Georgian Mathematical Journal
Volume 12 (2005), Number 2, 201–206

QUADRIC HYPERSURFACES CONTAINING A PROJECTIVELY NORMAL CURVE

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Abstract. Let $C \subset \mathbf{P}^n$ be a smooth projectively normal curve. Let Z be the scheme-theoretic base locus of $H^0(\mathbf{P}^n, \mathcal{I}_C(2))$ and Z' the connected component of Z containing C. Here we show that $Z' = C$ in certain cases (e.g., non-special line bundles with degree near to $2p_a(C)-2$ or certain special line bundles on general k-gonal curves).

2000 Mathematics Subject Classification: 14H45, 14H50.
Key words and phrases: Quadric hypersurfaces, k-gonal curves, gonality, projective normality, normally generated line bundle.

1. Introduction

Let $C \subset \mathbf{P}^n$ be a smooth projectively normal curve. Let Z denote the scheme-theoretic base locus of $H^0(\mathbf{P}^n, \mathcal{I}_C(2))$. Let Z' denote the connected component of Z containing C. A very natural problem is the description of Z and Z'. Set $d := \deg(C)$ and $g := p_a(C)$. If $g = n - 1$ and $d = 2n$, i.e., if C is canonically embedded, then a classical theorem of Enriques and Petri gives that $C = Z$ if and only if C is neither trigonal nor isomorphic to a smooth plane quintic, while if C is trigonal (resp. isomorphic to a smooth plane quintic), then Z is a degree $n - 1$ rational surface scroll (resp. the Veronese surface of $\mathbf{P}^5$). If $d \geq 2g + 2$, then the homogeneous ideal of C is generated by quadrics ([4]) and hence $Z = C$. If $d = 2g + 1$ F. Serrano described Z, up to possibly finitely many points and one line: either Z is a two-dimensional rational normal scroll or Z is the union of C and at most one line and finitely many points and in the former case C must be hyperelliptic or trigonal ([11]). The quoted result of F. Serrano completes the remark at the end of [2], covering the case " C linearly normal of degree $\delta = 2g + 1$ ". A cursory reading of [2] shows that even a partial description of Z' may have some geometric consequence. Notice that $D \subset Z'$ for any line $D \subset \mathbf{P}^n$ such that $\text{length}(C \cap D) \geq 3$. Let $\text{Sec}(C) \subseteq \mathbf{P}^n$ be the secant variety of C, i.e., the 3-dimensional irreducible variety which is the union of all secant lines and all tangent lines to C, i.e., the closure in $\mathbf{P}^n$ of the set of all secant lines to C ([9]). Hence $\text{Sec}(C)$ is irreducible. Since $n \geq 3$, $\dim(\text{Sec}(C)) = 3$ ([9]).

In Section 2 we will prove the following results.

Theorem 1. *Let X be a smooth projective curve of genus $g \geq 2$ and $L \in \text{Pic}^d(X)$, $2d \geq 3g + 1$, L very ample. Let $\phi_L : X \to \mathbf{P}^n$, $n := h^0(X, L) - 1$,*

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

be the complete embedding associated to L*. Set* $C := \phi_L(X)$*. Let* Z *be the scheme-theoretic base locus of* $H^0(\mathbf{P}^n, \mathcal{I}_C(2))$*.*

(a) *Assume that for every integer* x *such that* $1 \le x \le n-3$ *there is no* $A \in \mathrm{Pic}(X)$ *such that* $\deg(A) \le d-2x-3$ *and* $h^0(X,A) \ge n-x$*. Then* C *is scheme-theoretically a connected component of* Z*.*
(b) *Assume that for every integer* x *with* $1 \le x \le n-3$ *there are at most finitely many* $A \in \mathrm{Pic}(X)$ *such that* $\deg(A) \le d-2x-3$ *and* $h^0(X,A) \ge n-x$*. Then* C *is an irreducible component of* Z_{red} *and for a general* $P \in X$ *the schemes* Z *and* C *coincide in a neighborood of* $\phi_L(P)$*.*
(c) *Assume that there is no* $A \in \mathrm{Pic}(X)$ *such that* $\deg(A) \le d-2x-4$ *and* $h^0(X,A) \ge n-x-1$ *for some integer* x *such that* $1 \le x \le n-4$*. Then* $Z_{red} \cap \mathrm{Sec}(C) = C$*.*

