Georgian Mathematical Journal
Volume 12 (2005), Number 1, 189–200

WINTNER-TYPE OSCILLATION CRITERIA OF SEMILINEAR ELLIPTIC INEQUALITIES

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Abstract. Wintner-type oscillation criteria of semilinear elliptic inequalities are obtained by using partial Riccati technique. The results presented improve the oscillation criteria due to E. S. Noussair and C.A. Swanson [3].

2000 Mathematics Subject Classification: 35B06, 35J60, 34C10.
Key words and phrases: Oscillation, semilinear elliptic inequalities, second order, partial Riccati technique.

1. Introduction

In this paper, we are concerned with obtaining Wintner-type oscillation criteria for semilinear elliptic inequalities

$$(Ly)(x) + q(x)f(y(x)) \leq 0, \quad x \in \Omega(r_0), \tag{1.1}$$

where $\Omega(r_0) = \{x \in \mathbb{R}^N : |x| \geq r_0\}$ for $r_0 \geq 0$, $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^N$, and $N \geq 2$.

Throughout this paper it is always assumed that the following hypotheses are valid without further mention.

(H_1) L is an elliptic operator of the form

$$L = \sum_{i,j=1}^{N} D_i\big[\, a_{ij}(x) D_j \,\big] + \sum_{i=1}^{N} b_i(x) D_i, \quad x \in \Omega(r_0),$$

where $x = (x_i)$, $D_i = \partial/\partial x_i$, $a_{ij} \in C^{1+\nu}_{loc}(\Omega(r_0), \mathbb{R})$, $b_i \in C^{\nu}_{loc}(\Omega(r_0), \mathbb{R})$ for all i, j, ($\nu \in (0,1)$), and the symmetric matrix $A(x) = (a_{ij})$ is positive definite at each $x \in \overline{\Omega(r_0)}$. Let $\lambda_{\max}(x) \in C(\Omega(r_0), \mathbb{R})$ be the largest eigenvalue of the matrix $A(x)$. We suppose that there exists a function $\lambda \in C(\,[r_0, \infty), \mathbb{R}^+\,)$ such that

$$\lambda(r) \geq \max_{|x|=r} \lambda_{\max}(x), \quad \text{for } r \geq r_0;$$

(H_2) $f \in C(\mathbb{R}, \mathbb{R}) \cup C^1(\mathbb{R} - \{0\}, \mathbb{R})$, $yf(y) > 0$ and $f'(y) \geq k > 0$ for $y \neq 0$;
(H_3) $q \in C^{\nu}_{loc}(\Omega(r_0), \mathbb{R})$, ($\nu \in (0,1)$).

By a solution of (1.1), we mean a function $y \in C^{2+\nu}_{loc}(\Omega(r_0), \mathbb{R})$ ($\nu \in (0,1)$) satisfying (1.1) almost everywhere on $\Omega(r_0)$. For the question of the existence of a solution of (1.1) we refer the reader to the monograph [1]. Our attention is restricted to those solutions which do not vanish identically in any neighborhood of $|x|$. A nontrivial solution $y(x)$ of (1.1) is said to be oscillatory if, for any $R > 0$, $y(x)$ has zero on $\Omega(r_0) \cap \{\, x : |x| > R\}$, otherwise it is said to be nonoscillatory. (1.1) is called oscillatory if all its solutions are oscillatory.

ISSN 1072-947X / \$8.00 /

In the qualitative theory of nonlinear partial differential equations, one of the important problems is to determine whether solutions of the equation under consideration are or are not oscillatory. A number of results on the oscillation of (1.1) are obtained by imposing restrictions on the elliptic operator L. An important special case of (1.1) is the case $b_i(x) \equiv 0$ (for all i), for which (1.1) becomes

$$\sum_{i,\,j=1}^{N} D_i\big[\, a_{ij}(x)\, D_j y\,\big] + q(x)f(y) \le 0. \tag{1.2}$$

