

MINIMAL FREE MULTI-MODELS FOR CHAIN ALGEBRAS

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To the memory of G. Chogoshvili

Abstract. Let R be a local ring and A a connected differential graded algebra over R which is free as a graded R -module. Using homological perturbation theory techniques, we construct a minimal free multi-model for A having properties similar to those of an ordinary minimal model over a field; in particular the model is unique up to isomorphism of multialgebras. The attribute ‘multi’ refers to the category of multicomplexes.

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INTRODUCTION

Let R be a commutative ring with 1, and let A be a connected differential graded algebra over R which is free as a graded R -module, endowed with the obvious augmentation map. For example, A could be the chains on the loop space ΩX of a simply connected space X . As in [37], we refer to a differential graded algebra of the kind $(T[V], d)$, where $T[V]$ denotes the graded tensor algebra on a free graded R -module V , together with a morphism $(T[V], d) \rightarrow (A, d)$ of differential graded algebras which is also a chain equivalence, as a *free model for A* . The approach in [37] provides a small free model, and we recall briefly the construction: Let ΩBA be the cobar construction on the bar construction BA , let $FH(JBA)$ be a free resolution (in the category of R -modules) of the homology $H(JBA)$ of the coaugmentation coideal JBA of the bar construction BA , and consider the tensor algebra $T[s^{-1}FH(JBA)]$ on the desuspension $s^{-1}FH(JBA)$ of $FH(JBA)$. A suitable homological perturbation theory argument, applied to these data, enabled us to construct a differential d on $T[s^{-1}FH(JBA)]$ and a morphism

$$(T[s^{-1}FH(JBA)], d) \rightarrow (T[s^{-1}(JBA)], d_\Omega) = \Omega BA$$

of differential graded algebras which is also chain equivalence; the composite of this chain equivalence with the standard adjoint chain equivalence $\Omega BA \rightarrow A$ then yields a small free model for A .

In particular, when R is a local ring which is as well a principal ideal domain and when $FH(JBA)$ is a minimal resolution of the homology $H(JBA)$, the differential graded algebra $(T[s^{-1}FH(JBA)], d)$ together with the comparison

map into A is what has been called a *minimal free model for A* in [37]. According to [37] (5.11), such a minimal free model exists and is unique up to isomorphism of chain algebras. When the local ring is no longer a principal ideal domain, this approach still yields a small free model but not a free minimal one in the naive sense, cf. [37] (5.12). In the present paper we shall show that the resulting small free model is minimal as an algebra in the category of *multicomplexes* or, equivalently, as a *multialgebra* (precise definitions will be given in the next section) and, given an augmented connected differential graded algebra A that is free as a module over the local ring R , we shall in fact establish the existence and uniqueness of what we shall call a *minimal free multi-model* for A . See Theorem 3.10 below for details. The idea of using this additional structure is related with the more familiar one of using a filtration as an additional piece of structure, cf. e. g. [24]. Indeed, a multicomplex structure is equivalent to that of a filtered chain complex having the property that the associated (bi)graded object is free over the ground ring. Multicomplexes occur at various places in the literature; historical comments will be given in the next section. A special case of a multicomplex arises from an ordinary chain complex with the degree filtration, cf. (1.9) above.

Here is an outline of the contents of the paper. In Section 1 we recall the concept of a multicomplex and introduce that of a *multialgebra*. A special case of a multialgebra is an ordinary differential graded algebra with the degree filtration. We also introduce appropriate notions of morphism and of homotopy. In Section 2 we explore free multialgebras, and in Section 3 we study minimal free multialgebras over a local ring. In particular we shall show that, over an arbitrary local ring, a differential graded algebra that is free as a module over the ground ring, viewed as a multialgebra in the sense explained above, has a minimal free model in the category of multialgebras that is unique up to isomorphism. Details will be given in Theorem 3.10. Some comments about the significance of this result and about its relationship with the literature will be given in Remark 3.11.

The ground ring will be denoted by R throughout, and graded and bigraded modules will always be free over the ground ring R unless they are explicitly specified otherwise; the notions of chain equivalence and weak equivalence (i. e. isomorphism on homology) are then equivalent, and we shall use the term ‘weak equivalence’ only when there is a difference between the two. The same kind of remark applies to the concepts of *multiequivalence* and *weak multiequivalence* introduced in (1.11) and (1.12.1) below. The reader will have no trouble to replace ‘free over R ’ with ‘projective over R ’. We shall stick to the free case to avoid unnecessary complications with language and terminology. Our notation is the same as that in e. g. [29], [37] and [52]. Graded and bigraded algebras will always be assumed to be augmented.

This paper is dedicated to the memory of G. Chogoshvili. Within the tradition on algebraic and topological research in Georgia which goes back to him, the ideas which led to multicomplexes and multialgebras are well represented, cf. e. g. [4], [37], [40]–[42], [54]–[56]. This list is certainly not exhaustive.

1. MULTICOMPLEXES AND MULTIALGEBRAS

Let R be a commutative ring with 1, taken henceforth as a ground ring. A *multicomplex* is a bigraded R -module together with a differential on the associated graded module that preserves *column filtration* (see Definition 1.4.1 below for details). Taking components we arrive at the following.

Definition 1.1. A *multicomplex* X is a bigraded R -module $\{X_{p,q}\}_{p,q \in \mathbb{Z}}$, together with R -linear morphisms

$$d^i: X_{p,q} \longrightarrow X_{p-i,q+i-1}, \quad i = 0, 1, \dots,$$

such that, for each $n \geq 0$, $\sum_{i+j=n} d^i d^j = 0$.

Henceforth we shall refer to $d = \{d^0, d^1, d^2, \dots\}$ as a *multidifferential*. Notice that, for each ℓ , the operator d^0 is a differential $X_{\ell,*} \rightarrow X_{\ell,*-1}$ but, for $j \geq 1$, the operator d^j is not necessarily a differential. We shall refer to d^0 as the *vertical differential*. Likewise we shall occasionally refer to d^1 as a *horizontal operator*. When d^0 is zero, for each ℓ , the operator d^1 is manifestly a differential $X_{*,\ell} \rightarrow X_{*-1,\ell}$: we shall then refer to it as a *horizontal differential*. A bicomplex may be viewed as a multicomplex with $d^i = 0$ for $i \geq 2$.

The multicomplex terminology goes back at least to Liulevicius [44]; without reference to an explicit name, the structure has been exploited in [19], [45], [64]. A triangular complex in the sense of [25] is a special case of a multicomplex, and there is a close relationship between multicomplexes and the predifferential theory developed in [4], cf. the proof of Theorem 3.10 below as well as [54], [55]. Multicomplexes play a major role in *homological perturbation theory*, cf. e.g. Section 2 of [31] and Section 1 of [32]. More details and historical comments about homological perturbation theory can be found, e.g., in [37]. A “recursive structure of triangular complexes”, a concept isolated in Section 5 of [25], is in fact an example of what was later identified as a perturbation. In [36], certain algebraic structures behind the spectral sequence of a foliation are explored by means of a multialgebra version of the Maurer–Cartan algebra.

Given a bigraded R -module X , we shall refer to the graded R -module CX , where

$$CX_n = \sum_{i+j=n} X_{i,j},$$

as the corresponding *total object*.

For a multicomplex X , the formal infinite sum $d = \sum d^j$ defines an operator on the total object CX whenever the sum is finite in each degree, and (CX, d) is then a chain complex; we refer to this situation by saying that (CX, d) is *well defined*. This will manifestly be the case when the column filtration is *bounded below* (cf. e.g. [46]) in the sense that, for each degree n (of CX), there is an integer $s = s(n)$ such that

$$X_{p,q} = 0 \quad \text{whenever} \quad p < s.$$

Definition 1.2. Given a multicomplex X , its *total complex* is the chain complex (CX, d) , where

$$d = \sum d^j,$$

whenever the latter sum is finite in each degree.