Theorem 2. *Fix integers* g, k, d *such that* $g \ge 2k-1 \ge 7$*, and* $d > (3g-1)/2$*. Let* X *be a general* k*-gonal curve of genus* g*. Fix* $L \in \mathrm{Pic}^d(X)$ *such that* $h^1(X,L) > 0$ *and* L *is very ample. Let* B *be the scheme-theoretic base locus of the line bundle* $K_X \otimes L^*$*. Set* $b := \deg(B)$*,* $n := h^0(X,L)-1$*, and* $t := n+g-d$*. Assume* $b+1 \le ((d-1)-(3g-1)/2)(k-2)/2$*. Let* $R \in \mathrm{Pic}^k(X)$ *be the degree* k *line bundle computing the gonality of* X*. Let* $\phi_L : X \to \mathbf{P}^n$ *denote the linearly normal embedding of* X *associated to* L*. Set* $C := \phi_L(X)$*. Let* $Z \subset \mathbf{P}^n$ *denote the scheme-theoretic base locus of the linear system* $|\mathcal{I}_C(2)|$ *of all quadric hypersurfaces of* $\mathbf{P}^n$ *containing* C*. Then:*

(a) *Fix* $P \in C$*. There is an open neighborhood* U *of* P *in* $\mathbf{P}^n$ *such that* $Z \cap U = C \cap U$ *(scheme-theoretically) if and only if there is no line* $D \subset \mathbf{P}^n$ *such that* $P \in D$ *and the scheme* $C \cap D$ *has length at least* 3*.*
(b) *Assume that there is no degree* 2 *effective divisor* A *of* X *such that* $h^0(X, \mathcal{O}_X(B+A) \otimes R^*) > 0$*; this condition is always satisfied if* $b \le k-4$*. Then* C *is a connected component of* Z*.*
(c) *Fix non-negative integers* β *and* τ *such that* $2\beta+6+\tau k \le g+\tau$*. Let* $E \subset X$ *a general subset of* X *with* $\mathrm{card}(E) = \beta$*. Set* $d' := 2g-2-k\tau-\beta$ *and* $M := K_X \otimes (R^{\otimes\tau})^*(-E) \in \mathrm{Pic}^{d'}(X)$*. Then* $h^0(X, \mathcal{O}_X(E+A) \otimes R^*) = 0$*,* M *is very ample,* E *is the base locus of* $K_X \otimes M^*$ *and* $h^0(X, \mathcal{O}_X(E+A) \otimes R^*) = 0$*. We may apply parts* (a) *and* (b) *taking* $d' := d$*,* $L := M$ *and* $B := E$*.*

We will say that a line D as in part (a) of Theorem 2 is a trisecant line of C, although the set $(C \cap D)_{red}$ may just be one or two points and the scheme $C \cap D$ may have length 4 or more. Notice that $D \subset Z$ for any trisecant line of C.

To prove our results we will use a method due to M. Green and R. Lazarsfeld ([10], Proposition 2.5.2).

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

2. Proofs of Theorems 1, 2 and Related Results

Proof of Theorem 1. By [5], Theorem 0.3, L is normally generated, i.e., C is projectively normal. Fix $P \in X$. Let $Y \subset \mathbf{P}^{n-1}$ be the image of the linear projection of C from the point $\phi_L(P)$. Since L is very ample, $L(-P)$ is spanned and hence Y is just the image of X by the complete linear system associated to $L(-P)$. Since $h^0(X, L(-P-B)) = h^0(X, L) - 3 = h^0(X, L(-P)) - 2$ for every degree two effective divisor B on X, $L(-P)$ is very ample, $Y \cong X$ and $\deg(Y) = d-1$. By [5], Theorem 0.3, Y is projectively normal unless there is an integer x with $1 \le x \le n-3$ and an effective divisor D on X such that $\deg(D) \ge 2x+2$, $h^1(X, L^{\otimes 2}(-2P-D)) = 0$ and $\phi_{L(-P)}(D)$ spans a linear subspace of dimension x of $\mathbf{P}^{n-1}$ in which $\phi_{L(-P)}(D)$ fails to impose independent conditions to quadrics. We do not use the last assertion, but only that $\deg(P+D) \ge 2x+3$, while $\dim(\langle \phi_L(P+D) \rangle) \le x+1$, i.e., $h^0(X, L(-P-D)) \ge n-x$. Hence parts (a) and (b) follow. Now we will check part (c). Fix $Q \in (\mathbf{P}^n \backslash C) \cap Z_{red}$. First assume the existence of a line $T \subset \mathbf{P}^n$ and $P \in C$ such that $\mathrm{card}(C \cap T) \ge 2$ and T intersects quasi-transversally C at P. Hence $T \subset Z$ and we see that this case is impossible if the assumptions of part (a) are satisfied. Now assume that Q is contained in a tangent line of C, say the tangent line to C at $\phi_L(P)$. Apply part (a) to $L(-P)$ and use again the linear projection from $\phi_L(P)$ to see that this case is impossible under the assumptions of part (c), concluding the proof. □

