Georgian Mathematical Journal
Volume 12 (2005), Number 1, 131–138

UNIQUENESS OF NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING 1-POINTS

INDRAJIT LAHIRI AND PULAK SAHOO

Abstract. We prove two theorems on the uniqueness of nonlinear differential polynomials sharing 1-points which improve an earlier result of Fang and Hong.

2000 Mathematics Subject Classification: 30D35.
Key words and phrases: Uniqueness, meromorphic function, non-linear differential polynomial.

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let k be a positive integer or infinity and $a \in \{\infty\} \cup \mathbb{C}$. We denote by $E_{k)}(a; f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. If for some $a \in \{\infty\} \cup \mathbb{C}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$, then we say that f, g share the value a CM (counting multiplicities).

In [3, 5], the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points was studied. Also in [3] the following question was asked: *What can be said if two non-linear differential polynomials generated by two meromorphic functions share* 1 *CM ?*

In the meantime some works have been done in this direction (cf. [1, 7]). Recently, Fang and Hong [1] have proved the following result.

Theorem A ([1])**.** *Let f and g be two transcendental entire functions and $n(\geq 11)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share* 1 *CM, then $f \equiv g$.*

In this paper we prove the following two theorems which improve Theorem A.

Theorem 1.1. *Let f and g be two transcendental entire functions and $n(\geq 7)$ be an integer. If $E_{3)}(1; f^n(f-1)f') = E_{3)}(1; g^n(g-1)g')$, then $f \equiv g$.*

Theorem 1.2. *Let f and g be two transcendental meromrophic functions such that $\Theta(\infty; f) > 0$, $\Theta(\infty; g) > 0$, $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$, and $n(\geq 11)$ be an integer. If $E_{3)}(1; f^n(f-1)f') = E_{3)}(1; g^n(g-1)g')$, then $f \equiv g$.*

Remark 1.1. If we choose $n\,(\geq 12)$, then in Theorem 1.2 the condition $\Theta(\infty; f) > 0$ and $\Theta(\infty; g) > 0$ can be removed.

The following example shows that the condition $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$ is sharp for Theorem 1.2.

ISSN 1072-947X / $8.00 /

Example 1.1. Let

$$f = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad g = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})} \quad \text{and} \quad h = \frac{\alpha^2(e^z-1)}{e^z-\alpha},$$

where $\alpha = \exp\left(\frac{2\pi i}{n+2}\right)$ and n is a positive integer.

Then $T(r,f) = (n+1)T(r,h) + O(1)$ and $T(r,g) = (n+1)T(r,h) + O(1)$. Further we see that $h \neq \alpha, \alpha^2$ and a root of $h = 1$ is not a pole of f and g. Hence $\Theta(\infty; f) = \Theta(\infty; g) = 2/(n+1)$. Also, $f^{n+1}\left(\frac{f}{n+1} - \frac{1}{n+1}\right) \equiv g^{n+1}\left(\frac{g}{n+1} - \frac{1}{n+1}\right)$ and $f^n(f-1)f' \equiv g^n(g-1)g'$ but $f \not\equiv g$.

Though we do not explain the standard notation of the value distribution theory (see [2]) we give the following definitions.

Definition 1.1 ([4])**.** For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid= 1)$ the counting functions of simple a-points of f.

For a positive integer m we denote by $N(r,a;f \mid\leq m)$ ($N(r,a;f \mid\geq m)$) the counting function of those a-points of f whose multiplicities are not greater(less) than m, where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid\leq m)$ and $\overline{N}(r,a;f \mid\geq m)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f \mid< m)$, $N(r,a;f \mid> m)$, $\overline{N}(r,a;f \mid< m)$ and $\overline{N}(r,a;f \mid> m)$ are defined analogously.