A more rigorous description in the language of *assembly functors* [39] can be found in [47]. Henceforth a multicomplex X will be *assumed to be bounded below*; then the operation d is defined on the total object CX and (CX, d) is a chain complex.

An ordinary chain complex C can be viewed as a multicomplex in an obvious way. More precisely,

$$C_{i,0} = C_i, \quad C_{i,\neq 0} = 0, \quad d^1 = d, \quad d^j = 0 \text{ for } j \neq 1, \quad (1.3)$$

yields a multicomplex whose total complex is just C . We refer to (1.3) as the *associated multicomplex*.

Definition 1.4.1. The *column filtration* of a bigraded R -module is the ascending filtration $\{F_p\}$ given by

$$(F_p(X))_{i,j} = \begin{cases} X_{i,j} & \text{if } i \leq p, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.4.2. The *row filtration* of a bigraded R -module is the descending filtration $\{F^q\}$ given by

$$(F^q(X))_{i,j} = \begin{cases} X_{i,j} & \text{if } j \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

Given a multicomplex X , the row and column filtrations induce corresponding filtrations on the total complex CX ; in particular the filtrations are compatible with the differential on the latter. We then refer to these filtrations as *column* and *row filtrations* as well. Moreover, the sum

$$\partial = d^1 + d^2 + \dots$$

is then what is called a *perturbation* of the differential d^0 on CX with respect to the column filtration, that is, each d^j lowers column filtration by j .

Proposition 1.5. *Let X be a bigraded R -module, and let d be a differential on its total object CX that is compatible with the column filtration. Then the components*

$$d^i: X_{p,q} \longrightarrow X_{p-i,q+i-1}, \quad i = 0, 1, \dots,$$

of d endow X with multicomplex structure in such a way that totalization yields the original data.

We shall need appropriate notions of morphism of multicomplexes and of homotopy between such morphisms. To handle them concisely, we introduce the following terminology; our description differs from the notions of morphism given in [44], [45], and [47].

Definition 1.6. Given two bigraded R -modules X and Y , a *multimorphism of bigraded R -modules of degree η* , written as $f: X \rightarrow Y$, consists of a sequence $f = \{f^k\}_{k \geq \ell}$ of R -module morphisms

$$f^k: X_{p,q} \rightarrow Y_{p-k,q+k+\eta}$$

where ℓ is a (possibly negative) integer. We refer to the f^k 's as the *components* of f , and we denote the degree of f by $|f|$ as usual. We shall then write

$$f = f^\ell + \dots + f^0 + f^1 + f^2 + \dots : X \rightarrow Y.$$

Here the infinite sum is to be understood in a formal way. However, when CX is well defined, this infinite sum converges in the sense that in each degree only finitely many terms are non-zero. We note that a multimorphism $f: X \rightarrow Y$ preserves column filtrations if and only if it is of the form

$$f = f^0 + f^1 + f^2 + \dots : X \rightarrow Y.$$

Definition 1.7.1. Given two multimorphisms $f = f^\ell + \dots + f^0 + f^1 + f^2 + \dots : X \rightarrow Y$ and $g = g^\ell + \dots + g^0 + g^1 + g^2 + \dots : Y \rightarrow Z$ of bigraded modules of degree η and η' respectively, the *composite* $g \circ f$ is the multimorphism $g \circ f: X \rightarrow Z$ of bigraded R -modules of degree $\eta + \eta'$, where

$$(g \circ f)^k = \sum_{i+j=k} g^i f^j,$$

that is, $g \circ f$ is obtained by a formal evaluation of the ‘composition’

$$(f^\ell + \dots + f^0 + f^1 + f^2 + \dots)(g^\ell + \dots + g^0 + g^1 + g^2 + \dots).$$

This evaluation makes sense since, for each k , the sum $\sum_{i+j=k} g^i f^j$ is finite. For example, given a bigraded R -module X , a multimorphism

$$d = d^0 + d^1 + d^2 + \dots : X \rightarrow X$$

of degree -1 yields a multicomplex structure on X if and only if, as a multimorphism of bigraded R -modules, the composite $d \circ d$ is zero. This operation of composition of multimorphisms is plainly associative. Henceforth we shall discard the symbol ‘ \circ ’ and write $gf = g \circ f$ etc.

Remark 1.7.2. The bigraded R -modules together with a suitable choice of multimorphisms constitute a category in an obvious fashion. In particular, invertible multimorphisms of the kind $f = f^0 + f^1 + f^2 + \dots : X \rightarrow Y$, necessarily of degree 0, are *isomorphisms* in this category. Henceforth when we refer to isomorphisms in the multi-setting this kind of isomorphism will always be understood.

Proposition 1.7.3. *Let X and Y be bigraded R -modules, let $f: X \rightarrow Y$ be a multimorphism, and let $Cf: CX \rightarrow CY$ be the corresponding morphism of graded R -modules. Then f is an isomorphism if and only if Cf is an isomorphism of graded R -modules.*

Lemma 1.8. *A multimorphism $f: X \rightarrow Y$ of degree 0 of the kind $f = f^0 + f^1 + f^2 + \dots$ is an isomorphism if and only if f^0 is an isomorphism.*

Proof. It is obvious that the condition is necessary. To see that it is also sufficient, suppose that f^0 is an isomorphism, and let g^0 be its inverse. To extend g^0 to an inverse of f , all we have to do is to solve the equation

$$\text{Id} = fg = \sum f^i g^j$$

for g^1, g^2, \dots , which amounts to solving the series

$$0 = \sum_{i+j=k} f^i g^j, \quad k \geq 1,$$

of equations for g^1, g^2, \dots . This series of equations admits a unique solution g^1, g^2, \dots . □

Definition 1.9.1. Let X and Y be multicomplexes. A *morphism of multicomplexes* written as $f: X \rightarrow Y$, is a multimorphism

$$f = f^0 + f^1 + f^2 + \dots : X \rightarrow Y$$

of the underlying bigraded R -modules of degree zero having the property that

$$df + (-1)^{|f|}fd = 0$$

as multimorphisms of the underlying bigraded R -modules.

Thus in particular a morphism of multicomplexes preserves column filtrations.

Proposition 1.9.2. *Let X and Y be multicomplexes, and let $f: CX \rightarrow CY$ be a morphism of chain complexes that preserves column filtrations. Then the components*

$$f^k: X_{p,q} \rightarrow X_{p-k,q+k}, \quad k = 0, 1, \dots,$$

constitute a morphism of multicomplexes which, in turn, induces the original morphism $f: CX \rightarrow CY$ of filtered chain complexes.

Definition 1.10.1. Given two morphisms $f, g: X \rightarrow Y$ of multicomplexes, a *homotopy of morphisms of multicomplexes* or, more briefly, a *multihomotopy*, written as

$$h: f \simeq g: X \rightarrow Y,$$

is a multimorphism $h = h^{-1} + h^0 + h^1 + h^2 + \dots : X \rightarrow Y$ of degree 1 of the underlying bigraded R -modules satisfying the identity

$$dh + hd = g - f, \tag{1.10.2}$$

interpreted as one among multimorphisms of the underlying bigraded R -modules. The two morphisms f and g of multicomplexes will then be said to be *multihomotopic*.

Notice that a multihomotopy does not necessarily preserve column filtrations.