Concerning (1.2) there exists a well-elaborated oscillation theory. In 1980, Noussair and Swanson [3] first extended the well-known Wintner theorem [5] to (1.2) based on the partial Riccati transformation. The survey paper by Swanson [4] contains a complete bibliography till 1979. Recently, Xu [6] and Zhang et. al. [8] obtained Kamenev-type oscillation criteria [2]. On the other hand, using the averaging functions from a general class of parameter functions, Xu [7] gave new oscillation criteria for (1.2). It seems, however, that a very few results are established for (1.1) in general form. Motivated by this fact, in this paper we develop the technique exploited by Noussair and Swanson [3] to establish Wintner-type oscillation criteria for (1.1), which extend and improve the results in [3]. Finally, some examples are given to illustrate the advantages of our results. It is to be emphasized that we do not assume any condition on the functions b_i and q except the conditions $b_i,\ q \in C^{\nu}_{loc}(\Omega)$ (for all i). The results obtained here are new even for (1.2).

2. Main Results

Lemma 2.1. *Let $\alpha,\ \beta \in \mathbb{R}^N$, $C > 0$, then*

$$C\alpha\alpha^T + \alpha\beta^T \ge \frac{C}{2}\,\alpha\alpha^T - \frac{1}{2C}\,\beta\beta^T. \tag{2.1}$$

Lemma 2.1 is easy to verify and the proof is omitted.

We now introduce our principal notations. For any given functions $\rho \in C^1(\,[r_0,\infty),\mathbb{R}^+\,)$ and $(\,\lambda\eta\,) \in C^1(\,[\,r_0,\infty),\mathbb{R})$, we define

$$g(r) = \frac{k}{2\,\omega}\frac{r^{1-N}}{\lambda(r)\rho(r)}, \quad p(r) = \frac{\rho'(r)}{\rho(r)} + \frac{k}{\omega}\,\eta(r)r^{1-N},$$

and

$$\begin{aligned}\theta(r) =&\rho(r)\Big\{\int\limits_{S_r}\Big[\,q(x) - \frac{1}{2k}\lambda(x)\big|B^TA^{-1}\big|^2\,\Big]d\sigma + \frac{k}{2\omega}r^{1-N}\lambda(r)\eta^2(r)\\ &-\big[\,\lambda(r)\eta(r)\,\big]'\Big\} + \frac{1}{2}\Big(\frac{p(r)}{g(r)}\Big)' - \frac{p^2(r)}{4g(r)}\,,\end{aligned}$$

where $S_r = \{x \in \mathbb{R}^N : |x| = r\}$, ω and $d\sigma$ denote the surface measure of unit sphere and the spherical integral element in $\mathbb{R}^N$, respectively, and $B^T = (\,b_1(x),\ldots,b_N(x)\,)$.

Theorem 2.1. *Suppose that there exist functions* $\rho \in C^1([r_0,\infty),\mathbb{R}^+)$, $(\lambda\eta) \in C^1([r_0,\infty),\mathbb{R})$, *satisfying*

$$\int_{r_0}^{\infty} g(r)\,dr = \infty \tag{2.2}$$

and

$$\int_{r_0}^{\infty} \theta(r)\,dr = \infty, \tag{2.3}$$

then (1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of (1.1). Without loss of generality we assume that $y(x) > 0$ for $|x| > r_0$. Put

$$W(x) = \frac{1}{f(y(x))}\,(A^{\nabla}y)(x),$$

where ${}^{\nabla}y$ denotes the gradient of y. Differentiation of the i-th component of $W(x)$ with respect to x_i gives

$$D_iW(x)_i = -\frac{f'(y)}{f^2(y)}\,D_iy\Big[\sum_{j=1}^{N} a_{ij}(x)D_jy\Big] + \frac{1}{f(y)}\,D_i\Big[\sum_{j=1}^{N} a_{ij}(x)D_jy\Big]$$

for all i. Summation over i and the use of (1.1) lead to

$$\begin{aligned}
\operatorname{div} W(x) &\le -f'(y)(W^TA^{-1}W)(x) - (B^TA^{-1}W)(x) - q(x)\\
&\le -k\,(W^TA^{-1}W)(x) - (B^TA^{-1}W)(x) - q(x) \quad (\,by\,(H_2)\,)\\
&\le -\frac{k}{\lambda(x)}(W^TW)(x) - (B^TA^{-1}W)(x) - q(x) \quad (\,by\,(H_1)\,)\\
&\le -\frac{k}{2\lambda(x)}(W^TW)(x) + \frac{1}{2k}\lambda(x)|B^TA^{-1}|^2 - q(x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1)}.
\end{aligned} \tag{2.4}$$