Definition 1.2 (cf. [12])**.** For $a \in \mathbb{C} \cup \{\infty\}$ we put

$$N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2).$$

2. Lemmas

In this section we present some lemmas which are needed in the sequel. We denote by h the function

$$h = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right).$$

Lemma 2.1. *If $E_{1)}(1;f) = E_{1)}(1;g)$ and $h \not\equiv 0$, then*

$$N(r,1;f \mid\leq 1) \leq N(r,0;h) \leq N(r,h) + S(r,f) + S(r,g).$$

Proof. By a simple calculation we see that a simple zero of f is a zero of h and the lemma follows. □

Lemma 2.2. *If $E_{3)}(1;f) = E_{3)}(1;g)$ and $h \not\equiv 0$, then*

$$\begin{aligned} N(r,h) \leq{}& \overline{N}(r,\infty;f \mid\geq 2) + \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,\infty;g \mid\geq 2) \\ &+ \overline{N}(r,0;g \mid\geq 2) + \overline{N}(r,1;f \mid\geq 4) + \overline{N}(r,1;g \mid\geq 4) \\ &+ \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ and $\overline{N}_0(r,0;g')$ are the reduced counting functions of the zeros of f' and g' which are not the zeros of $f(f-1)$ and $g(g-1)$, respectively.

The proof is omitted.

Lemma 2.3 ([13]). *If $h \equiv 0$, then f and g share 1 CM.*

Lemma 2.4 ([8, 10]). *If f and g share 1 CM, then one of the following cases holds:*

(i) $T(r,f)+T(r,g) \leq 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\}+S(r,f)+S(r,g)$,

(ii) $f \equiv g$;

(iii) $fg \equiv 1$.

Lemma 2.5. *If $E_{3)}(1;f) = E_{3)}(1;g)$, then the conclusion of Lemma 2.4 holds.*

Proof. If $h \equiv 0$, then the result follows from Lemmas 2.3 and 2.4. So we suppose that $h \not\equiv 0$. Then by the second fundamental theorem, Lemmas 2.1 and 2.2 we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) - \overline{N}_0(r,0;f') + S(r,f) \\ &\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;f \mid\geq 2) \\ &\quad + \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,\infty;g \mid\geq 2) + \overline{N}(r,0;g \mid\geq 2) \\ &\quad + \overline{N}(r,1;f \mid\geq 4) + \overline{N}(r,1;g \mid\geq 4) - N(r,1;f \mid\leq 1) \\ &\quad + \overline{N}_0(r,0;g'). \end{aligned} \tag{2.1}$$

Again, by the second fundamental theorem we get

$$T(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;g) - \overline{N}_0(r,0;g') + S(r,g). \tag{2.2}$$

Also, we note that

$$\overline{N}(r,1;f) - \frac{1}{2}N(r,1;f \mid\leq 1) + \overline{N}(r,1;f \mid\geq 4) \leq \frac{1}{2}N(r,1;f) \leq \frac{1}{2}T(r,f) \tag{2.3}$$

and

$$\overline{N}(r,1;g) - \frac{1}{2}N(r,1;g \mid\leq 1) + \overline{N}(r,1;g \mid\geq 4) \leq \frac{1}{2}N(r,1;g) \leq \frac{1}{2}T(r,g). \tag{2.4}$$

Adding (2.1) and (2.2) and using (2.3) and (2.4), we obtain

$$\begin{aligned} T(r,f) + T(r,g) \leq 2\{&N_2(r,0;f) + N_2(r,\infty;f) + N_2(r,0;g) + N_2(r,\infty;g)\} \\ &+ S(r,f) + S(r,g). \end{aligned}$$

This proves the lemma. □

Lemma 2.6. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{n+1},$$

where $n(\geq 2)$ is an integer. Then

$$f^{n+1}(af+b) \equiv g^{n+1}(ag+b)$$

implies $f \equiv g$, where a, b are finite nonzero constants and n is an integer.

We omit the proof because it can be carried out that of Lemma 6 [6].

Lemma 2.7. *Let f and g be nonconstant meromorphic functions. Then*

$$f^n(f-1)f'g^n(g-1)g' \not\equiv 1,$$

where n (≥ 5) is an integer.

