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GELFAND PAIRS AND GENERALIZED D'ALEMBERT'S AND CAUCHY'S FUNCTIONAL EQUATIONS

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Abstract. We show that Cauchy's, d'Alembert's functional equations and their generalizations are the functional equations for bounded spherical functions associated to some Gel'fand pairs. Our results are very close to the ones obtained by Stetkær in [17].

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1. Introduction

Set-up 1: We adapt here our point of view to the results obtained by Stetkær H. in [17] and [18]. Let G be a locally compact group, and let K be a compact subgroup of G such that (G, K) is a Gel'fand pair ([9] and [11]), the associated K-spherical functions are non-zero continuous functions on G such that $\int_K f(xky)d\omega_K(k) = f(x)f(y)$, $x, y \in G$ ([9] and [11]). Let $K \propto G$ be the semidirect product group, then $(K \propto G, K)$ is a Gel'fand pair (see [5]), and the associated K-spherical functions satisfy the functional equation

$$\int_K f(xk \cdot y)\, d\omega_K(k) = f(x)f(y), \quad x, y \in G, \tag{1.1}$$

where ω_K is the normalized Haar measure of K. If G is an abelian locally compact group and K is a compact subgroup of $\mathrm{Aut}(G)$ consisting of automorphisms and homemorphisms of G, then $(K \propto G, K)$ is a Gel'fand pair and the associated bounded K-spherical functions satisfy the functional equation

$$\int_K f(x + k \cdot y)\, d\omega_K(k) = f(x)f(y), \quad x, y \in G, \tag{1.2}$$

and have the form

$$f(x) = \int_K \chi(k \cdot x)\, d\omega_K(k), \quad x \in G \tag{1.3}$$

for some unitary character χ of G ([3], [8]). For the Gel'fand pair $(\{I\} \propto G, \{I\})$, the $\{I\}$-spherical functions are solutions of Cauchy's functional equation

$$f(x + y) = f(x)f(y), \quad x, y \in G. \tag{1.4}$$

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Another interesting instance is the case of the Gel'fand pair $(\{I,-I\} \propto G, \{I,-I\})$, where the $\{I,-I\}$-spherical functions are solutions of d'Alembert's functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \quad x,y\in G. \tag{1.5}$$

In the next set-up, we generalize the results mentioned above.

Set-up 2: Let G be a locally compact group and let K be a compact subgroup of G such that (G,K) is a Gel'fand pair ([5], [8], [10]). Let Φ be a finite subgroup of $\mathrm{Aut}(G)$ such that $\varphi(K)\subset K, \forall\ \varphi\in\Phi$. The number of the elements of a finite group Φ will be denoted by $|\Phi|$. $C(G)$ (resp. $C_b(G)$) denotes the space of continuous (resp. continuous and bounded) complex-valued functions. By $L_1(G)$ we denote the Banach algebra of all integrable functions on G and by $L_1(G\setminus\setminus K)$ the subalgebra of functions in $L_1(G)$ that are bi-K-invariant (i.e. $f(kx)=f(xk)=f(x), x\in G, k\in K$). One can form a semi-direct product group $G_1=\Phi\propto G$ and a compact subgroup $K_1=\Phi\propto K$ using the group law

$$(\varphi,x)(\psi,y)=(\varphi\psi,x\varphi(y)), \quad \varphi,\psi\in\Phi, \quad x,y\in G.$$

