Georgian Mathematical Journal
Volume 12 (2005), Number 1, 45–52

ON THE UNIQUENESS OF THE TWO-SIDED ERGODIC MAXIMAL FUNCTION

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Abstract. It is proved that the two-sided ergodic maximal operator is one-to-one.

2000 Mathematics Subject Classification: 28D05.
Key words and phrases: Two-sided ergodic maximal function, uniqueness theorem.

1. Introduction. Let $(X, \mathbb{S}, \mu)$ be a finite measure space, $\mu(X) < \infty$, and let $T : X \to X$ be an invertible measure-preserving ergodic transformation.

For an integrable function f, $f \in L(X)$, the ergodic maximal function is denoted by $\mathbf{M}_+ f$ (the subscript "+" indicates that the operator is right-sided):

$$\mathbf{M}_+ f(x) = \sup_{m \geq 0} \frac{1}{m} \sum_{k=0}^{m-1} f(T^k x), \quad x \in X.$$

In [3] we prove that the ergodic maximal operator has the injectivity property, i.e. $\mathbf{M}_+ f = \mathbf{M}_+ g$ a.e. implies that $f = g$ a.e. (A different proof of this theorem is proposed in [4].) The same uniqueness theorem is proved for various one-sided maximal operators in [1], [2]. As mentioned in the introduction of [3], the essential idea of proving these theorems is contained in the proof of the uniqueness theorem for the one-sided Hardy–Littlewood maximal operator but the problem remains still open for non-one-sided maximal operators. As an approach to the solution of this problem, in the present paper we propose the proof of the uniqueness theorem for two-sided ergodic maximal operator $\mathbf{M}$,

$$\mathbf{M} f(x) = \sup_{n \leq 0 < m} \frac{1}{m-n} \sum_{k=n}^{m-1} f(T^k x), \quad x \in X.$$

Theorem 1. *Let $f, g \in L(X)$ and*

$$\mathbf{M} f = \mathbf{M} g \quad a.e. \tag{1}$$

Then

$$f = g \quad a.e. \tag{2}$$

An extension of the proof to the continuous case still requires to overcome some technical difficulties.

A simple example illustrates that the theorem is not valid for the symmetric ergodic maximal operator (see Section 5). The discrete nature of the operator plays a significant role in this situation (see [7]), and the continuous analogue of this theorem should in our opinion be correct.

ISSN 1072-947X / \$8.00 /

2. An auxiliary lemma. An analogue of the following lemma for the operator $\mathbf{M}_+$ is proved in [3].

Lemma 1. *Let $f \in L(X)$ and let*

$$F_f = \Big\{ x \in X : \mathbf{M}f(x) = \frac{1}{m-n} \sum_{k=n}^{m-1} f(T^k x) \ \text{ for some } \ n \le 0 \ \text{ and } \ m > 0 \Big\}.$$

Then

$$\mu(F_f) = \mu(X), \tag{3}$$

and, consequently,

$$\mu\big\{ x \in X : T^k x \in F_f \quad \text{for all} \quad k \in \mathbb{Z} \big\} = \mu(X). \tag{4}$$

Proof. Let

$$\lambda_0 = \frac{1}{\mu(X)} \int_X f \, d\mu.$$

The Individual Ergodic Theorem,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} f \circ T^k = \lambda_0 \quad \text{a.e.} \tag{5}$$

(see [6]) implies $\mu(\mathbf{M}f \ge \lambda_0) = \mu(X)$.

If $\lim\limits_{m\to\infty} \frac{1}{m} \sum\limits_{k=0}^{m-1} f(T^k x) = \lambda_0$ and $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} f(T^{-k} x) = \lambda_0$, then

$$\lim_{n,m\to\infty} \frac{1}{n+m} \sum_{k=-n}^{m-1} f(T^k x) = \lambda_0. \tag{6}$$

Thus (6) holds for a.a. $x \in X$, and this implies that a.a. $x \in (\mathbf{M}f > \lambda_0)$ belongs to F_f.

If $\mathbf{M}f(x) = \lambda_0$, then also $\mathbf{M}_+ f(x) = \lambda_0$ and it is proved in [3] (see Corollary 1 therein) that for a.a. $x \in X$ there exist $m = m(x)$ such that $\frac{1}{m} \sum\limits_{k=0}^{m-1} f(T^k x) = \lambda_0$. Consequently a.a. $x \in (\mathbf{M}f = \lambda_0)$ belongs to F_f.