Proposition 1.10.3. *Let X and Y be multicomplexes, let $f, g: CX \rightarrow CY$ be morphisms of chain complexes that preserve column filtrations, and let $h: CX \rightarrow CY$ be a chain homotopy between f and g that raises column filtration at most by one. Then the components*

$$h^k: X_{p,q} \rightarrow X_{p-k,q+k+1}, \quad k = -1, 0, 1, \dots,$$

constitute a multihomotopy between f and g , viewed as morphisms of multicomplexes, and this multihomotopy induces the original chain homotopy between f and g , viewed as chain maps.

Definition 1.11. A *multiequivalence* is a morphism $f: X \rightarrow Y$ of multicomplexes having an inverse with respect to the notion of multihomotopy; in other words, f is a multiequivalence provided there are a morphism $g: Y \rightarrow X$ of multicomplexes and multihomotopies $fg \simeq \text{Id}$ and $gf \simeq \text{Id}$.

Definition 1.12.1. A *weak multiequivalence* is a morphism $f: X \rightarrow Y$ of multicomplexes inducing an isomorphism

$$f_*: (E_{*,*}^r(X), d^r) \rightarrow (E_{*,*}^r(Y), d^r)$$

of column spectral sequences for $r \geq 2$.

Lemma 1.12.2. *Let X and Y be multicomplexes that are free as bigraded R -modules. Then a weak multiequivalence $f: X \rightarrow Y$ is a genuine multiequivalence.*

Proof. This is left to the reader. □

Before we spell out the next observation we remind the reader that $\{F_p\}$ refers to the column filtration reproduced in Definition 1.4.1 above.

Lemma 1.12.3. *A morphism $f: X \rightarrow Y$ of multicomplexes is a weak multiequivalence if and only if, for each $q \geq 0$, the restriction $Cf|: F_q(CX) \rightarrow F_q(CY)$ is a chain equivalence, that is, if and only if $Cf: CX \rightarrow CY$ is a filtered chain equivalence with respect to the column filtration.*

Proof. This comes down to the standard identification of $E^2(X)$ etc. with $H(F_*(CX), F_{*-1}(CX))$ etc. Details are left to the reader. □

Definition 1.13. Let X and Y be multicomplexes. Then the *tensor product* $X \otimes Y$ in the category of multicomplexes is defined by

$$(X \otimes Y)_{p,q} = \sum_{i+k=p, j+l=q} X_{i,j} \otimes Y_{k,l}, \tag{1.13.1}$$

$$d^r(x_{i,j} \otimes y_{k,l}) = d^r(x_{i,j}) \otimes y_{k,l} + (-1)^{(i+j)} x_{i,j} \otimes d^r(y_{k,l}).$$

Notice that when C and C' are chain complexes, the associated multicomplex of their tensor product $C \otimes C'$ as chain complexes coincides with the tensor product of the associated multicomplexes.

Definition 1.14.1. The *horizontal suspension* of a bigraded R -module X is the bigraded R -module sX given by

$$(sX)_{p,q} = X_{p-1,q}; \tag{1.14.2}$$

abusing notation somewhat, we write $s: X \rightarrow sX$ for the corresponding (*horizontal*) *suspension operator*, which is the identity when we neglect bigrading and which, in the above language, is a multimorphism of degree $\eta = 1$ of the kind

$$s = s^\ell: X_{*,*} \rightarrow (sX)_{*+1,*},$$

with $\ell = -1$, that is, s has a single component.

Definition 1.14.3. The *suspension* of a multicomplex X is the multicomplex sX which as a bigraded R -module is the horizontal suspension and whose multidifferential is given by

$$sd^j + d^j s = 0; \quad (1.14.4)$$

here $s: X \rightarrow sX$ denotes the corresponding (horizontal) suspension operator, and we do not distinguish in notation between the constituents of the multidifferential on X and sX .

Notice that when C is a chain complex, the associated multicomplex of its suspension sC as a chain complex coincides with the suspension of the associated multicomplex.

Definition 1.15. Given two multicomplexes X and Y , their *direct sum* $X \oplus Y$ is the multicomplex given by

$$(X \oplus Y)_{p,q} = X_{p,q} \oplus Y_{p,q},$$

with the obvious multidifferential induced by those on X and Y .

Definition 1.16. A *multialgebra* is a bigraded algebra A together with a multicomplex structure so that the structure map $m: A \otimes A \rightarrow A$ is a morphism of multicomplexes.

Definition 1.17. Given a multialgebra A , a *multi left A -module* X is a multicomplex X together with the structure $m: A \otimes X \rightarrow X$ of a left bigraded A -module on X that is a morphism of multicomplexes. Multi right A -modules are defined accordingly.

Let A and B be multialgebras, and let f_1 and f_2 be morphisms $A \rightarrow B$ of multialgebras. Then B admits an obvious structure of a bigraded A -bimodule which we write as $(a, b) \mapsto a \cdot b$ where

$$a \cdot b = (f_1(a))b, \quad b \cdot a = b(f_2(a)), \quad a \in A, \quad b \in B. \quad (1.17.1)$$

We shall refer to a multimorphism $h: A \rightarrow B$ of the underlying bigraded R -modules as an f_1 - f_2 -*multiderivation* provided it is a derivation with respect to the bigraded A -bimodule structure (1.17.1), i. e. if

$$m(f_1 \otimes h + h \otimes f_2) = h m, \quad (1.17.2)$$

where m refers to the structure maps.

Definition 1.18. A *homotopy* $f_1 \simeq f_2$ of morphisms of multialgebras is a multihomotopy $h: A \rightarrow B$ (in the sense of (1.10)) that is also a f_1 - f_2 -multiderivation. More briefly, we shall refer to such a homotopy as a *multihomotopy* (in the context of morphisms of multialgebras).

1.19. Given an ordinary differential graded algebra A , viewed as an ordinary chain complex, the associated multicomplex (1.3) plainly inherits a multialgebra structure which we refer to as the *associated multialgebra* structure.

2. FREE MULTIALGEBRAS

Definition 2.1. A multialgebra A is *free* if its underlying bigraded algebra is (isomorphic to) the tensor algebra $T[V]$ on some free bigraded R -module V , with the obvious bigrading, cf. (1.13).

A free multialgebra A admits an obvious *augmentation map* $\varepsilon: T[V] \rightarrow R$, and we shall say it is *connected* (as an augmented algebra) if CV is non-negative or if CV is non-positive and zero in degree zero.

For convenience we recollect some properties of free connected multialgebras. Henceforth $V, V',$ and W denote free connected bigraded R -modules that are non-negative or non-positive and zero in degree zero, and free bigraded algebras will always be assumed connected.

(2.2) Multiderivations and multidifferentials. Let $A = T[V]$ be a free connected multialgebra, and let M be a bigraded A -bimodule. As in Section 1 above, we refer to a multimorphism

$$d = d^0 + d^1 + d^2 + \dots : T[V] \rightarrow M$$

of degree -1 that is also a derivation (with respect to the bigraded A -bimodule structure) as a *multiderivation*. Each multiderivation is plainly determined by its restriction

$$\beta = d| : V \rightarrow M$$

to V , and β is a multimorphism. When M itself is the bigraded tensor algebra $T[W]$ on some bigraded R -module W , the multimorphism β has *components*

$$\beta_i : V \rightarrow W^{\otimes i}$$

which are itself multimorphisms and, conversely, each sequence $\{\beta_i\}$ of multimorphisms $V \rightarrow W^{\otimes i}$ determines a multiderivation $T[V] \rightarrow T[W]$.