A function $F:\Phi\propto G\to\mathbb{C}$, that is bi-$\Phi\propto K$-invariant, can be regarded as a function f_Φ on G such that f_Φ is both bi-K-invariant and Φ-invariant (i.e. $f\circ\varphi=\varphi, \forall\varphi\in\Phi$), therefore $F(\varphi,x)=f_\Phi(x), \varphi\in\Phi, x\in G$. Accordingly, we obtain the isomorphism

$$L_1(\Phi\propto G\setminus\setminus\Phi\propto K)\longrightarrow L_1(G\setminus\setminus K)\cap L_1^\Phi(G),$$
$$F\mapsto f_\Phi,$$

where $L_1^\Phi(G)=\{f\in L_1(G): f\circ\varphi=f,\ \varphi\in\Phi\}$. Then $(\Phi\propto G,\Phi\propto K)$ is a Gel'fand pair. Let $x_1,\ y_1\in G_1=\Phi\propto G$ and let $k_1\in K_1=\Phi\propto K$, then we have

$$x_1k_1y_1=(\theta,x)(\varphi,k)(\psi,y)=(\theta\varphi\psi,x\theta(k)\,\theta(\varphi(y))).$$

Let $F\in C_b(\Phi\propto G)\setminus\{0\}$, then F is a $\Phi\propto K$-spherical function if and only if f_Φ satisfies the functional equation

$$\Sigma_{\varphi\in\Phi}\int_K f(xk\varphi(y))d\omega_K(k)=|\Phi|f(x)f(y), \quad x,y\in G. \tag{1.6}$$

This equation generalizes the following functional equations:

− the K-spherical functional equation

$$\int_K f(xky)\,d\omega_K(k)=f(x)f(y), \quad x,y\in G; \tag{1.7}$$

− the generalized d'Alembert's functional equation

$$\int_K f(xky)\,d\omega_K(k)+\int_K f(xk\sigma(y))\,d\omega_K(k)=2f(x)f(y) \quad x,y\in G, \tag{1.8}$$

where $\sigma\in\mathrm{Aut}(G)$ is such that $\sigma\circ\sigma=\sigma$ (see [1] and [10]);

– the Cauchy's functional equation (1.4) and the d'Alembert's functional equations (1.5) and

$$f(x+y)+f(x+\sigma(y))=2f(x)f(y)\quad x,y\in G, \tag{1.9}$$

where $\sigma\in \mathrm{Aut}(G)$ is such that $\sigma\circ\sigma=\sigma$ (see [19]).

In the previous paper [7], we have dealt with the stability of the functional equation (1.6). Here we study the properties of solutions of the functional equation (1.6). The results obtained generalize the ones obtained in [10], [17], [18] and [19].

In the first section, we give general properties in Proposition 2.3, we prove that if a measurable and essentially bounded function f on G (i.e., $f\in L^\infty(G)$) satisfies the functional equation (1.6), then $f\in C_b(G)$, which explains why we restrict our-selves to solutions $f\in C_b(G)$. In the second section, we give a description of the bounded solutions of (1.6). These solutions are expressed by the formula

$$f(x)=\frac{1}{|\Phi|}\,\Sigma_{\varphi\in\Phi}\, l(\varphi(x)),\quad x\in G, \tag{1.10}$$

where l is a bounded K-spherical function associated to the Gel'fand pair (G,K).

In Section 3, we consider a Riemannian symmetric pair (G,K) ([12]) and characterize solutions of the functional equation (1.6) as eigenfunctions for some invariant operators on G.

In the last section we give some applications of our study.

2. General Properties

In what follows, we study general properties of solutions of (1.6). Let G be a locally compact group, K be a compact subgroup of G, and let Φ be a finite group of K-invariant automorphisms of G.

Proposition 2.1 (7)**.** *For an arbitrary fixed $\tau\in\Phi$, and $f\in C(G)$, we have*
i) *the mapping $\varphi\longrightarrow\varphi\circ\tau$ is an automorphism of Φ;*
ii) $\Sigma_{\varphi\in\Phi}\int_K f(xk\varphi(\tau(y)))d\omega_K(k)=\Sigma_{\varphi\in\Phi}\int_K f(xk\varphi(y))d\omega_K(k)$; $x,y\in G$.
iii) *If f is bi-K-invariant, then for all $z,y,x\in G$, we have*

$$\int_K f(ykx)\,d\omega_K(k)=\int_K f(xky)\,d\omega_K(k),$$

and

$$\int_K\int_K f(zhykx)\,d\omega_K(h)\,d\omega_K(k)=\int_K\int_K f(zhxky)\,d\omega_K(h)\,d\omega_K(k).$$