Let

$$w(r) = \rho(r)\Big[\int_{S_r} W(x)\cdot\nu(x)\,d\sigma + \lambda(r)\eta(r)\Big] \quad \text{for} \quad r \ge r_0, \tag{2.5}$$

where $\nu(x) = x/|x|$, $(\,|x| \ne 0\,)$, denotes the outward unit normal. Using the divergence theorem in (2.5), we obtain by (2.4)

$$\begin{aligned}
w'(r) &= \frac{\rho'(r)}{\rho(r)}\,w(r) + \rho(r)\Big\{\int_{S_r} \operatorname{div} W(x)\,d\sigma + \big[\,\lambda(r)\eta(r)\,\big]'\Big\}\\
&\le \frac{\rho'(r)}{\rho(r)}\,w(r) - \rho(r)\Big\{\frac{k}{2\lambda(r)}\int_{S_r}(W^TW)(x)\,d\sigma
\end{aligned}$$

$$+\int_{S_r}\Big[q(x)-\frac{1}{2k}\,\lambda(x)\big|B^TA^{-1}\big|^2\Big]d\sigma-\big[\lambda(r)\eta(r)\big]'\Bigg\}.$$

By the Schwartz inequality

$$\int_{S_r}\big|W(x)\big|^2\,d\sigma\ge\frac{r^{1-N}}{\omega}\Big[\int_{S_r}W(x)\cdot\nu(x)\,d\sigma\Big]^2.$$

Thus for $r\ge r_0$

$$\begin{aligned}
w'(r)\le{}&\frac{\rho'(r)}{\rho(r)}\,w(r)-\rho(r)\Bigg\{\frac{k}{2\omega}\,\frac{r^{1-N}}{\lambda(r)}\Big[\int_{S_r}W(x)\cdot\nu(x)\,d\sigma\Big]^2\\
&+\int_{S_r}\Big[q(x)-\frac{1}{2k}\,\lambda(x)\big|B^TA^{-1}\big|^2\Big]d\sigma-\big[\lambda(r)\eta(r)\big]'\Bigg\}\\
={}&\frac{\rho'(r)}{\rho(r)}\,w(r)-\rho(r)\Bigg\{\frac{k}{2\omega}\,\frac{r^{1-N}}{\lambda(r)}\Big[\frac{w(r)}{\rho(r)}-\lambda(r)\eta(r)\Big]^2\\
&+\int_{S_r}\Big[q(x)-\frac{1}{2k}\,\lambda(x)\big|B^TA^{-1}\big|^2\Big]d\sigma-\big[\,\lambda(r)\eta(r)\,\big]'\Bigg\}\\
={}&-g(r)w^2(r)+p(r)w(r)-\rho(r)\Bigg\{\int_{S_r}\Big[q(x)-\frac{1}{2k}\lambda(x)\big|B^TA^{-1}\big|^2\Big]d\sigma\\
&+\frac{k}{2\omega}\,r^{1-N}\lambda(r)\eta^2(r)-\big[\,\lambda(r)\eta(r)\,\big]'\Bigg\},
\end{aligned}$$

that is

$$Z'(r)\le-g(r)Z^2(r)-\theta(r),\tag{2.6}$$

where

$$Z(r)=w(r)-\frac{1}{2}\,\frac{p(r)}{g(r)}.$$

Hence, for all $r\ge r_0$, we have

$$Z(r)\le Z(r_0)-\int_{r_0}^{r}g(s)Z^2(s)\,ds-\int_{r_0}^{r}\theta(s)\,ds.\tag{2.7}$$