Proof. If possible, let

$$f^n(f-1)f'g^n(g-1)g' \equiv 1.$$

Let z_0 be an 1-point of f with multiplicity $p(\geq 1)$. Then z_0 is a pole of g with multiplicity $q(\geq 1)$ such that

$$2p-1=(n+2)q+1,$$

i.e.,

$$2p=(n+2)q+2\geq n+4,$$

i.e.,

$$p\geq \frac{n+4}{2}.$$

Let z_0 be a zero of f with multiplicity $p(\geq 1)$ and it be a pole of g with multiplicity $q(\geq 1)$. Then

$$(n+1)p-1=(n+2)q+1. \tag{2.5}$$

From (2.5) we get

$$\begin{aligned} & q+2=(n+1)(p-q)\geq n+1 \\ \text{i.e.,}\quad & q\geq n-1. \end{aligned}$$

Again, from(2.5) we get

$$\begin{aligned} & (n+1)p=(n+2)q+2\geq (n+2)(n-1)+2 \\ \text{i.e.,}\quad & p\geq \frac{(n+2)(n-1)+2}{n+1}=n. \end{aligned}$$

Since a pole of f is either a zero of $g(g-1)$ or a zero of g', we see that

$$\begin{aligned} \overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}_0(r,0;g') \\ &\leq \frac{1}{n}N(r,0;g)+\frac{2}{n+4}N(r,1;g)+\overline{N}_0(r,0;g') \\ &\leq \left(\frac{1}{n}+\frac{2}{n+4}\right)T(r,g)+\overline{N}_0(r,0;g'). \end{aligned}$$

Now by the second fundamental theorem we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f) \\ &\leq \frac{1}{n}N(r,0;f)+\frac{2}{n+4}N(r,1;f)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f), \end{aligned}$$

i.e.,

$$\left(1-\frac{1}{n}-\frac{2}{n+4}\right)T(r,f)\le\left(\frac{1}{n}+\frac{2}{n+4}\right)T(r,g)+\overline{N}_0(r,0;g')$$
$$-\overline{N}_0(r,0;f')+S(r,f). \tag{2.6}$$

Similarly, we get

$$\left(1-\frac{1}{n}-\frac{2}{n+4}\right)T(r,g)\le\left(\frac{1}{n}+\frac{2}{n+4}\right)T(r,f)+\overline{N}_0(r,0;f')$$
$$-\overline{N}_0(r,0;g')+S(r,g). \tag{2.7}$$

Adding (2.6) and (2.7) we get

$$\left(1-\frac{2}{n}-\frac{4}{n+4}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

which is a contradiction because $1-\frac{2}{n}-\frac{4}{n+4}>0$. This proves the lemma. ☐

Lemma 2.8 ([9]). *Let f be a nonconstant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\cdots+a_nf^n$, where $a_0,a_1,\ldots,a_n$ are constants and $a_n\ne 0$. Then*

$$T(r,P(f))=nT(r,f)+O(1).$$

Lemma 2.9 ([11]). *Let f be a nonconstant meromorphic function. Then*

$$N(r,0;f^{(k)})\le k\overline{N}(r,\infty;f)+N(r,0;f)+S(r,f).$$

Lemma 2.10. *Let f and g be two nonconstant meromorphic functions and*

$$F=f^{n+1}\left(\frac{f}{n+2}-\frac{1}{n+1}\right)\quad\text{and}\quad G=g^{n+1}\left(\frac{g}{n+2}-\frac{1}{n+1}\right),$$

where $n(\ge 4)$ is an integer. Then $F'\equiv G'$ implies $F\equiv G$.