In case $W = V$, for each multiderivation d of degree -1 , the composite dd is a multiderivation d of degree -2 , whence $dd = 0$ if and only if the restriction $dd|$ to V vanishes. Hence if $(T[V], d)$ is a multialgebra, (V, β_1) is a multicomplex.

(2.3) Multimorphisms. Let $A = T[V]$ be a free multialgebra, and let B be a bigraded algebra. Each multimorphism $f: T[V] \rightarrow B$ of bigraded algebras is determined by its restriction

$$\alpha = f| : V \rightarrow B.$$

When B is the bigraded tensor algebra $T[W]$ on some bigraded R -module W , the multimorphism α has *components*

$$\alpha_i : V \rightarrow W^{\otimes i},$$

and, conversely, each sequence $\{\alpha_i\}$ of multimorphisms $V \rightarrow W^{\otimes i}$ determines a multimorphism $T[V] \rightarrow T[W]$ of bigraded algebras.

When d and d' endow $T[V]$ and $T[V']$, respectively, with multialgebra structures and when $f: T[V] \rightarrow T[V']$ is a multimorphism of bigraded algebras, $Df = d'f - fd$ is a multiderivation of degree -1 (with respect to the obvious bigraded $T[V]$ -bimodule structure on $T[V']$). Hence $d'f = fd$ if and only if

$d'f|V = fd|V$. In degree 1 this condition gives $\alpha_1\beta_1 = \beta'_1\alpha'_1$. Consequently, if f is a morphism of multicomplexes, so is α_1 .

The proof of the following is straightforward and left to the reader.

Lemma 2.3.1. *A multimorphism $f: T[V] \rightarrow T[V']$ of bigraded algebras is an isomorphism if and only if its first component $\alpha_1: V \rightarrow V'$ is an isomorphism.*

(2.4) Multihomotopies. Let A and B be multialgebras, and let f and f' be morphisms $A \rightarrow B$ of multialgebras. Recall from Section 1 that an f - f' -multiderivation $h: A \rightarrow B$ of degree 1 is called a *multihomotopy* $f \simeq f'$ of morphisms of multialgebras provided $Dh(= dh + hd) = f' - f$. When A , viewed as a bigraded algebra, is a tensor algebra on some bigraded R -module, this notion of multihomotopy can be conveniently described in terms of a suitable cylinder construction, cf. e. g. [3] and [37] (3.3)

Let V be a multicomplex and let $A = T[V]$. Then the *cylinder* $A \times I$ is characterized as follows:

- As a bigraded algebra, $A \times I$ is the tensor algebra $T[V' \oplus V'' \oplus sV]$ on the direct sum of two copies V' and V'' of V and the horizontal suspension sV of V – this is just the tensor algebra on the corresponding cylinder $V \times I$; we write $i': A \rightarrow A \times I$ and $i'': A \rightarrow A \times I$ for the obvious injections of bigraded algebras which identify V with V' and V'' respectively;
- up to the obvious change in notation, the multidifferential on V' and V'' is the same as that in V ;
- to define the multidifferential on sV , let $S: T[V] \rightarrow A \times I$ be the $i'-i''$ -multiderivation determined by $Sv = sv$, so that, for $a, b \in T[V]$

$$S(ab) = (S(a))b'' + (-1)^{|a|}a'S(b), \tag{2.4.1}$$

and define the multidifferential d on $sV \subseteq A \times I$ by

$$\begin{aligned} d^0(sv) &= -Sd^0v, \\ d^1(sv) &= v'' - v' - Sd^1v, \\ d^j(sv) &= -Sd^jv, \quad j \geq 2. \end{aligned} \tag{2.4.2}$$

We note that (2.4.2) implies that S is a multihomotopy $S: i' \simeq i''$ of morphisms of multialgebras. Moreover, i' and i'' are multiequivalences, and it is manifest that the module of indecomposables $Q(A \times I)$ is just the corresponding cylinder on the indecomposables QA .

Proposition 2.4.3. *Let V be a multicomplex, let $A = T[V]$, let B be a multialgebra, and let f' and f'' be morphisms $A \rightarrow B$ of multialgebras. Then the formulas*

$$Hi' = f', \quad Hi'' = f'', \quad HS = h$$

determine a natural bijection between multihomotopies $h: f' \simeq f''$ of morphisms of multialgebras and morphisms $H: A \times I \rightarrow B$ of multialgebras with the property that

$$Hi' = f', \quad Hi'' = f''.$$

Proof. This is straightforward and left to the reader. □

Remark 2.4.4. It is not hard to deduce from (2.4.3) that the above notion of multihomotopy of morphisms between multialgebras A and B is an equivalence relation, provided the A underlying bigraded algebra is a tensor algebra; cf. e. g. [3] for the more conventional case of chain algebras. However, for arbitrary multialgebras A , this need *not* be the case.

Let $H: T[V] \times I \rightarrow B$ be a morphism of multialgebras. Then

$$H(v'') = H(v') + Hdsv + HS(dv). \tag{2.4.5}$$

Inspection shows that H is determined by its values on V' and sV . In fact, write the multidifferential on $T[V]$ $T[V]$ in the form $d = d^{(0)} + \partial$ so that $d^{(0)}$ is the multidifferential that comes from the multidifferential on V and so that the “multi” operator ∂ lowers augmentation filtration. We note that this filtration has nothing to do with the corresponding row or column filtrations; however, ∂ corresponds to a perturbation of the differential induced by $d^{(0)}$ on the total complex $CT[V] = T[CV]$ with respect to the augmentation filtration. Then

$$Sdv = Sd^0v + S\partial v = (d^0v)' + (d^0v)'' + \text{terms of lower filtration} \tag{2.4.6}$$

whence by induction on degree we see that H is determined by its values on V' and sV . This proves the following:

Lemma 2.4.7. *Let $f: T[V] \rightarrow B$ be a morphism of multialgebras, and let $\gamma: V \rightarrow B$ be a multimorphism of degree 1 of the underlying bigraded modules. Then there is a morphism $f': T[V] \rightarrow B$ of multialgebras and a multihomotopy $h_\gamma: f \simeq f'$ of morphisms of multialgebras so that the restriction of h_γ to V coincides with γ , and f' and h_γ are uniquely determined by the given data.*

Corollary 2.4.8. *Let $f: (T[V], d) \rightarrow (T[W], d)$ be a morphism between free multialgebras, let $\alpha_1: (V, \beta_1) \rightarrow (W, \tilde{\beta}_1)$ be its first component, and let $\alpha'_1: (V, \beta_1) \rightarrow (W, \tilde{\beta}_1)$ be a morphism of multicomplexes that is multihomotopic to α_1 . Then there is a morphism*

$$g: (T[V], d) \rightarrow (T[W], d)$$

of multialgebras with first component α'_1 and multihomotopic to f .

Theorem 2.5 (Multi-Version of the Adams–Hilton Theorem). *Let A and A' be multialgebras, not necessarily free as bigraded modules over the ground ring, let*

$$\begin{array}{ccc} & A & \\ & \downarrow g & \\ (T[V], d) & \xrightarrow{f'} & A' \end{array} \tag{2.5.1}$$

be a diagram in the category of multialgebras, and suppose that g is a weak multiequivalence. Then there is a morphism $f: (T[V], d) \rightarrow A$ of multialgebras so that gf is homotopic to f' as morphisms of multialgebras and, furthermore, the multihomotopy class of f is uniquely determined by this condition.

Another way to spell this out is to say that, for each weak multiequivalence g , the induced morphism

$$g^\sharp: [(T[V], d), A] \longrightarrow [(T[V], d), A'] \quad (2.5.2)$$

on the sets of homotopy classes of morphisms of multialgebras is a bijection.