By (2.3) and (2.7) we can find a number $a\ge r_0$ such that for all $r\ge a$

$$Z(r)\le-\int_{r_0}^{r}g(s)Z^2(s)\,ds=:-H(r).$$

Thus

$$H'(r)=g(r)Z^2(r)\ge g(r)H^2(r).$$

This yields

$$\int_{r_0}^{r} g(s)\,ds \le \int_{r_0}^{r} \frac{d\,H(r)}{H^2(r)} \le \frac{1}{H(r_0)} \quad \text{for} \quad r \ge r_0,$$

which contradicts (2.2), and this completes the proof. □

Remark 2.1. For (1.2), let $\eta(r) \equiv 0$, then Theorem 2.1 improves Theorem 4 in [3].

The following oscillation criteria (Theorems 2.2–2.4) treat the cases when it is not possible to verify easily conditions (2.2) or (2.3).

Theorem 2.2. *Suppose that there exist functions $\rho \in C^1([r_0,\infty),\mathbb{R}^+)$, $(\lambda\eta) \in C^1([r_0,\infty),\mathbb{R})$, $\varphi \in C(\mathbb{R},[0,\infty))$ with φ nondecreasing on $[0,\infty)$, and $\phi \in C([r_0,\infty),\mathbb{R}^+)$ satisfying*

$$\int_{r_0}^{\infty} \Big[\int_{r_0}^{s} \frac{\phi^2(\tau)}{g(\tau)}\,d\tau \Big]^{-1} \phi(s)\,\varphi\Big(\int_{r_0}^{s} \phi(\tau)\,d\tau \Big) ds = \infty, \tag{2.8}$$

$$\int_{r_0}^{\infty} \frac{\varphi(s)}{s^2}\,ds < \infty. \tag{2.9}$$

If

$$\lim_{r\to\infty} \Big[\int_{r_0}^{r} \phi(s)\,ds \Big]^{-1} \int_{r_0}^{r} \phi(s) \int_{r_0}^{s} \theta(\tau)\,d\tau\,ds = \infty, \tag{2.10}$$

then (1.1) *is oscillatory.*

Proof. By using the same argument as in the proof of Theorem 2.1, we get (2.7) holds for $r \ge r_0$. Multiplying (2.7) by $\phi(s)$ and integrating from r_0 to r, we have

$$\begin{aligned}&\int_{r_0}^{r} \phi(s) Z(s)\,ds + G(r)\\ &\quad\le \Big[\int_{r_0}^{r} \phi(s)\,ds \Big] \Big[Z(r_0) - \Big(\int_{r_0}^{r} \phi(s)\,ds \Big)^{-1} \int_{r_0}^{r} \phi(s) \int_{r_0}^{s} \theta(\tau)\,d\tau\,ds \Big],\end{aligned}$$

where $G(r) = \int_{r_0}^{r} \phi(s) \int_{r_0}^{s} g(\tau) Z^2(\tau)\,d\tau\,ds$. In view of condition (2.10), there exists a constant $a > r_0$ such that

$$Z(r_0) - \Big[\int_{r_0}^{r} \phi(s)\,ds \Big]^{-1} \int_{r_0}^{r} \phi(s) \int_{r_0}^{s} \theta(\tau)\,d\tau ds < 0, \quad \text{for all} \quad r \ge a.$$

So, for every $r \geq a$

$$G(r) \leq -\int_{r_0}^{r} \phi(s)Z(s)\,ds.$$

In virtue of the Schwartz inequality

$$G^2(r) \leq \bigg[\int_{r_0}^{r} \phi(s)Z(s)\,ds\bigg]^2 \leq \bigg[\int_{r_0}^{r} \frac{\phi^2(s)}{g(s)}\,ds\bigg]\bigg[\int_{r_0}^{r} g(s)Z^2(s)\,ds\bigg],$$

that is

$$\bigg[\int_{r_0}^{r} \frac{\phi^2(s)}{g(s)}\,ds\bigg]^{-1}\phi(r) \leq \frac{G'(r)}{G^2(r)}\,. \tag{2.11}$$