Proof. If $F'\equiv G'$ then $F\equiv G+c$, where c is a constant. If possible, let $c\ne 0$. Then by the second fundamental theorem and *Lemma 2.8* we get

$$\begin{aligned}(n+2)T(r,f)&\le\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,c;F)+S(r,F)\\&=\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,\frac{n+2}{n+1};f)+\overline{N}(r,0;g)\\&\quad+\overline{N}(r,\frac{n+2}{n+1};g)+S(r,f)\\&\le 3T(r,f)+2T(r,g)+S(r,f),\end{aligned}$$

i.e.,

$$(n-1)T(r,f)\le 2T(r,g)+S(r,f).$$

Similarly, we get

$$(n-1)T(r,g)\le 2T(r,f)+S(r,g).$$

This shows that

$$(n-3)T(r,f)+(n-3)T(r,g)\le S(r,f)+S(r,g),$$

which is a contradiction. Therefore $c = 0$ and so $F \equiv G$. This proves the lemma. □

3. Proofs of the Theorems

Proof of Theorem 1.2. Let F and G be defined as in Lemma 2.10. Now in view of the first fundamental theorem and Lemma 2.8 we get

$$\begin{aligned} T(r,F) &= T(r,\frac{1}{F}) + O(1) \\ &= N(r,0;F) + \left(r,\frac{1}{F}\right) + O(1) \\ &\le N(r,0;F) + m\left(r,\frac{F'}{F}\right) + m\left(r,\frac{1}{F'}\right) \\ &= T(r,F') + N(r,0;F) - N(r,0;F') + S(r,F) \\ &= T(r,F') + (n+1)N(r,0;f) + N\left(r,\frac{n+2}{n+1};f\right) - nN(r,0;f) \\ &\quad - N(r,1;f) - N(r,0;f') + S(r,f) \\ &= T(r,F') + N(r,0;f) + N\left(r,\frac{n+2}{n+1};f\right) \\ &\quad - N(r,1;f) - N(r,0;f') + S(r,f). \end{aligned}$$

If possible, suppose that

$$\begin{aligned} T(r,F') + T(r,G') \le 2\{&N_2(r,0;F') + N_2(r,0;G') + N_2(r,\infty;F') \\ &+ N_2(r,\infty;G')\} + S(r,F') + S(r,G'). \end{aligned} \tag{3.1}$$

Then we get by Lemma 2.9

$$\begin{aligned} T(r,F) + T(r,G) &\le T(r,F') + T(r,G') + N(r,0;f) + N\left(r,\frac{n+2}{n+1};f\right) \\ &\quad - N(r,1;f) - N(r,0;f') + N(r,0;g) + N\left(r,\frac{n+2}{n+1};g\right) \\ &\quad - N(r,1;g) - N(r,0;g') + S(r,f) + S(r,g) \\ &\le 2N_2(r,0;F') + 2N_2(r,0;G') + 2N_2(r,\infty;F') \\ &\quad + 2N_2(r,\infty;G') + N(r,0;f) + N\left(r,\frac{n+2}{N+1};f\right) \\ &\quad - N(r,1;f) - N(r,0;f') + N(r,0,g) + N\left(r,\frac{n+2}{n+1};g\right) \\ &\quad - N(r,1;g) - N(r,0;g') + S(r,f) + S(r,g) \\ &\le 4\overline{N}(r,0;f) + 2N(r,1;f) + 2N(r,0;f') \\ &\quad + 4\overline{N}(r,0;g) + 2N(r,1;g) + 2N(r,0;g') \\ &\quad + 4\overline{N}(r,\infty;f) + 4\overline{N}(r,\infty;g) + N(r,0;f) \end{aligned}$$

$$
\begin{aligned}
&+ N\left(r, \frac{n+2}{n+1}; f\right) - N(r,1;f) - N(r,0;f') \\
&+ N(r,0;g) + N\left(r, \frac{n+2}{n+1}; g\right) - N(r,1;g) - N(r,0;g') \\
&+ S(r,f) + S(r,g) \\
\le\ & 6N(r,0;f) + N(r,1;f) + 5\overline{N}(r,\infty;f) + N\left(r, \frac{n+2}{n+1}; f\right) \\
&+ 6N(r,0;g) + N(r,1;g) + 5\overline{N}(r,\infty;g) + N\left(r, \frac{n+2}{n+1}; g\right) \\
&+ S(r,f) + S(r,g) \\
\le\ & 8T(r,f) + 8T(r,g) + 5\overline{N}(r,\infty;f) + 5\overline{N}(r,\infty;g) \\
&+ S(r,f) + S(r,g).
\end{aligned}
$$