Proof of Theorem 2. By [1], Theorem 1, C is projectively normal. By Riemann–Roch we have $t = h^1(X, L) - 1$.

(i) *Proof of part* (a). Set $P' := \phi_L^{-1}(P)$. It is easy to check that $L(-P')$ is very ample if and only there is no trisecant line to C passing through P. Hence we may assume $L(-P')$ very ample. Fix a hyperplane H of $\mathbf{P}^n$ such that $P \notin H$ and identify the curve $\phi_{L(-P')}(X) \cong X$ with the linear projection, C', of C from P into H. Since the base locus of $K_X \otimes (L(-P'))^*$ is contained in $B + P'$ and $b+1 \le ((d-1)-(3g-1)/2)(k-2)/2$, $L(-P')$ is normally generated ([1], Theorem 1), i.e., the projective space Γ of all quadric cones of $\mathbf{P}^n$ with vertex containg P has dimension $-1 + h^0(H, \mathcal{O}_H(2)) - h^0(C', \mathcal{O}_{C'}(2)) = \binom{n+1}{2} - 2(d-1) + g - 2$; here we use the vanishing of $h^1(X, L^{\otimes 2}(-2P'))$ which is true because $2d > 2g$. Since $h^1(X, L^{\otimes 2}) = 0$ and L is normally generated, the projective space Ψ of all quadric hypersurfaces of $\mathbf{P}^n$ containing C has dimension $-1 + \binom{n+2}{2} - 2d + g - 1$. Hence Γ has codimension $n-1$ in Ψ. Since every singular quadric hypersurface is a cone, this equality is equivalent to the fact that for every tangent vector v to $\mathbf{P}^n$ at P different from the tangent vector to C at P there is $S \in |\mathcal{I}_C(2)|$ smooth at P and such that its Zariski tangent space at P does not contain v. Since C is smooth at P, this is equivalent to the existence of an open neighborhood U of P in $\mathbf{P}^n$ such that $Z \cap U = C \cap U$.

(ii) *Proof of part* (b). By part (a) it is sufficient to show that there is no line $D \subset \mathbf{P}^n$ such that the scheme $D \cap C$ has length at least three. By [3], Proposition 1.1, we have $K_X \otimes L^* \cong R^{\otimes t}(B)$ and in particular $b = 2g - 2 - d - tk$. Fix any degree 3 effective divisor Z of X. By assumption we have $h^0(X, \mathcal{O}_X(B+Z) \otimes R^*) = 0$. Since $2(b+3) + t(k-1) \le g$, we obtain $h^0(X, L(-Z)) = h^0(X, L) - 3$

(use Riemann–Roch and Serre duality). Hence the degree 3 subscheme $\phi_L(Z)$ of C is not contained in a line.

(iii) *Proof of part* (c). By the generality of E no two points of it are contained in the same fiber of the degree k morphism $\phi_R : X \to \mathbf{P}^1$ and no point of E is one of the ramification points of ϕ_R; here we use the uniqueness of R, i.e., a very weak form of the generality of X. Since $k \geq 4$ this implies that for any degree $x \leq 3$ effective divisor F no fiber of ϕ_R is contained in $B + F$. Since $2\beta + 6 \leq g$ we obtain $h^0(X, \mathcal{O}_X(E + F)) = 1$ ([3], Propoposition 1.1). This equality for $x = 2$ gives the very ampleness of M (use Serre duality and Riemann–Roch). The last assertion follows from the case $x = 3$ of the first one. □

In [7] S. Kim and Y. R. Kim studied the very ampleness and the projective normality of the line bundle $K_X \otimes (R^*)^{\otimes r}$ for any k-gonal curve X whose degree k pencil $|T|$ is simple, i.e., it does not factor through a covering of a curve of genus > 0. It is very easy to extend some of their results to our set-up and obtain the following result.