A proof of the corresponding classical Theorem of Adams and Hilton may be found in [1] (3.1); see also [3](1.4).

Proof of Theorem 2.5. For intelligibility, we reproduce first the argument for the classical Theorem of Adams and Hilton:

In degree 0, the restriction $f|: (T[V])_0 = R \rightarrow A_0$ is taken to be the obvious morphism that sends $1 \in R$ to $1 \in A$, and the morphism $f: T[V] \rightarrow A$ and homotopy $h: T[V] \rightarrow A$ are then constructed by induction on the degree of the generating module V . More precisely, appropriate morphisms $f_j: V_j \rightarrow A_j$ and $h_j: V_j \rightarrow A'_{j+1}$ are constructed by induction in such a way that, for $j \geq 1$,

$$dh_j + h_{j-1}d = g_j f_j - f'_j. \quad (2.5.3)$$

Here are the details; we write $Z_k(-)$ etc. for cycles in degree k : Let

$$\zeta_1: V_1 \rightarrow Z_1(A)$$

be a morphism so that

$$g_1 f_1 - f'_1: V_1 \longrightarrow Z_1(A')$$

goes into the boundaries. The existence of such a morphism ζ_1 is guaranteed by the hypothesis that g is a weak equivalence. Let

$$f_1 = \zeta_1: V_1 \rightarrow Z_1(A).$$

Since V_1 is free, there is a morphism

$$h_1: V_1 \rightarrow A'_2$$

so that

$$dh_1 = g_1 f_1 - f'_1: V_1 \longrightarrow Z_1(A').$$

Next, let $n \geq 1$, and suppose by induction that the components f_1, \dots, f_n and h_1, \dots, h_n have already been constructed in such a way that (2.5.3) holds for $1 \leq j \leq n$. Then the composite

$$g_n f_n d: V_{n+1} \longrightarrow A'_n$$

goes into the n -cycles $Z_n(A')$ of A' , in fact, in view of (2.5.3), we have

$$g_n f_n d = f'_n d + dh_n d = d(f'_{n+1} + h_n d),$$

whence $g_n f_n d$ goes into the n -boundaries of A' . But g is a weak equivalence, and hence $f_n d$ goes into the n -boundaries of A , that is, there is a morphism

$$\tilde{f}_{n+1}: V_{n+1} \longrightarrow A_{n+1}$$

so that

$$d\tilde{f}_{n+1} = f_n d.$$

Moreover,

$$d(g_{n+1} \tilde{f}_{n+1} - f'_{n+1} - h_n d) = g_n f_n d - f'_n d - dh_n d = dh_n d - dh_n d = 0.$$

Hence

$$g_{n+1}\tilde{f}_{n+1} - f'_{n+1} - h_n d: V_{n+1} \longrightarrow A'_{n+1}$$

goes into the $(n + 1)$ -cycles $Z_{n+1}(A')$ of A' . Since g is a weak equivalence, there is a morphism

$$\zeta_{n+1}: V_{n+1} \rightarrow Z_{n+1}(A)$$

so that, with $f_{n+1} = \tilde{f}_{n+1} + \zeta_{n+1}$, the morphism

$$g_{n+1}f_{n+1} - f'_{n+1} - h_n d: V_{n+1} \longrightarrow A'_{n+1}$$

goes into the $(n + 1)$ -boundaries $Z_{n+1}(A')$ of A' . Since V_{n+1} is free, there is thus a morphism

$$h_{n+1}: V_{n+1} \rightarrow A'_{n+2}$$

so that

$$dh_{n+1} + h_n d = g_{n+1}f_{n+1} - f'_{n+1}.$$

This completes the inductive step.

Proceeding thus, as n tends to infinity, we obtain the desired morphism f and homotopy h .

We now explain the necessary modifications for a complete argument for Theorem 2.5, the multi-version of the Adams–Hilton Theorem. We shall show that, in the situation of Theorem 2.5, the morphism $f: CT[V] \rightarrow CA$ of differential graded algebras can be constructed compatibly with the column filtrations and, furthermore, that the homotopy h can be constructed so that it raises column filtration by 1. By Proposition 1.9.2, the morphism f then determines a corresponding morphism of multialgebras and, by Proposition 1.10.3, the homotopy then determines a corresponding multihomotopy. Here are the details:

As before, we denote the column filtrations by $\{F_q\}$. For each $q \geq 0$, we then have a diagram

$$\begin{array}{ccc} & & F_q(A) \\ & & \downarrow g|_{F_q(A)} \\ (T[F_q(V)], d) & \xrightarrow{f'} & F_q(A') \end{array}$$

in the category of differential algebras; furthermore, cf. (1.12.3), since g is a weak multiequivalence, for each q , the restriction $g|_{F_q(A)}$ is a weak equivalence. By the corresponding classical Adams–Hilton Theorem [1], for each q , there is a morphism $f^{(q)}: (T[F_q(V)], d) \rightarrow F_q(A)$ of differential graded algebras so that $(g|_{F_q(A)})f^{(q)}$ is homotopic to $f'|_{F_q(A)}$ as morphisms of differential graded algebras and, furthermore, the homotopy class of $f^{(q)}$ is uniquely determined by this condition. It remains to show that the morphisms $f^{(q)}$ and the corresponding homotopies can be constructed compatibly with the column filtrations. This is seen by a slightly more complicated induction than the one that came into play above. Here are the details for the inductive step.

Let $q \geq 0$, and suppose that the morphism

$$f^{(q)}: F_q(T[V]) = T[F_q(V)] \longrightarrow F_q(A)$$

and chain homotopy

$$h^{(q)}: F_q(\mathbb{T}[V]) = \mathbb{T}[F_q(V)] \longrightarrow F_q(A)$$

have already been constructed. Furthermore, let $n \geq 1$, and suppose by induction that the components $f_1^{(q+1)}, \dots, f_n^{(q+1)}$ and $h_1^{(q+1)}, \dots, h_n^{(q+1)}$ have already been constructed in such a way that the appropriate replacement for (2.5.3) holds for $1 \leq j \leq n$, that is, that

$$dh_j^{(q+1)} + h_{j-1}^{(q+1)}d = g_j^{(q+1)}f_j^{(q+1)} - (f')_j^{(q+1)} \quad (2.5.4)$$

for $1 \leq j \leq n$. Then the composite

$$g_n^{(q+1)}f_n^{(q+1)}d: (F_{q+1}(V))_{n+1} \longrightarrow (F_{q+1}(A'))_n$$

goes into the n -cycles $Z_n(F_{q+1}(A'))$ of $F_{q+1}(A')$, in fact, in view of (2.5.4), we have

$$g_n^{(q+1)}f_n^{(q+1)}d = (f')_n^{(q+1)}d + dh_n^{(q+1)}d = d((f')_{n+1}^{(q+1)} + h_n^{(q+1)}d),$$

whence $g_n^{(q+1)}f_n^{(q+1)}d$ goes into the n -boundaries of $F_{q+1}(A')$. But g is a weak equivalence which, by virtue of (1.12.3), is compatible with the filtrations and hence $f_n^{(q+1)}d$ goes into the n -boundaries of $F_{q+1}(A)$, that is to say, the morphism

$$f_{n+1}^{(q)}: (F_q(V))_{n+1} \longrightarrow F_q(A_{n+1})$$

admits an extension

$$\tilde{f}_{n+1}^{(q+1)}: (F_{q+1}(V))_{n+1} \longrightarrow F_{q+1}(A_{n+1})$$

so that

$$d\tilde{f}_{n+1}^{(q+1)} = f_n^{(q+1)}d.$$

Moreover,

$$\begin{aligned} d(g_{n+1}^{(q+1)}\tilde{f}_{n+1}^{(q+1)} - (f')_{n+1}^{(q+1)} - h_n^{(q+1)}d) &= g_n^{(q+1)}f_n^{(q+1)}d - (f')_n^{(q+1)}d - dh_n^{(q+1)}d \\ &= dh_n^{(q+1)}d - dh_n^{(q+1)}d = 0. \end{aligned}$$