So by Lemma 2.8 we obtain

$$
\begin{aligned}
&(n-6)T(r,f) + (n-6)T(r,g) \\
&\qquad \le 5\overline{N}(r,\infty;f) + 5\overline{N}(r,\infty;g) + S(r,f) + S(r,g). \qquad (3.2)
\end{aligned}
$$

Let us choose ε such that

$$0 < \varepsilon < n - 11 + \min\{\Theta(\infty;f), \Theta(\infty;g)\}.$$

Then from (3.2) we get

$$(n-11+\Theta(\infty;f)-\varepsilon)T(r,f)+(n-11+\Theta(\infty;g)-\varepsilon)T(r,g) \le S(r,f)+S(r,g),$$

which is a contradiction.

Therefore inequality (3.1) does not hold. Since $E_{3)}(1;F') = E_{3)}(1;G')$, by Lemmas 2.5, 2.6, 2.7 and 2.10 we get $f \equiv g$. This proves the theorem. ☐

Proof of Theorem 1.1. If (3.1) holds, then from (3.2) we get

$$(n-6)T(r,f) + (n-6)T(r,g) \le S(r,f) + S(r,g),$$

which is a contradiction.

Therefore inequality (3.1) does not hold. Since $E_{3)}(1;F') = E_{3)}(1;G')$, by Lemmas 2.5, 2.6, 2.7 and 2.10 we get $f \equiv g$. This proves the theorem. ☐

References

1. M.-L. Fang and W. Hong, A unicity theorem for entire functions concerning differential polynomials. *Indian J. Pure Appl. Math.* **32**(2001), No. 9, 1343–1348.
2. W. K. Hayman, Meromorphic functions. *Oxford Mathematical Monographs Clarendon Press, Oxford,* 1964.
3. I. Lahiri, Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. *Ann. Polon. Math.* **71**(1999), No. 2, 113–128.
4. I. Lahiri, Value distribution of certain differential polynomials. *Int. J. Math. Math. Sci.* **28**(2001), No. 2, 83–91.

5. I. Lahiri, Linear differential polynomials sharing the same 1-points with weight two. *Ann. Polon. Math.* **79**(2002), No. 2, 157–170.
6. I. Lahiri, On a question of Hong Xun Yi. *Arch. Math. (Brno)* **38**(2002), No. 2, 119–128.
7. W. C. Lin, Uniqueness of differential polynomials and a problem of Lahiri. (Chinese) *Pure Appl. Math. (Xi'an)* **17**(2001), No. 2, 104–110.
8. E. Mues and M. Reinders, Meromorphic functions sharing one value and unique range sets. *Kodai Math. J.* **18**(1995), No. 3, 515–522.
9. C. C. Yang, On deficiencies of differential polynomials. II. *Math. Z.* **125**(1972), 107–112.
10. C. C. Yang and X. H. Hua, Uniqueness and value-sharing of meromorphic functions. *Ann. Acad. Sci. Fenn. Math.* **22**(1997), No. 2, 395–406.
11. H. X. Yi, Uniqueness of meromorphic functions and a question of C. C. Yang. *Complex Variables Theory Appl.* **14**(1990), No. 1-4, 169–176.
12. H. X. Yi, On characteristic function of a meromorphic function and its derivative. *Indian J. Math.* **33**(1991), No. 2, 119–133.
13. H. X. Yi, Some further results on uniqueness of meromorphic functions. *Complex Variables Theory Appl.* **38**(1999), No. 4, 375–385.

(Received 9.03.2003)

Authors' address:

Department of Mathematics
University of Kalyani
West Bengal 741235
India
E-mails: ilahiri@hotmail.com
ilahiri@vsnl.com