Proposition 2.2. *Let $f\in C(G)\setminus\{0\}$ be a solution of* (1.6)*, then we have $f(e)=1$, $f\circ\tau=f$ for all $\tau\in\Phi$ and f is bi-K-invariant.*

Proof. By easy computation. □

Proposition 2.3. *Let $f\in L_1^\infty(G)$ be a measurable and essentially bounded function on G satisfying the functional equation* (1.6)*, then $f\in C_b(G)$.*

Proof. For all $h \in L_1(G \setminus \setminus K)$ and for almost all $y \in G$, we have

$$\begin{aligned}
|\Phi|\langle \check{h}, f\rangle f(y) &= |\Phi| \int_G f(y)\, \check{h}(x) f(x)\, dx \\
&= \Sigma_{\varphi\in\Phi} \int_K \int_G f(xk\varphi(y))\, h(x^{-1})\, d\omega_K(k)\, dx \\
&= \Sigma_{\varphi\in\Phi} \int_K \int_G f(x^{-1}k\varphi(y))\, h(x)\, d\omega_K(k)\, dx \\
&= \Sigma_{\varphi\in\Phi} \int_K (h * f)(k\varphi(y))\, d\omega_K(k) = \Sigma_{\varphi\in\Phi}(h * f)(\varphi(y))
\end{aligned}$$

which finishes the proof of our proposition. □

3. Solutions of Equation (1.6)

In the following theorem, we determine the bounded solutions of the functional equation (1.6).

Theorem 3.1. *Let G be a locally compact group and let K be a compact subgroup such that (G, K) is a Gel'fand pair. Let Φ be a finite subgroup of the group* Aut(G) *such that K is Φ-invariant. Let $f \in C_b(G) \setminus \{0\}$. Then f is a solution of equation* (1.6) *if and only if there exists a K-spherical function l such that*

$$f(x) = \frac{1}{|\Phi|} \Sigma_{\varphi\in\Phi}\, l(\varphi(x)), \quad x \in G.$$

Proof. By using some ideas of [7] and [2], we define for each fixed $y \in G$, the function χ_y on Σ (the Gel'fand spectrum of the commutative Banach algebra $L_1(G \setminus \setminus K)$ (see [11]) to be

$$\chi_y(l) = \frac{1}{|\Phi|} \Sigma_{\varphi\in\Phi}\, l(\varphi(y)), \quad l \in \Sigma.$$

Let $F(L_1(G \setminus \setminus K))$ be the set of Fourier transforms of functions in $L_1(G \setminus \setminus K)$. For all $g \in F(L_1(G \setminus \setminus K))$, we have $\chi_y g \in F(L_1(G \setminus \setminus K))$. Indeed, if $g = Fh$, $h \in L_1(G \setminus \setminus K)$, then $\chi_y g = Fz$, where $z \in L_1(G \setminus \setminus K)$ has the form

$$z(x) = \frac{1}{|\Phi|} \Sigma_{\varphi\in\Phi} \int_K h(xk\varphi(y^{-1}))\, d\omega_K(k).$$