But, for all $r \geq a$

$$G(r) \geq \bigg[\int_{a}^{r} \phi(s)\,ds\bigg]\bigg[\int_{r_0}^{a} g(s)Z^2(s)\,ds\bigg] = C\int_{a}^{r} \phi(s)\,ds,$$

where $C = \int_{r_0}^{a} g(s)Z^2(s)ds$. Since $g(s)Z^2(s)$ is continuous and not identically zero on $[r_0, a]$, we have $C > 0$. Thus, for all $r \geq a$,

$$\varphi\bigg(\int_{a}^{r} \phi(s)\,ds\bigg) \leq \varphi(C^{-1}G(r)). \tag{2.12}$$

From (2.11) and (2.12) it follows that

$$\begin{aligned}
&\int_{a}^{r}\bigg[\int_{r_0}^{s} \frac{\phi^2(\tau)}{g(\tau)}\,d\tau\bigg]^{-1}\phi(s)\varphi\bigg(\int_{a}^{s}\phi(\tau)\,d\tau\bigg)ds \\
&\quad\leq \int_{a}^{r} \frac{\varphi(C^{-1}G(s))}{G^2(s)}\,dG(s) = \int_{C^{-1}G(a)}^{C^{-1}G(r)} \frac{\varphi(s)}{s^2}\,ds < \infty,
\end{aligned}$$

which contradicts condition (2.8). □

Lemma 2.2. *Suppose that there exist functions* $\rho \in C^1([r_0, \infty), \mathbb{R})$ *and* $(\lambda\eta) \in C^1([r_0,\, \infty), \mathbb{R})$ *such that* (2.2) *and*

$$\Theta(r) := \int_{r}^{\infty} \theta(r)\,dr < \infty \quad \textit{for} \quad r \geq r_0, \tag{2.13}$$

hold. If (1.1) *is nonoscillatory, then there exist a constant* $a > r_0$ *and a function* $Z \in C^1([a, \infty), \mathbb{R})$ *satisfying*

$$Z(r) \geq \int_{r}^{\infty} \theta(s)\,ds + \int_{r}^{\infty} g(s)Z^2(s)ds \quad \textit{for} \quad r \geq a. \tag{2.14}$$

Proof. Proceeding as in the proof of Theorem 2.1, there exist a number $a > r_0$ and a function $Z \in C^1([a,\infty),\mathbb{R})$ satisfying (2.6). Therefore for $b \ge r \ge a$

$$Z(b) + \int_r^b \theta(\tau)\,d\tau + \int_r^b g(s)Z^2(s)\,ds \le Z(r). \tag{2.15}$$

Now we claim that

$$\int_r^\infty g(s)Z^2(s)\,ds < \infty. \tag{2.16}$$

Otherwise

$$\int_r^\infty g(s)Z^2(s)\,ds = \infty,$$

then there is a number $a_1 \ge a$ such that, taking into account (2.13) and (2.15),

$$Z(b) \le -\int_{a_1}^b g(s)Z^2(s)\,ds \quad \text{for} \quad b \ge a_1.$$

As in the proof of Theorem 2.1, it is easy to show that

$$\int_{a_1}^\infty g(s)\,ds < \infty,$$

which contradicts (2.2). Thus (2.16) holds. Therefore, from (2.15), for $r \ge a$

$$Z(r) \ge \limsup_{b\to\infty} Z(b) + \int_r^\infty \theta(s)\,ds + \int_r^\infty g(s)Z^2(s)\,ds. \tag{2.17}$$