The proof of (3) is completed.

Since $\{x \in X : T^k x \in F_f \text{ for all } k \in \mathbb{Z}\} = \bigcap_{k=-\infty}^{\infty} T^k(F_f)$, (4) holds as well. □

3. Discrete maximal operator. Let Γ denote the set of all two-sided sequences of real numbers indexed by integers $\mathbb{Z}$. The maximal operator M is defined by

$$M\alpha(q) = \sup_{n \le q < m} \frac{1}{m-n} \sum_{k=n}^{m-1} \alpha(k), \quad q \in \mathbb{Z}, \quad \alpha \in \Gamma.$$

Thus, if $\alpha(q) = f(T^q x)$, then

$$M\alpha(q) = \mathbf{M}f(T^q x). \tag{7}$$

We consider the one-sided maximal operators M_+ and M_- as well:

$$M_+\alpha(q) = \sup_{m\ge q} \frac{1}{m-q+1}\sum_{k=q}^{m}\alpha(k) \ \text{ and } \ M_-\alpha(q) = \sup_{n\le q} \frac{1}{q-n+1}\sum_{k=n}^{q}\alpha(k).$$

Let us introduce some brief notations. Sets of the type $\{q \in \mathbb{Z}: M\alpha(q) > \lambda\}$, $\{x \in X : \mathbf{M}f(x) = \mathbf{M}g(x)\}, \dots$ will be denoted by $(M\alpha > \lambda)$, $(\mathbf{M}f = \mathbf{M}g), \dots$

For $\alpha \in \Gamma$ and $I \subset \mathbb{Z}$, let $A_I = (1/\operatorname{card}(I)) \sum_{k\in I} \alpha(k)$ and if $I_{p,r} = \{p, p+1, \dots, r\}$, $p, r \in \mathbb{Z}$, $p \le r$, is an interval of integers, then $A_{p,r} = A_{I_{p,r}}$.

We say that $I_{p,r}$ is a finite connected component of $N \subset \mathbb{Z}_0$ if $I_{p,r} \subset N$ and $p-1, r+1$ do not belong to N.

The proof of the following lemma is very easy but we formulate it for further reference.

Lemma 2. *Let $I, J \subset \mathbb{Z}$ be disjoint and $K = I \cup J$.*
(i) *If $A_K = \lambda$ and $A_J = \lambda$, then $A_I = \lambda$;*
(ii) *If $A_K = \lambda$ and $A_J < \lambda$, then $A_I > \lambda$;*
(iii) *If $A_K > \lambda$ and $A_J \le \lambda$, then $A_I > \lambda$;*
(iv) *If $A_I = \lambda$ and $A_J > \lambda$, then $A_K > \lambda$.*

Proof. If $\sum_{k\in K}\alpha(k) = \lambda \operatorname{card}(K)$ and $\sum_{k\in J}\alpha(k) = \lambda\operatorname{card}(J)$, then $\sum_{k\in I}\alpha(k) = \lambda(\operatorname{card}(K) - \operatorname{card}(J)) = \lambda\operatorname{card}(I)$ and (i) follows.

If $\sum_{k\in I}\alpha(k) \le \lambda\operatorname{card}(I)$ and $\sum_{k\in J}\alpha(k) < \lambda\operatorname{card}(J)$, then $\sum_{k\in K}\alpha(k) < \lambda(\operatorname{card}(I)+ \operatorname{card}(J)) = \lambda\operatorname{card}(K)$ which is a contradiction and (ii) follows.

In a similar way one can show (iii) and (iv). □

For $\alpha \in \Gamma$, let $N_\alpha \subset \mathbb{Z}$ be the set of integers for which the supremum is achieved after finitely many steps, i.e., $q \in N_\alpha$ if and only if $M\alpha(q) = A_{n,m}$ for some $n, m \in \mathbb{Z}$, $n \le q \le m$. Observe that if $\alpha_x(k) = f(T^k x)$, $f \in L(X)$, then $k \in N_{\alpha_x} \Leftrightarrow T^k x \in F_f$. Hence, for a.a. $x \in X$, we have (see (4))