Now let Ff (the Fourier transform of f) be regarded as a pseudomeasure on $L_1(G \setminus \setminus K)$ (see [5]) defined by

$$\langle Ff, g\rangle = \int_G f(x)h(x)\, dx,$$

where $Fh = g$. We show that for each $y \in G$ and $g \in F(L_1(G \setminus \setminus K))$,

$$[(\chi_y - f(y))g]\, Ff = 0.$$

Indeed, for each $g \in F(L_1(G \setminus \setminus K))$, we have

$$\langle [(\chi_y - f(y))g]Ff, A\rangle = \int\limits_G f(x)(z * q)(x)\,dx - f(y)\int\limits_G f(x)(h*q)(x)\,dx,$$

where $Fh = g$, $Fq = A$ and $Fz = \chi_y(g)$. Then we obtain

$$\begin{aligned}\langle [(\chi_y - f(y))g]Ff, A\rangle &= \frac{1}{|\Phi|}\int\limits_G\int\limits_K f(x)\,\Sigma_{\varphi\in\Phi}(h*q)(x\varphi(y^{-1})k)\,d\omega_K(k)\,dx\\ &\quad - f(y)\int\limits_G f(x)(h*q)(x)\,dx\\ &= \frac{1}{|\Phi|}\int\limits_G\int\limits_K f(x)\,\Sigma_{\varphi\in\Phi}(h*q)(xk\varphi(y^{-1}))\,d\omega_K(k)\,dx\\ &\quad - f(y)\int\limits_G f(x)(h*q)(x)\,dx\\ &= \frac{1}{|\Phi|}\int\limits_G\int\limits_K f(x)\Sigma_{\varphi\in\Phi}(h*q)(xk\varphi(y))d\omega_K(k)\,dx\\ &\quad - f(y)\int\limits_G f(x)(h*q)(x)\,dx\\ &= f(y)\int\limits_G f(x)(h*q)(x)\,dx - f(y)\int\limits_G f(x)(h*q)(x)\,dx = 0.\end{aligned}$$

Since f is non-zero, the support of the pseudomeasure Ff is nonempty. Let l be in the support of Ff. Then we obtain that, for all $y \in G$ and for all $g \in F((L_1(G \setminus \setminus K))$, the function $(\chi_y - f(y))g$ vanishes at l (see [5] Theorem 1.3.1). Consequently, we have $\chi_y(l) = f(y)$, which ends the proof. □

4. On Riemannian Symmetric Pairs

In this section, we suppose that G is a connected compact Lie group and K is a compact subgroup of G such that (G, K) is a Riemannian symmetric pair, then (G, K) is a Gel'fand pair ([13]). Let Φ be a finite subgroup of $\mathrm{Aut}(G)$ such that $\varphi(K) \subset K$ for all $\varphi \in \Phi$. We shall characterize solutions of (1.6) in terms of eigenfunctions of some left invariant differential operators.

For each fixed $a \in G$, we define the left (resp. the right) translation operators as $(L_a f)(x) = f(a^{-1}x)$ (resp. $(R_a f)(x) = f(xa)$) and say that an operator T is left (resp. right) invariant if $(L_a T)f = T(L_a f)$ (resp. $(R_a T)f = T(R_a f)$). Let $\mathbb{D}(G)$ denote the algebra of left invariant differential operators on G. For all $f \in C(G)$, one puts $f^\natural(x) = \int_K \int_K f(kxh)d\omega_K(k)d\omega_K(h)$, for all $x \in G$.

Proposition 4.1. *For any differential operator D, the operator D_K defined by*

$$(D_K f)(x) = \frac{1}{|\Phi|}\Sigma_{\varphi\in\Phi} D\{(L_{x^{-1}}f)^{\natural}\circ\varphi\}(e),$$

for all $f \in C^{\infty}(G)$ and $x \in G$, has the following properties:

i) *D_K is left invariant;*

ii) *D_K is right invariant under K. Furthermore one has $R_h D_K = D_K R_h = D_K$;*

iii) *$(D_K f)(e) = \frac{1}{|\Phi|}\Sigma_{\varphi\in\Phi} D\{f^{\natural}\circ\varphi\}(e)$. In particular if f is bi-K-invariant and $f\circ\tau = f$ for all $\tau\in\Phi$ then $(D_K f)(e) = (Df)(e)$;*

vi) *If f is a solution of (1.6), then $D_K f = (Df)(e)f$.*

Proof. By easy computations, we have iii) and vi).