If $\limsup_{r\to\infty} Z(b) < 0$, then there exist two numbers $\delta < 0$ and $a_2 \ge a_1$ such that $Z(b) < \delta$ for $b \ge a_2$. It follows from (2.2) that, for $r \ge b$,

$$\int_r^\infty g(s)Z^2(s)\,ds \ge \delta^2 \int_r^\infty g(s)\,ds = \infty,$$

which contradicts (2.16). Thus $\limsup_{b\to\infty} Z(b) \ge 0$. It follows from (2.17) that (2.14) holds. □

Theorem 2.3. *Suppose that there exist functions $\rho \in C^1([r_0,\infty),\mathbb{R}^+)$ and $(\lambda\eta) \in C^1([r_0,\infty),\mathbb{R})$ such that* (2.2) *and* (2.13) *hold. If*

$$\int_{r_0}^\infty g(s)\Theta_+^2(s)\,\exp\left[2\int_{r_0}^s g(\tau)\Theta(\tau)\,d\tau\right]ds = \infty, \tag{2.18}$$

then equation (1.1) *is oscillatory.*

Proof. Suppose that (1.1) is nonoscillatory. Then it follows from Lemma 2.2 that there exist a constant $a > r_0$ and a function $Z \in C^1([a,\infty),\mathbb{R})$ satisfying (2.14) for $r \geq a$. Define

$$v(r) = \int_r^\infty g(s)Z^2(s)\,ds,$$

then

$$v'(r) = -g(r)Z^2(r). \tag{2.19}$$

Multiplying (2.19) by $\exp[\,2\int_r^s g(\tau)\Theta(\tau)\,d\tau\,]$ and integrating from r to b, we obtain

$$\begin{aligned} v(r) = {}& \exp\left[2\int_r^b g(s)\Theta(s)\,ds\right]v(b) \\ & + \int_r^b g(s)[Z^2(s) - 2\Theta(s)v(s)]\,\exp\left[2\int_r^s g(\tau)\Theta(\tau)d\tau\right]ds. \end{aligned} \tag{2.20}$$

It follows from (2.14) that $Z(r) \geq \Theta(r) + v(r)$ for $r \geq a$, which implies $Z^2(r) - 2\Theta(r)v(r) \geq \Theta_+^2(r) + v^2(r)$ for $r \geq a$. This and (2.20) imply

$$\begin{aligned} v(r) \geq {}& \int_r^\infty g(s)\Theta_+^2(s)\exp\left[2\int_r^s g(\tau)\Theta(\tau)\,d\tau\right]ds \\ & + \int_r^\infty g(s)v^2(s)\exp\left[2\int_r^s g(\tau)\Theta(\tau)\,d\tau\right]ds. \end{aligned} \tag{2.21}$$

This contradicts (2.18). The proof is completed. □

Theorem 2.4. *Suppose that there exist functions $\rho \in C^1([r_0,\infty),\mathbb{R}^+)$ and $(\lambda\eta) \in C^1([r_0,\infty),\mathbb{R})$ such that (2.2), (2.13) and*

$$\Theta_1(r) := \int_r^\infty g(s)\Theta_+^2(s)\,\exp\left[2\int_r^s g(\tau)\Theta(\tau)\,d\tau\right]ds < \infty \quad \textit{for} \quad r \geq r_0$$

hold. If

$$\int_{r_0}^\infty g(s)\Theta_1^2(s)\,\exp\left[\,2\int_{r_0}^s g(\tau)(\,\Theta(\tau) + \Theta_1(\tau)\,)d\tau\right]ds = \infty, \tag{2.22}$$

then (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.3, we can find a constant $a > r_0$ and a function $v \in C^1([a,\infty),\mathbb{R})$ satisfying (2.21) for all $r \geq a$. Define

$$u(r) = \int_r^\infty g(s)v^2(s)\exp\left[2\int_r^s g(\tau)\Theta(\tau)d\tau\right]ds.$$

Then

$$\begin{aligned} u'(r) &= -g(r)\big[\,v^2(r) + 2\Theta(r)u(r)\,\big] \\ &\leq -g(r)\Big\{\,\big[\,\Theta_1^2(r) + u^2(r)\,\big] + 2\big[\,\Theta(r) + \Theta_1(r)\,\big]\,u(r)\Big\}. \end{aligned} \tag{2.23}$$