$$N_{\alpha_x} = \mathbb{Z}. \tag{8}$$

Lemma 3. *Let $\alpha \in \Gamma$, $q \in N_\alpha$ and*

$$M\alpha(q) = \lambda,$$

and let $I_{p,q-1}$ and $I_{q+1,r}$ be finite connected components of $(M_-\alpha > \lambda)$ and $(M_+\alpha > \lambda)$, respectively, then

$$M\alpha(q) = \frac{1}{r-p+1}\sum_{k=p}^{r}\alpha(k) = \lambda. \tag{9}$$

We assume that if $M_-\alpha(q-1) \le \lambda$ (i.e., $I_{p,q-1} = \varnothing$), then $p = q$ and if $M_+\alpha(q+1) \le \lambda$ (i.e., $I_{q+1,r} = \varnothing$), then $r = q$ in (9) and in the proof below.

Proof. Let

$$\lambda = M\alpha(q) = \frac{1}{q_2 - q_1 + 1}\sum_{k=q_1}^{q_2}\alpha(k), \tag{10}$$

where $q_1 \le q \le q_2$. We can assume that I_{q_1,q_2} is minimal in a sense that it contains no proper subset $I_{p_1,p_2} \ni q$ for which $A_{p_1,p_2} = \lambda$, and we will show that $q_1 = p$ and $q_2 = r$.

If $q_2 > r$, then $A_{r+1,q_2} \le \lambda$, since $r+1 \notin M_+\alpha > \lambda$. If now

$$A_{r+1,q_2} = \lambda, \tag{11}$$

then Lemma 2 (i), (10) and (11) imply that $A_{q_1,r} = \lambda$, which contradicts the minimality of I_{q_1,q_2}, and if

$$A_{r+1,q_2} < \lambda, \tag{12}$$

then Lemma 2 (ii), (10) and (12) imply that $A_{q_1,r} > \lambda$, which is a contradiction, since $q \in I_{q_1,r}$ and (10) holds.

Analogously, q_1 cannot be smaller than p.

If $q_2 < r$, then

$$A_{q_2+1,r} > \lambda, \tag{13}$$

(see [3], Lemma 4) and Lemma 2 (iv), (10) and (13) imply that $A_{q_1,r} > \lambda$, which is a contradiction, since $q \in I_{q_1,r}$ and (10) holds.

Analogously, q_1 cannot be larger than p. □

Lemma 4. *Let $\alpha \in \Gamma$ and let $I_{p,r}$ be a finite connected component of $(M\alpha > \lambda_0)$ for some $\lambda_0 < \infty$. Then for each $q \in I_{p,r}$ there exists an interval of integers $J \subset I_{p,r}$ containing q such that $M\alpha(q) = A_J$.*

Proof. Let $\epsilon < M\alpha(q) - \lambda_0$ and $A_{q_1,q_2} > \lambda > M\alpha(q) - \epsilon$, where $q_1 \le q \le q_2$. If $r < q_2$, then $A_{r+1,q_2} \le \lambda$ since $r+1 \notin (M\alpha > \lambda_0)$ and we can apply Lemma 2 (iii) to conclude that $A_{q_1,r} > \lambda$. Similarly, we can deal with q_1 and it follows that for each $\epsilon > 0$ there exists an interval $I \subset I_{p,r}$ containing q such that $A_I \ge M(q) - \epsilon$. Consequently, $M(q) = A_J$ for some interval $J \subset I_{p,r}$, $J \ni q$, since the number of such intervals is finite and the lemma follows. □

Lemma 5. *Let $\alpha \in \Gamma$, let $I_{p,r}$ be a finite connected component of $(M\alpha > \lambda_0)$ for some $\lambda_0 < \infty$, and let $\lambda \ge \lambda_0$. If we know the values of α on $(M\alpha > \lambda) \cap I_{p,r}$, then we can identify the sets $(M_-\alpha > \lambda) \cap I_{p,r}$ and $(M_+\alpha > \lambda) \cap I_{p,r}$.*

Proof. Obviously, $((M_+\alpha > \lambda) \cap I_{p,r}) \subset ((M\alpha > \lambda) \cap I_{p,r})$. Thus we should determine for each $q \in ((M\alpha > \lambda) \cap I_{p,r})$ whether it belongs to $(M_+\alpha > \lambda) \cap I_{p,r}$. Assume $s \ge q$ is the minimal integer outside $(M\alpha > \lambda)$ (note that $s \le r$ and we know all values of α on $I_{q,s-1}$). If now $q_2 \ge s$ is such that $A_{q,q_2} > \lambda$, then $A_{s,q_2} \le \lambda$ and $A_{q,s-1} > \lambda$ because of Lemma 2 (iii). Hence $q \in ((M_+\alpha > \lambda) \cap I_{p,r})$ if and only if $\sup_{q_2 \in I_{q,s-1}} A_{q,q_2} > \lambda$.