Proposition 1. *Fix integers g, k, r such that $k \geq 4$, $r > 0$ and $rk \leq 2g/(k-1) - 2k - 2$, Let X be a smooth genus g curves equipped with a base point free $R \in \mathrm{Pic}^k(X)$ such that the associated pencil $|R|$ is simple. Set $L := K_X \otimes (R^*)^{\otimes r}$. Then:*

(a) *L is very ample and the associated embedded curve $\phi_L(X)$ has no trisecant line.*
(b) *$L(-P)$ is very ample for every $P \in X$.*

Proof. The ampleness of L was proved in [7], Theorem B, under the weaker assumptions $k \geq 3$ and $rk \leq 2g/(k-1) - 2k - 1$. To check part (b) and the second assertion of part (a) following the proof of [7] or the discussion of quadric cones made in part (i) of the proof of Theorem 2, it is sufficient to show that for all $P_1, P_2 \in X$ we have $h^0(X, R^{\otimes r}(P + P_1 + P_2)) = r + 1$. This is true by [7], Theorem A. □

Remark 1. Let X be a smooth genus g curve, $L \in \mathrm{Pic}^d(X)$ such that $h^1(X, L) = 0$ and $A \subset$X an effective divisor. We have $h^0(X, L) - h^0(X, L(-A)) <$ $\deg(A)$ if and only if there is an effective divisor $B \subset X$ such that $L(-A) \cong K_X(-B)$. Notice that $\deg(B) = 2g - 2 - d + \deg(A)$ and that if $\deg(A) + \deg(B) < g$, then for a general $L \in \mathrm{Pic}^d(X)$ there is no such pair (A, B). Now assume L very ample. Let $\phi_L : X \to \mathbf{P}^{d-g}$ denote the associated embedding. Fix an integer m such that $0 < m < d - g - 1$. There is an m-dimensional linear space $U \subset \mathbf{P}^{d-g}$ such that the scheme-theoretic intersection $\phi_L(X) \cap U$ has length at least $m + 2$ if and only if there are effective divisors A, B on X such that $\deg(A) = m + 2$ and $L(-A) \cong K_X(-B)$. Now take $m = 1$. There is a trisecant line of $\phi_L(X)$ passing through $\phi_L(P)$ if and only if there is an effective $2g + 1 - d$ divisor B and an effective degree two divisor E such that $L(-P - E) \cong K_X(-B)$. We see in this way that for any X and any d such that $2g - 5 \leq d \leq 2g - 2$ some of the non-special very ample and normally generated degree d line bundles considered in [6] and [8] have images with trisecant lines and hence their images are not set-theoretically cut out by quadrics.

Proposition 2. *Fix an integer c such that $0 \le c \le 2$. Let X be a smooth curve of genus $g > [(c+6)(c+5)/2]$, $P \in X$ and $L \in Pic^{2g-2-c}(X)$ such that $h^1(X,L) = 0$ and L is very ample. Let Z be the scheme-theoretic intersection of all quadric hypersurfaces of $\mathbf{P}^{g-2-c}$ containing $\phi_L(X)$. Assume that there is no trisecant line to $\phi_L(X)$ containing $\phi_L(P)$. Then there is an open neighborhood of $\phi_L(P)$ in $\mathbf{P}^{g-2-c}$ such that $Z \cap U = \phi_L(X) \cap U$ except in the cases of pairs (X,L) considered in* [6] *and* [8]*, Theorem* 1.1.

Proof. Since L is very ample, the two conditions on $\phi_L(P)$ are equivalent and equivalent to the very ampleness of $L(-P)$ (Remark 1). Apply the quoted results [6] and [8], Theorem 1.1, to the pair $(X, L(-P))$ and use Remark 1. □

Acknowledgements

The first author was partially supported by MIUR and GNSAGA of INdAM (Italy) and the second author partially supported by KOSEF #2002-000-00051-0.

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(Received 8.07.2004)

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