Using $\exp[\,2\int_r^s g(\tau)[\,\Theta(\tau) + \Theta_1(\tau)\,]\,d\tau$ as an integrating factor, we integrate (2.23) from r to b and obtain

$$\begin{aligned} u(r) &\geq \exp\left[2\int_r^b g(s)\,\big(\,\Theta(s) + \Theta_1(s)\,\big)\,ds\right]u(b) \\ &\quad + \int_r^b g(s)\big[\,\Theta_1^2(s) + u^2(s)\,\big]\exp\left[2\int_r^s g(\tau)\big(\,\Theta(\tau) + \Theta_1(\tau)\,\big)\,d\tau\right]ds \\ &\geq \int_r^b g(s)\Theta_1^2(s)\exp\left[2\int_r^s g(\tau)\big(\,\Theta(\tau) + \Theta_1(\tau)\,\big)\,d\tau\right]ds \\ &\quad + \int_r^b g(s)u^2(s)\exp\left[2\int_r^s g(\tau)\big(\,\Theta(\tau) + \Theta_1(\tau)\,\big)\,d\tau\right]ds. \end{aligned}$$

Letting $b \to \infty$ in the above inequality, we get a contradiction to (2.22). This completes the proof. □

Remark 2.2. If we assume further that $\Theta(r)$, $\Theta_1(r), \ldots$ are integrable, similarly to Theorems 2.3 and 2.4, we can establish a number of oscillation criteria for equation(1.1).

3. Corollaries and Examples

The results in Section 2 are of high degree of generality. With an appropriate choice of the functions ρ, η and φ. Some interesting corollaries can be obtain from Theorems 2.1–2.4, for example.

The following is an oscillation result for the linear equations

$$\Delta y + \sum_{i=1}^N b_i(x)\frac{\partial y}{\partial x_i} + q(x)y = 0, \quad x \in \Omega(r_0), \tag{3.1}$$

where b_i, $q \in C_{loc}^\nu(\Omega)$, $\nu \in C(0,1)$.

Corollary 3.1. *Suppose that the conditions*

$$\lim_{r\to\infty}\int_{r_0}^{r}\Big[s\ln s\,q_M(s)-\frac{1}{2s\ln s}\Big]ds=\infty,\qquad N=2,\tag{3.2}$$

and

$$\lim_{r\to\infty}\int_{r_0}^{r}\Big[s\,q_M(s)-\frac{(2-N)^2}{2s}\Big]ds=\infty,\qquad N\ge 3,\tag{3.3}$$

where

$$q_M(r)=\frac{1}{\omega r^{N-1}}\int_{S_r}\Big[q(x)-\frac{1}{2}\sum_{i=1}^{N}b_i^2(x)\Big]d\sigma,$$

hold. Then (3.1) *is oscillatory.*

Proof. The assertion of Corollary 3.1 follows from that of Theorem 2.1 if we choose $\eta(r)\equiv 0$ and

$$\rho(r)=\begin{cases}\ln r, & \text{for}\quad N=2,\\ r^{2-N}, & \text{for}\quad N\ge 3.\end{cases}$$

□

Remark 3.1. Corollary 3.1 improves some results in [3,4].

Example 3.1. Consider the elliptic equation

$$\Delta y+\frac{\sin|x|}{|x|}\frac{\partial y}{\partial x_1}+\frac{\cos|x|}{|x|}\frac{\partial y}{\partial x_2}+\frac{\mu}{|x|^2}\,y=0,\tag{3.4}$$

for $|x|\ge e$, where μ is a constant with $2\mu>1$, and $N=2$.

A direct calculation gives

$$\int_{e}^{r}\Big[s\ln s q_M(s)-\frac{1}{2s\ln s}\Big]ds$$
$$=\frac{2\mu-1}{2}\ln^2 r-\frac{1}{2}\ln(\ln r)-\frac{2\mu-1}{2}\to\infty\quad\text{as}\quad r\to\infty.$$

Hence (3.4) is oscillatory by Corollary 3.1.