In a similar way one can identify $(M_-\alpha > \lambda)$. □

Lemma 6. *Let $\alpha \in \Gamma$ and let $I_{p,r}$ be a finite connected component of $(M\alpha > \lambda)$ for some $\lambda < \infty$. Then the values $M\alpha(q), q \in I_{p,r}$, uniquely define the values $\alpha(q), q \in I_{p,r}$. Thus if $M\alpha(q) = M\beta(q)$, $q \in \mathbb{Z}$, for some $\beta \in \Gamma$, then $\alpha(q) = \beta(q), q \in I_{p,r}$.*

Proof. Note that $q \in N_\alpha$ for each $q \in I_{p,r}$ because of Lemma 4.

Set the values $M\alpha(q), q \in I_{p,r}$, in descending order, i.e., assume $\lambda_1 > \lambda_2 > \cdots > \lambda_j > \lambda$, where

$$I_i = \left\{q \in I_{p,r} : M\alpha(q) = \lambda_i\right\} \neq \varnothing$$

and

$$\bigcup_{i=1}^{j} I_i = I_{p,r}.$$

Define the values $\alpha(q)$ by induction with respect to i. For $i = 1$, α is equal to λ_1 on I_1, i.e.,

$$\alpha(q) = \lambda_1$$

for all $q \in I_1$. Indeed, it follows from Lemma 3 that $M\alpha(q) = \alpha(q)$ for each $q \in I_1$, since $q - 1$ and $q + 1$ do not belong to $(M\alpha > \lambda_1)$.

Let us now assume that α is already defined on $I_1 \cup I_2 \cup \cdots \cup I_i$, $i < j$; we will define it on I_{i+1}.

For $q \in I_{i+1}$ (which implies that $M\alpha(q) = \lambda_{i+1}$), since $((M\alpha > \lambda_{i+1}) \cap I_{p,r}) = I_1 \cup I_2 \cup \cdots \cup I_i$ and we know the values of α on this set, we can identify $(M_-\alpha > \lambda_{i+1}) \cap I_{p,r}$ and $(M_+\alpha > \lambda_{i+1}) \cap I_{p,r}$, because of Lemma 5.

Consequently, we can apply Lemma 3 and all the values in equation (9) are known except $\alpha(q)$ which can be determined. □

Corollary 1. *Let $\alpha \in \Gamma$ and let $M\alpha(p) < M\alpha(q_0) > M\alpha(r)$ for some $p, r \in \mathbb{Z}$, $p \leq q_0 \leq r$. Then the values $M\alpha(q)$, $q \in \mathbb{Z}$, uniquely define the value $\alpha(q_0)$. Thus if some other $\beta \in \Gamma$ is given such that $M\alpha(q) = M\beta(q)$, $q \in \mathbb{Z}$, then $\alpha(q_0) = \beta(q_0)$.*

Proof. If we take λ strictly between $M(q_0)$ and $\max((M\alpha(p), M\alpha(r))$, then there is a finite connected component of $(M\alpha > \lambda)$ containing q_0. □

Lemma 7. *Let $\alpha \in \Gamma$, let $q \in N_\alpha$ and $M\alpha(q) \geq \lambda_0$ for each $q \in \mathbb{Z}$, and let*

$$\operatorname{card}\{k \geq 0 : \alpha(k) = \lambda_0\} = \operatorname{card}\{k \leq 0 : \alpha(k) = \lambda_0\} = \infty. \tag{14}$$

Then the values $M\alpha(q)$, $q \in \mathbb{Z}$, uniquely define the values $\alpha(q)$, $q \in \mathbb{Z}$. Thus if $M\alpha(q) = M\beta(q)$, $q \in \mathbb{Z}$, for some $\beta \in \Gamma$ such that $q \in N_\beta$ and $M\beta(q) \geq \lambda_0$ for each $q \in \mathbb{Z}$ and

$$\operatorname{card}\left\{k \geq 0 : \beta(k) = \lambda_0\right\} = \operatorname{card}\left\{k \leq 0 : \beta(k) = \lambda_0\right\} = \infty,$$

then $\alpha(q) = \beta(q), q \in \mathbb{Z}$.

