline{\mathbb{G}\mathrm{r}}$; this gives the functors $\overline{T}$ and $\overline{C}$.

Corollary 3.10. *For free objects in $\overline{\mathbb{G}\mathrm{r}}$ the left and right directional diagrams in* (3.6) *commute, i.e., if G is a free object in $\overline{\mathbb{G}\mathrm{r}}$, then*

$$
\begin{aligned}
L\,\overline{A}(G) &= \overline{S}_2 \cdot \overline{LL}(G) = S_2 \cdot LL\big|_{\overline{\mathbb{G}\mathrm{r}}}(G),\\
W\overline{Q}_1 &= \overline{S}_1 \cdot \overline{LL}(G) = S_1 \cdot LL\big|_{\overline{\mathbb{G}\mathrm{r}}}(G).
\end{aligned}
$$

It may be useful to formulate the result concerning free Leibniz algebras in the following form.

Corollary 3.11. *The composition of functors $\overline{S}_2 \circ \overline{LL}$ in the commutative diagram*

$$
\begin{tikzcd}
\overline{\mathbb{G}\mathrm{r}} \arrow[r, "\overline{LL}"] \arrow[dr, "LL|_{\overline{\mathbb{G}\mathrm{r}}}"'] & \overline{\mathbb{L}\mathbb{L}} \arrow[r, "\overline{S}_2"] \arrow[d, "\mathrm{I}"] & \mathbb{L}\mathrm{eibniz} \\
 & \mathbb{L}\mathbb{L} \arrow[ur, "S_1"'] &
\end{tikzcd}
$$

takes free objects from $\overline{\mathbb{G}\mathrm{r}}$ to free Leibniz algebras and for any free Leibniz algebra A there is a free object $G \in \overline{\mathbb{G}\mathrm{r}}$ with $\overline{S}_2 \cdot \overline{LL}(G) \approx A$.

Let $V : \overline{\mathbb{LL}} \to \mathbb{L}\mathrm{eibniz}$ be the obvious forgetful functor. The following commutative diagram is due to the referee:

$$\begin{tikzcd}
\mathbb{G}\mathrm{r} \arrow[r, "\overline{C}"] \arrow[d, "W"'] & \overline{\mathbb{G}\mathrm{r}} \arrow[d, "V\circ\overline{\mathbb{LL}}"] \\
\mathbb{L}\mathrm{ie} \arrow[r] & \mathbb{L}\mathrm{eibniz}.
\end{tikzcd}$$

These results together with Theorem 3.6 of [2] give Witt's well-known construction for groups with action and prove an analogue of Witt's theorem for this special kind of groups and Leibniz algebras.

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