For i) let $f \in C^{\infty}(G)$ and let $a \in G$, for all $x \in G$, we have

$$\begin{aligned} L_a(D_K f)(x) =&(D_K f)(a^{-1}x) = \frac{1}{|\Phi|}\Sigma_{\varphi\in\Phi} D\{(L_{(x^{-1}a)}f)^{\natural}\circ\varphi\}(e)\\ =&\frac{1}{|\Phi|}\Sigma_{\varphi\in\Phi} D\{(L_{x^{-1}}(L_a f))^{\natural}\circ\varphi\}(e) = D_K(L_a f)(x).\end{aligned}$$

For ii) let $f \in C^{\infty}(G)$ and let $h \in K$, for all $x \in G$, we have

$$\begin{aligned} R_h(D_K f)(x) =&(D_K f)(xh) = \frac{1}{|\Phi|}\Sigma_{\varphi\in\Phi} D\{(L_{(xh)^{-1}}f)^{\natural}\circ\varphi\}(e)\\ =&\frac{1}{|\Phi|}\Sigma_{\varphi\in\Phi} D\{(L_{x^{-1}}(R_h f))^{\natural}\circ\varphi\}(e) = D_K(R_h f)(x).\end{aligned}$$

For vi)

$$\begin{aligned} \Sigma_{\varphi\in\Phi}(L_{x^{-1}}f)^{\natural}(\varphi(y)) =&\Sigma_{\varphi\in\Phi}\int_K (L_{x^{-1}}f)(k\varphi(y))\,d\omega_K(k)\\ =&\Sigma_{\varphi\in\Phi}\int_K f(xk\varphi(y))\,d\omega_K(k) = |\Phi| f(x)f(y).\end{aligned}$$

For $y = e$ we get

$$(D_K f)(x) = \frac{1}{|\Phi|}\Sigma_{\varphi\in\Phi} D\{(L_{x^{-1}}f)^{\natural}\circ\varphi\}(e) = (Df)(e)f(x). \qquad \square$$

Lemma 4.1. *If f is a solution of equation (1.6), then f is analytic.*

Proof. Since F is a $\Phi\propto K$-spherical function and the space $\Phi\propto G\setminus\Phi\propto G$ is connected, using the proof in [13] we derive the rest. $\square$

Theorem 4.1. *Let G be a connected Lie group and let K be a compact subgroup of G such that (G, K) is a Rimannian symmetric pair. Let Φ be a finite subgroup of $\mathrm{Aut}(G)$ such that K is Φ-invariant. Let $f \in C(G)$. Then the following statements are equivalent:*

(1) *f is a solution of (1.6).*

(2) i) *$f(e) = 1$, $f\circ\tau = f$ for all $\tau\in\Phi$ and f is bi-K-invariant;*

ii) *$f \in C^\infty(G)$;*
iii) *f is analytic;*
vi) *f is a joint eigenfunction of the operators D_K for all $D \in \mathbb{D}(G)$.*

Proof. (1) $\Longrightarrow$ (2) follows directly from Propositions 4.1 and 2.2 and Lemma 4.1. Conversely, suppose that $D_K f = (Df)(e)f$ for all $D \in \mathbb{D}(G)$. For a fixed $x \in G$ we define the function

$$F(y) = \frac{1}{|\Phi|} \Sigma_{\varphi\in\Phi} \int_K f(xk\varphi(y))\, d\omega_K(k), \quad y \in G.$$

Since $f \circ \tau = f$, for all $\tau \in \Phi$ we get

$$F(y) = \frac{1}{|\Phi|} \Sigma_{\varphi\in\Phi} \int_K (L_{(\varphi(x)k)^{-1}} f)(y)\, d\omega_K(k).$$