Proof. Relation (14) implies that the set $(M\alpha > \lambda_0)$ consists of finite connected components. Hence we can determine the values $\alpha(q)$, $q \in (M\alpha > \lambda_0)$, by Lemma 6. It also follows from Lemma 5 that we can identify finite connected components of $(M_+\alpha > \lambda_0)$ and $(M_-\alpha > \lambda_0)$, and if $M\alpha(q) = \lambda_0$, then we can use Lemma 3 to find the only unknown quantity $\alpha(q)$ of equation (9). □

4. Proof of Theorem 1. Equation (1) implies that $\mathbf{M}f(T^k x) = \mathbf{M}g(T^k x)$, $k \in \mathbb{Z}$, for a.a. $x \in X$ (more exactly, for all $x \notin \bigcap_{k=-\infty}^{\infty} T^k(\mathbf{M}f \neq \mathbf{M}g)$). Thus

$$M\alpha_x(k) = M\beta_x(k), \quad k \in \mathbb{Z}, \tag{15}$$

for a.a. $x \in X$, where

$$\alpha_x(k) = f(T^k x) \quad \text{and} \quad \beta_x(k) = g(T^k x)$$

(see (7)), and we will show that

$$\alpha_x(k) = \beta_x(k), \quad k \in \mathbb{Z}, \tag{16}$$

for a.a. $x \in X$, which completes the proof of (2).

Relation (8) implies that, for a.a. $x \in X$,

$$\mathbb{Z} = N_{\alpha_x} = N_{\beta_x}. \tag{17}$$

Let λ_0 be $\operatorname{ess\,inf} \mathbf{M}f = \operatorname{ess\,inf} \mathbf{M}g$. Then

$$\mathbf{M}f(T^k x) = \mathbf{M}g(T^k x) \geq \lambda_0, \quad k \in \mathbb{Z},$$

for a.a. $x \in X$ (for all $x \notin \bigcap_{k=-\infty}^{\infty} T^k(\mathbf{M}f = \mathbf{M}g < \lambda_0)$ and, consequently,

$$M\alpha_x(k) = M\beta_x(k) \geq \lambda_0, \quad k \in \mathbb{Z}, \tag{18}$$

for a.a. $x \in X$.

We consider two cases:

(i) $\mu(\mathbf{M}f = \lambda_0) = \mu(\mathbf{M}g = \lambda_0) = 0$. Then

$$M\alpha_x(k) = M\beta_x(k) > \lambda_0, \quad k \in \mathbb{Z}, \tag{19}$$

for a.a. $x \in X$.

Choose any decreasing sequence λ_i, $i = 1, 2, \ldots$, convergent to λ_0. Then

$$\mu(\mathbf{M}f < \lambda_i) = \mu(\mathbf{M}g < \lambda_i) > 0, \quad i = 1, 2, \ldots,$$

and the Individual Ergodic Theorem implies that

$$\begin{aligned} &\operatorname{card}\big\{k \leq 0 : T^k x \in (\mathbf{M}f < \lambda_i)\big\} \\ &\qquad = \operatorname{card}\big\{k \geq 0 : T^k x \in (\mathbf{M}g < \lambda_i)\big\} = \infty, \quad i \geq 1, \end{aligned} \tag{20}$$

for a.a. $x \in X$. Now, for each x satisfying (15), (19) and (20) and for any $q_0 \in \mathbb{Z}$, since $M\alpha_x(q_0) > \lambda_0$, there exist $p \leq q_0$ and $r \geq q_0$ such that $M\alpha_x(p) < M\alpha_x(q_0) > M\alpha_x(r)$. Thus we can apply Corollary 1 of Lemma 6 to conclude that

$$\alpha_x(q_0) = \beta_x(q_0)$$

and, since q_0 is an arbitrary integer, (16) is proved.