Hence

$$g_{n+1}^{(q+1)}\tilde{f}_{n+1}^{(q+1)} - (f')_{n+1}^{(q+1)} - h_n^{(q+1)}d: (F_{q+1}(V))_{n+1} \longrightarrow (F_{q+1}(A'))_{n+1}$$

goes into the $(n+1)$ -cycles $Z_{n+1}(F_{q+1}(A'))$ of $F_{q+1}(A')$. Since g is a filtered weak equivalence, cf. what was said above, there is a morphism

$$\zeta_{n+1}^{(q+1)}: (F_{q+1}(V))_{n+1} \rightarrow Z_{n+1}(F_{q+1}(A))$$

so that, with $f_{n+1}^{(q+1)} = \tilde{f}_{n+1}^{(q+1)} + \zeta_{n+1}^{(q+1)}$, the morphism

$$g_{n+1}^{(q+1)}f_{n+1}^{(q+1)} - (f')_{n+1}^{(q+1)} - h_n^{(q+1)}d: (F_{q+1}(V))_{n+1} \longrightarrow (F_{q+1}(A'))_{n+1}$$

goes into the $(n+1)$ -boundaries $Z_{n+1}(F_{q+1}(A'))$ of $F_{q+1}(A')$. Since $F_{q+1}(V_{n+1})$ is free, the morphism

$$h_{n+1}^{(q)}: F_q(V_{n+1}) \rightarrow (F_q(A'))_{n+2}$$

admits an extension

$$h_{n+1}^{(q+1)}: F_{q+1}(V_{n+1}) \rightarrow (F_{q+1}(A'))_{n+2}$$

so that

$$dh_{n+1}^{(q+1)} + h_n^{(q+1)}d = g_{n+1}^{(q+1)}f_{n+1}^{(q+1)} - (f')_{n+1}^{(q+1)}.$$

This completes the inductive step.

Proceeding thus, as n tends to infinity, we obtain the desired extensions $f^{(q+1)}$ and $h^{(q+1)}$. Likewise, as q tends to infinity, we obtain the desired filtered morphism f and homotopy h . This completes the proof. \square

3. MINIMAL FREE MULTIALGEBRAS

Let R be a local ring, with maximal ideal $\mathfrak{m} \subseteq R$ and residue field k and let M be an R -module. Recall that a free resolution

$$0 \longleftarrow M \xleftarrow{\varepsilon} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} \dots \tag{3.1}$$

of M in the category of R -modules is called *minimal* if $\delta_j(F_j) \subseteq \mathfrak{m}F_{j-1}$ for $j \geq 1$, cf. [12], [57]. In particular, a minimal resolution exists and is unique up to a (non-canonical) isomorphism of chain complexes [12]. We also recall the following

Definition 3.2. A chain complex (V, d) over (a local ring) R that is free as a graded module over R (as always) is *minimal* provided $d(V) \subseteq \mathfrak{m}V$.

For us the key concepts will be those given in (3.3) and (3.5) below.

Definition 3.3. A *minimal free multicomplex* is a multicomplex (X, d) that is free as a bigraded module over R (as always) and has the properties that $d^0 = 0$ and that (X, d^1) is a minimal chain complex.

Note that a minimal free chain complex (V, d) over R is a minimal free multicomplex with respect to the obvious multicomplex structure on (V, d) .

Proposition 3.4. *Let (V, d) and (V', d') be minimal free multicomplexes of finite type. Then a weak multiequivalence*

$$f: (V, d) \longrightarrow (V', d')$$

is an isomorphism of multicomplexes.

Proof. Since (V, d) and (V', d') are assumed to be minimal (free) multicomplexes, the vertical differentials d^0 and $(d')^0$ are zero, the operations d^1 and $(d')^1$ are horizontal differentials, and the “component” f^0 of f is a chain map

$$f^0: (V, d^1) \longrightarrow (V', (d')^1).$$

Furthermore, f^0 induces an isomorphism on homology. However, (V, d^1) and $(V', (d')^1)$ are minimal free chain complexes of finite type in the sense of (3.2) whence f^0 is an isomorphism of chain complexes and in particular admits an inverse g^0 ; cf. [37] (5.3). By (1.8), the morphism g^0 extends to an inverse g of f in the category of multicomplexes. \square

Definition 3.5. A *minimal free multialgebra* over R is a free multialgebra $(T[V], d)$ such that the generating multicomplex (V, β_1) is a minimal free multicomplex.

Proposition 3.6. *Let $(\mathbb{T}[V], d)$ and $(\mathbb{T}[V'], d')$ be minimal free multi R -algebras, assume that V and V' are of finite type, and let $f: (\mathbb{T}[V], d) \longrightarrow (\mathbb{T}[V'], d')$ be a multiequivalence of multi R -algebras. Then f is homotopic to an isomorphism $g: (\mathbb{T}[V], d) \longrightarrow (\mathbb{T}[V'], d')$ of multi R -algebras by a homotopy of morphisms of multi R -algebras.*

Proof. Let

$$f = f^0 + f^1 + f^2 + \cdots : \mathbb{T}[V] \longrightarrow \mathbb{T}[V']$$

be the given morphism of multialgebras, let

$$\alpha = f|_V : V \longrightarrow \mathbb{T}[V']$$

be the restriction of f to V , and write

$$\alpha_i : V \longrightarrow (V')^{\otimes i}$$

for its components, in the category of bigraded R -modules. For each $i \geq 1$, α_i is then itself a multimorphism

$$\alpha_i = \alpha_i^0 + \alpha_i^1 + \alpha_i^2 + \cdots : V \longrightarrow (V')^{\otimes i},$$

with the notion of tensor product (1.13) understood. Furthermore, f^0 is a morphism

$$f^0 : (\mathbb{T}[V], d^1) \longrightarrow (\mathbb{T}[V'], (d')^1)$$

of differential graded algebras, and its first component

$$\alpha_1^0 : (V, \beta_1^1) \longrightarrow (V', (\beta')_1^1)$$

is a morphism of chain complexes. However, since f is a multiequivalence of multi R -algebras, f^0 is a chain equivalence; in view of [37] (3.2.2), this implies that the first component α_1^0 is a chain equivalence, in fact, by virtue of [37] (5.3), α_1^0 is an isomorphism of chain complexes. In view of (1.8), the morphism

$$\alpha_1 : (V, \beta_1) \longrightarrow (V', \beta'_1)$$

is an isomorphism of multicomplexes. By Corollary 2.4.8, α_1 can be extended to a morphism

$$g : (\mathbb{T}[V], d) \longrightarrow (\mathbb{T}[V'], d')$$

of multialgebras in such a way that (i) its first component coincides with α_1 and (ii) the morphisms f and g are homotopic as morphisms of multialgebras. By virtue of Lemma 2.3.1, the morphism g is an isomorphism since so is α_1 . \square

Definition 3.7. Let C be a chain complex over R that is free as a graded R -module (as always). Then a *minimal free multimodel for C* is a minimal free multicomplex U over R together with a multiequivalence $\alpha: U \longrightarrow C$; here C is viewed as a multicomplex in the obvious way.