Example 3.2. Consider the elliptic equation

$$\frac{\partial}{\partial x_1}\Big(\frac{1}{|x|^2}\frac{\partial y}{\partial x_1}\Big)+\frac{\partial}{\partial x_2}\Big(\frac{1}{|x|^2}\frac{\partial y}{\partial x_2}\Big)+\frac{2+\cos|x|-2|x|\sin|x|}{4|x|^{\frac{5}{2}}}(4y+\sqrt[3]{y})=0\tag{3.5}$$

for $|x|\ge e$, where $N=2$.

Here, we choose $\rho(r)=r$, $\eta(r)=-2\pi$, $\phi(r)=1/r$, and $\varphi(r)=\sqrt{r}$; then

$$p(r)=0,\quad g(r)=\frac{1}{\pi},\quad \theta(r)=\frac{\pi(2+\cos r-2r\sin r)}{2\sqrt{r}},$$

$$\int_e^r \theta(s)\,ds = \pi\big[\sqrt{r}(2+\cos r) - \sqrt{e}(2+\cos e)\big] \geq \sqrt{r} \quad \text{for} \quad r \geq e,$$

$$\lim_{r\to\infty}\Big[\int_e^r \phi(s)\,ds\Big]^{-1}\int_e^r \phi(s)\int_e^s \theta(\tau)\,d\tau\,ds \geq \lim_{r\to\infty}\frac{1}{\ln r}\int_e^r s^{-\frac{1}{2}}\,ds = \infty,$$

$$\lim_{r\to\infty}\int_e^r\Big[\int_e^s \frac{\phi^2(\tau)}{g(\tau)}\,d\tau\Big]^{-1}\phi(s)\,\varphi\Big(\int_e^s \phi(\tau)\,d\tau\Big)\,ds$$

$$= \lim_{r\to\infty}\frac{e}{\pi}\int_e^r \frac{(\ln s-1)^{1/2}}{s-e}\,ds = \infty.$$

Thus, all conditions of Theorem 2.2 are satisfied and hence (3.5) is oscillatory.

Example 3.3. Consider the elliptic equation

$$\frac{\partial}{\partial x_1}\Big(\frac{1}{|x|}\frac{\partial y}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(\frac{1}{|x|}\frac{\partial y}{\partial x_2}\Big) + \frac{1}{|x|^2}\Big(\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2}\Big) + \frac{\upsilon}{|x|^3}(y+y^3) = 0 \quad (3.6)$$

for $|x| \geq 1$, where υ is a constant with $\upsilon \geq 7/2$ and $N = 2$.

Taking $\rho(r) = 1/r$, $\eta(r) = 2\pi$, we have

$$p(r) = 0, \quad g(r) = \frac{r}{4\pi}, \quad \theta(r) = \frac{\pi(2\upsilon+1)}{r^3},$$

$$\Theta(r) = \int_r^\infty \theta(s)ds = \frac{\pi(2\upsilon+1)}{2}\,\frac{1}{r^2},$$

$$\lim_{r\to\infty}\int_1^r g(s)\Theta_+^2(s)\exp\Big[2\int_1^s g(\tau)\Theta(\tau)\,d\tau\Big]ds$$

$$= \lim_{r\to\infty}\frac{(2\upsilon+1)^2\pi}{16}\int_1^r s^{\frac{2\upsilon-11}{4}}\,ds = \infty.$$

Consequently, by Theorem 2.3, (3.6) is oscillatory.

Remark 3.2. The results obtained in this paper hold true if we replace condition (H_2) by

$$f \in C(\mathbb{R},\mathbb{R}), \quad yf(y) > 0 \quad \text{and} \quad \frac{f(y)}{y} \geq k > 0 \quad \text{for} \quad y \neq 0,$$

but the function $q(x)$ should be nonnegative in this case.

Acknowledgement

The author would like to thank the referee for his valuable suggestions and useful comments.

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(Received 28.12.2003)

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