(ii) $\mu(\mathbf{M}f = \lambda_0) = \mu(\mathbf{M}g = \lambda_0) > 0$. Then, by the Individual Ergodic Theorem,

$$\operatorname{card}\big\{k \leq 0 : T^k x \in (\mathbf{M}f = \lambda_0)\big\} = \operatorname{card}\big\{k \geq 0 : T^k x \in (\mathbf{M}g = \lambda_0)\big\} = \infty$$

and

$$\operatorname{card}\big\{k \leq 0 : M\alpha_x(k) = \lambda_0\big\} = \operatorname{card}\big\{k \geq 0 : M\alpha_x(k) = \lambda_0\big\} = \infty \tag{21}$$

for a.a. $x \in X$. If now x satisfies (15), (17), (18) and (21), then we can apply Lemma 7 and establish the validity of (16). □

5. Counterexample for the symmetric maximal operator. For a measurable function f, the symmetric ergodic maximal function $\mathbf{M}_s f$ is defined by

$$\mathbf{M}_s f(x) = \sup_{m \geq 0} \frac{1}{2m+1} \sum_{k=-m}^{m} f(T^k x), \quad x \in X.$$

Let the measure space X be $\{0, 1, 2\}$ with counting measure μ and let T be the transformation $T(x) = x + 1 (\mathrm{mod}\ 3)$. Define the functions f and g as follows: $f(x) = x$, $x \in X$ and $g(0) = 1$, $g(1) = 0$, $g(2) = 2$. Then $\mathbf{M}_s f(0) = \mathbf{M}_s g(0) = \mathbf{M}_s f(1) = \mathbf{M}_s g(1) = 1$ and $\mathbf{M}_s f(2) = \mathbf{M}_s g(2) = 2$, i.e., $\mathbf{M}_s f = \mathbf{M}_s g$, while $f \neq g$.

6. Infinite measure case. For infinite measure spaces, $\mu(X) = \infty$, the uniqueness theorem is not any longer valid. For example, every integrable negative function f has the maximal function $\mathbf{M}f$ equal identically to 0 since the limit of ergodic averages of every integrable function converges to 0 almost everywhere,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} f(T^k x) = 0$$

for a.a. $x \in X$, $f \in L(X)$ (see [5]).

However, for non-negative functions, the uniqueness theorem is correct:

Theorem 2. *Let T be an invertible measure-preserving ergodic transformation of a σ-finite measure space $(X, \mathbb{S}, \mu)$. If $0 \leq f, g \in L$ and $\mathbf{M}f = \mathbf{M}g$ almost everywhere, then $f = g$ almost everywhere.*

This theorem can be proved in the same way as for the one-sided maximal operator $\mathbf{M}_+$ in Section 4 of [3]. Moreover, the exact analog of Theorem 2 in [3] is correct for the two-sided operator $\mathbf{M}$.

Theorem 3. *Let T be an invertible measure-preserving ergodic transformation of a σ-finite measure space $(X, \mathbb{S}, \mu)$ with $\mu(X) = \infty$.*

(i) *If $f \in L$ and*

$$\mathbf{M}f = \mathbf{M}g \quad a.e. \ on \quad X, \tag{22}$$

then $f = g$ a.e. on $(\mathbf{M}f > 0)$;

(ii) *If $f \in L$ and $\mu(\mathbf{M}f = 0) > 0$, then (22) holds for each $g \in L$ such that $g = f$ on $(\mathbf{M}f > 0)$ and $g \leq f$ on $(\mathbf{M}f = 0)$.*

Since this theorem can be proved as in [3], we omit its proof here.

Acknowledgement

The research was supported by the CNR-NATO grant of Italy No. 217.35 S.

References

1. L. Ephremidze, On the uniqueness of maximal functions. *Georgian Math. J.* **3**(1996), 49–52.
2. L. Ephremidze, On the uniqueness of maximal operators for ergodic flows. *Rev. Mat. Complut.* **15**(2002), No. 1, 75–84.
3. L. Ephremidze, On the uniqueness of the ergodic maximal function. *Fund. Math.* **174**(2002), No. 3, 217–228.
4. R. L. Jones, On the uniqueness of the ergodic maximal function. *Proc. Amer. Math. Soc.* **132**(2004), No. 4, 1087–1090.
5. P. R. Halmos, Lectures on ergodic theory. *Publications of the Mathematical Society of Japan,* No. 3. *The Mathematical Society of Japan,* 1956.
6. K. E. Petersen, Ergodic theory. *Cambridge Studies in Advanced Mathematics,* 2. *Cambridge University Press, Cambridge,* 1983.
7. P. Sjogren, How to recognize a discrete maximal function. *Indiana Univ. Math. J.* **37**(1988), No. 4, 891–898.

(Received 29.09.2004)

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