Definition 3.8. Let A be an augmented differential graded algebra over R that is free as a graded R -module (as always). Then a *minimal free multimodel for A* is a minimal free multialgebra $(\mathbb{T}[V], d)$ over R together with a morphism $g: (\mathbb{T}[V], d) \longrightarrow A$ of multi R -algebras that is also a multiequivalence; here A is identified with its associated multialgebra (cf. 1.19).

Theorem 3.9. *Let A be a differential graded R -algebra that is free as a module over R . If $(\mathbb{T}[V], d, g)$ and $(\mathbb{T}[V'], d', g')$ are minimal free multimodels for A , and if V and V' are of finite type, then $(\mathbb{T}[V], d)$ and $(\mathbb{T}[V'], d')$ are isomorphic multialgebras.*

Proof. By Theorem 2.5, there is a multiequivalence $\phi': (\mathbb{T}[V], d) \longrightarrow (\mathbb{T}[V'], d')$. In view of Proposition 3.6, this multiequivalence is multihomotopic to an isomorphism $\phi: (\mathbb{T}[V], d) \longrightarrow (\mathbb{T}[V'], d')$ of multi R -algebras. \square

Theorem 3.10. *Let A be a connected augmented differential graded R -algebra whose underlying graded R -module is free. Then A has a minimal free multimodel. When the homology of A is of finite type, a minimal free multimodel is unique up to isomorphism of multialgebras.*

Proof. For $k \geq 1$, pick a minimal resolution of $H_k(JBA)$ and assemble these resolutions to a minimal chain complex (FH, δ) of the kind (4.2.b) in [37]. The argument for the proof of [37] (4.4) yields a multidifferential $\vartheta = \{\delta, \vartheta^2, \vartheta^3, \dots\}$ on FH and a multiequivalence between (FH, ϑ) and JBA , and hence (FH, ϑ) is a minimal free multimodel for JBA ; the kind of reasoning used at this stage can be found in III.1 of [4] (for the special case where the base space is a point). The construction in [37] (4.7) then yields a free multimodel

$$(\mathbb{T}[s^{-1}FH(JBA)], d') \rightarrow (\mathbb{T}[s^{-1}(JBA)], d_\Omega) \rightarrow A$$

for A which is minimal by construction.

The uniqueness of the minimal free multimodel follows from Theorem 3.9. \square

Remark 3.11. In the situation of (the proof of) Theorem 3.10, the differential d' endows $M = s^{-1}FH(JBA)$ with the structure of a *s(trongly) h(omotopy) a(ssociative) coalgebra*; such a structure is dual to that of an $A(\infty)$ -algebra introduced in [60] and christened *s(trongly) h(omotopy) a(ssociative) algebra* in [61]. In the special case where C is the coalgebra of normalized chains on a simply connected space X and $A = \Omega C$, the cobar construction on C , Theorem 3.10 above yields a minimal model for the chain algebra of the loop space on X . For the special case where the ground ring is (local as above and) a principal ideal domain, a minimal model was obtained in [37]. Related models, not necessarily over a local ring and not necessarily minimal, have been developed in [37]. A special case of the kind of models in [37] can be found in [15]. For the dual situation, i.e. where, instead of C , a differential graded algebra A is considered, related models are given in [22] (not necessarily minimal ones) and in [40]. More comments can be found in Section 2 of [37] (see the discussion before (2.3) in the quoted reference).

REFERENCES

1. J. F. ADAMS and P. J. HILTON, On the chain algebra of a loop space. *Comment. Math. Helv.* **30**(1956), 305–330.
2. L. L. AVRAMOV and S. HALPERIN, Through the looking glass: a dictionary between rational homotopy theory and local algebra. *Algebra, algebraic topology and their interactions (Stockholm, 1983)*, 1–27, *Lecture Notes in Math.*, 1183, Springer, Berlin, 1986.

3. H. J. BAUES and J.-M. LEMAIRE, Minimal models in homotopy theory. *Math. Ann.* **225**(1977), No. 3, 219–242.
4. N. A. BERIKASHVILI, Differentials of a spectral sequence. (Russian) *Trudy Tbiliss. Math. Inst. Razmadze* **51**(1976), 5–106.
5. R. BROWN, The twisted Eilenberg–Zilber theorem', *Celebrazioni Archimedi de Secolo XX, Syracuse*, 1964, *Simposi di topologia* (1967), 33–37.
6. H. CARTAN, Algèbres d'Eilenberg–Mac Lane et homotopie, exposés 2–11. *Séminaire H. Cartan*, 1954–55. *École Normale Supérieure, Paris*, 1956.
7. H. CARTAN and S. EILENBERG Homological algebra. *Princeton University Press, Princeton, N. J.*, 1956.
8. K. T. CHEN, Iterated path integrals. *Bull. Amer. Math. Soc.* **83**(1977), No. 5, 831–879.
9. P. DELIGNE, P. GRIFFITHS, J. MORGAN, and D. SULLIVAN, Real homotopy theory of Kähler manifolds. *Invent. Math.* **29**(1975), No. 3, 245–274.
10. A. DOLD, Zur Homotopietheorie der Kettenkomplexe. *Math. Ann.* **140**(1960), 278–298.
11. A. DOLD, Halbexakte Homotopiefunktor. *Lecture Notes in Mathematics*, 12 *Springer-Verlag, Berlin-New York*, 1966.
12. S. EILENBERG, Homological dimension and syzygies. *Ann. of Math. (2)* **64**(1956), 328–336.
13. S. EILENBERG and S. MAC LANE, On the groups $H(\Pi, n)$. I. *Ann. of Math. (2)* **58**(1953), 55–106; II. Methods of computation. *Ann. of Math. (2)* **60**(1954), 49–139.
14. V.K.A.M. GUGENHEIM, On the chain-complex of a fibration. *Illinois J. Math.* **16**(1972), 398–414.
15. V.K.A.M. GUGENHEIM, On a perturbation theory for the homology of the loop-space. *J. Pure Appl. Algebra* **25**(1982), No. 2, 197–205.
16. V.K.A.M. GUGENHEIM and L. A. LAMBE, Perturbation theory in differential homological algebra. I. *Illinois J. Math.* **33**(1989), No. 4, 566–582.
17. V.K.A.M. GUGENHEIM, L. A. LAMBE, and J.D. STASHEFF, Algebraic aspects of Chen's twisting cochain. *Illinois J. Math.* **34**(1990), No. 2, 485–502.
18. V.K.A.M. GUGENHEIM, L. A. LAMBE, and J.D. STASHEFF, Perturbation theory in differential homological algebra. II. *Illinois J. Math.* **35**(1991), No. 3, 357–373.
19. V.K.A.M. GUGENHEIM and J.P. MAY On the theory and applications of differential torsion products. *Memoirs of the American Mathematical Society*, No. 142. *American Mathematical Society, Providence, R.I.*, 1974.
20. V.K.A.M. GUGENHEIM and R. J. MILGRAM, On successive approximations in homological algebra. *Trans. Amer. Math. Soc.* **150**(1970), 157–182.
21. V.K.A.M. GUGENHEIM and H. J. MUNKHOLM, On the extended functoriality of Tor and Cotor. *J. Pure Appl. Algebra* **4**(1974), 9–29.
22. V.K.A.M. GUGENHEIM and J. D. STASHEFF, On perturbations and A_∞ -structures. *Bull. Soc. Math. Belg. Sér. A* **38**(1986), 237–246 (1987).
23. S. HALPERIN, Lectures on minimal models. *Mém. Soc. Math. France (N.S.)* No. 9-10 (1983).
24. S. HALPERIN and J. D. STASHEFF, Obstructions to homotopy equivalences. *Adv. in Math.* **32**(1979), No. 3, 233–279.
25. A. HELLER, Homological resolutions of complexes with operators. *Ann. of Math. (2)* **60**(1954), 283–303.
26. P. J. HILTON, Homotopy theory and duality. *Gordon and Breach Science Publishers, New York–London–Paris*, 1965.

27. J. HUEBSCHMANN, Perturbation theory and small models for the chains of certain induced fibre spaces. *Habilitationsschrift Universität Heidelberg*, 1984; **Zbl.** 576.55012.
28. J. HUEBSCHMANN, The homotopy type of $F\Psi^q$. *The complex and symplectic cases. Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II* (Boulder, Colo., 1983), 487–518, *Contemp. Math.*, 55, Amer. Math. Soc., Providence, RI, 1986.
29. J. HUEBSCHMANN, Perturbation theory and free resolutions for nilpotent groups of class 2. *J. Algebra* **126**(1989), No. 2, 348–399.
30. J. HUEBSCHMANN, Cohomology of nilpotent groups of class 2. *J. Algebra* **126**(1989), No. 2, 400–450.
31. J. HUEBSCHMANN, The mod- p cohomology rings of metacyclic groups. *J. Pure Appl. Algebra* **60**(1989), No. 1, 53–103.
32. J. HUEBSCHMANN, Cohomology of metacyclic groups. *Trans. Amer. Math. Soc.* **328** (1991), No. 1, 1–72.
33. J. HUEBSCHMANN, Cohomology of finitely generated abelian groups. *Enseign. Math. (2)* **37**(1991), No. 1-2, 61–71.
34. J. HUEBSCHMANN, Berikashvili's functor D and the deformation equation. *Proc. A. Razmadze Math. Inst.* **119**(1999), 59–72.
35. J. HUEBSCHMANN, On the cohomology of the holomorph of a finite cyclic group. *J. Algebra* **279**(2004), No. 1, 79–90.
36. J. HUEBSCHMANN, Higher homotopies and Maurer-Cartan algebras: quasi-Lie-Rinehart, Gerstenhaber, and Batalin–Vilkovisky algebras. To appear in: *The Breadth of Symplectic and Poisson Geometry, Festschrift in honor of A. Weinstein's 60th birthday*; J. Marsden and T. Ratiu, eds.; *Progress in Mathematics*, Birkhäuser Verlag, Boston–Basel–Berlin, 2004; [math.DG/0311294](https://arxiv.org/abs/math/0311294)
37. J. HUEBSCHMANN and T. KADEISHVILI, Small models for chain algebras. *Math. Z.* **207**(1991), No. 2, 245–280.
38. J. HUEBSCHMANN and J. D. STASHEFF, Formal solution of the master equation via HPT and deformation theory. *Forum Math.* **14**(2002), No. 6, 847–868.
39. D. HUSEMOLLER, J. C. MOORE, and J. D. STASHEFF, Differential homological algebra and homogeneous spaces. *J. Pure Appl. Algebra* **5**(1974), 113–185.
40. T. V. KADEISHVILI, On the theory of homology of fiber spaces. (Russian) *International Topology Conference (Moscow State Univ., Moscow, 1979)*. *Uspekhi Mat. Nauk* **35**(1980), No. 3(213), 183–188; English transl.: *Russian Math. Surveys* **35**(1980), No. 3, 231–238.
41. T. KADEISHVILI, The predifferential of a fiber bundle. (Russian) *Uspekhi Mat. Nauk* **41**(1986), No. 6(252), 109–119; English transl.: The predifferential of a twisted product. *Russian Math. Surveys* **41**(1986), No. 6, 135–147.
42. T. KADEISHVILI, The $A(\infty)$ -algebra structure in cohomology, and rational homotopy type. (Russian) *Proc. A. Razmadze Math. Inst.* **107**(1993), 3–94.
43. L. LAMBE and J. STASHEFF, Applications of perturbation theory to iterated fibrations. *Manuscripta Math.* **58**(1987), No. 3, 363–376.
44. A. LIULEVICIUS, Multicomplexes and a general change of rings theorem. Mimeographed notes, University of Chicago.
45. A. LIULEVICIUS, A theorem in homological algebra and stable homotopy of projective spaces. *Trans. Amer. Math. Soc.* **109**(1963), 540–552.
46. S. MAC LANE, Homology. *Die Grundlehren der mathematischen Wissenschaften*, Bd. 114 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin–Göttingen–Heidelberg, 1963.
47. J. P. MEYER, Acyclic models for multicomplexes. *Duke Math. J.* **45**(1978), No. 1, 67–85.

48. J. C. MOORE, Algèbres d'Eilenberg-Mac Lane et homotopie, exposés 12 et 13. *Séminaire H. Cartan*, 1954–55, *École Normale Supérieure, Paris*, 1956.
49. J. C. MOORE, Differential homological algebra, *Actes du Congr. Intern. des Mathématiciens*, 1970, 335–339.
50. J. C. MOORE, Cartan's constructions, the homology of $\mathcal{K}(\pi, n)$'s, and some later developments. *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, 173–212. *Asterisque*, No. 32-33, *Soc. Math. France, Paris*, 1976.
51. J. C. MOORE and L. SMITH, Hopf algebras and multiplicative fibrations. I. *Amer. J. Math.* **90**(1968), 752–780.
52. H. J. MUNKHOLM, The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps. *J. Pure Appl. Algebra* **5**(1974), 1–50.
53. D. QUILLEN, Rational homotopy theory. *Ann. of Math. (2)* **90**(1969), 205–295.
54. S. A. SANEBLIDZE, Homology classification of differential algebras. (Russian) *Soobshch. Akad. Nauk Gruzín. SSR* **129**(1988), No. 2, 241–243.
55. S. SANEBLIDZE, Filtered model of a fibration and rational obstruction theory. *Manuscripta Math.* **76**(1992), No. 2, 111–136.
56. S. SANEBLIDZE, On the homotopy classification of spaces by the fixed loop space homology. *Proc. A. Razmadze Math. Inst.* **119**(1999), 155–164.
57. J. P. SERRE, Algèbre locale. Multiplicités. *Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics*, 11. Springer-Verlag, Berlin–New York, 1965.
58. W. SHIH, Homologie des espaces fibrés. *Inst. Hautes Études Sci. Publ. Math.* No. 13 (1962).
59. V.A. SMIRNOV, Homology of fibre spaces. *Uspekhi Mat. Nauk* **35**(1980), No. 3(213), 227–230; English transl.: *Russian Math. Surveys* **35**(1980), No. 3, 294–298.
60. J.D. STASHEFF, Homotopy associativity of H -spaces. I, II. *Trans. Amer. Math. Soc.* **108**(1963), 275–292; **108**(1963), 293–312.
61. J.D. STASHEFF and S. HALPERIN, Differential algebra in its own rite. *Proceedings of the Advanced Study Institute on Algebraic Topology (Aarhus Univ., Aarhus 1970)*, Vol. III, 567–577. *Various Publ. Ser.*, No. 13, *Mat. Inst., Aarhus Univ., Aarhus*, 1970.
62. D. SULLIVAN, Differential forms and the topology of manifolds. *Manifolds–Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)*, 37–49. *Univ. Tokyo Press, Tokyo*, 1975.
63. D. SULLIVAN, Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.* No. 47 (1977), 269–331 (1978).
64. C.T.C. WALL, Resolutions for extensions of groups. *Proc. Cambridge Philos. Soc.* **57**(1961), 251–255.

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