Consequently, for all $D \in \mathbb{D}(G)$ we have

$$(D_K F)(y) = \frac{1}{|\Phi|} \Sigma_{\varphi\in\Phi} \int_K D_K(L_{(\varphi(x)k)^{-1}} f)(y)\, d\omega_K.$$

Since D_K is left invariant, we obtain

$$(D_K F)(y) = Df(e)F(y).$$

In particular for $y = e$ we have

$$(D_K F)(e) = Df(e)F(e).$$

Hence, by Proposition 4.1 iii), it follows that

$$(DF)(e) = D(f)(e)F(e),$$

i.e.,

$$D(F - F(e)f)(e) = 0$$

for all $D \in \mathbb{D}(G)$.

Since $F - F(e)f$ is an analytic function on the connected Lie group G, by [13] we obtain

$$F - F(e)f \equiv 0$$

on G. We conclude that

$$\frac{1}{|\Phi|} \Sigma_{\varphi\in\Phi} \int_K f(xk\varphi(y))\, d\omega_K(k) = f(x)f(y), \quad x, y \in G. \qquad \square$$

5. Applications

5.1. Let G be an abelian, locally compact group and let Φ be a finite subgroup of $\mathrm{Aut}(G)$, then $(\Phi \propto G, \Phi)$ is a Gel'fand pair and the associated bounded Φ-spherical functions are solutions of the functional equation

$$\Sigma_{\varphi\in\Phi} f(x + \varphi(y)) = |\Phi| f(x)f(y), \quad x, y \in G, \tag{1.10}$$

and have the form

$$f(x) = \frac{1}{|\Phi|} \Sigma_{\varphi \in \Phi} \chi(\varphi(x)), \quad x \in G, \tag{1.11}$$

for some unitary character χ of G.

If $\sigma : G \longrightarrow G$ is a continuous automorphism for G such that $\sigma \circ \sigma = I$, then the bounded $\{I, \sigma\}$-spherical functions associated to the Gel'fand pair $(\{I, \sigma\} \propto G, \{I, \sigma\})$ satisfy d'Alembert's functional equation

$$\frac{f(x+y) + f(x + \sigma(y))}{2} = f(x)f(y), \quad x, y \in G, \tag{1.12}$$

and have the form

$$f = \frac{\chi + \chi \circ \sigma}{2}$$

for some character χ of G (see [4], [8], [14], [15] and [18]).

5.2. Let (G, K) be a compact Gel'fand pair and let Φ be a finite subgroup of $\mathrm{Aut}(G)$ such that $\varphi(K) \subset K$ for all $\varphi \in \Phi$. Then the $\Phi \propto K$-spherical functions associated to the compact Gel'fand pair $(\Phi \propto G, \Phi \propto K)$ are solutions of the functional equation (1.6) and each bounded one has the form

$$f(x) = \frac{1}{|\Phi|} \Sigma_{\varphi \in \Phi} \langle \pi(x)\xi, \xi \rangle, \quad x \in G, \tag{1.13}$$

where (π, H_π) is an irreducible continuous and unitary representation on a Hilbert space H_π, $\langle\ ,\ \rangle$ is the inner product of the space H_π and ξ is a unit vector satisfying $\pi(k)\xi = \xi$ for all $k \in K$ (see [1]).

5.3. Let G be a noncompact connected semi-simple Lie group with finite center and K be a maximal compact subgroup of G. Then (G, K) is a Gel'fand pair [9]. By Iwasawa decomposition, we may write $G = KS$, where S is a closed solvable subgroup. Let Φ be a finite subgroup of $\mathrm{Aut}(G)$. Then the bounded $\Phi \propto K$-spherical functions associated to the Gel'fand pair $(\Phi \propto G, \Phi \propto K)$ are solutions of the functional equation (1.6) and have the form

$$f(x) = \frac{1}{|\Phi|} \Sigma_{\varphi \in \Phi} \int_K \chi(k\varphi(x))\, d\omega_K(k), \quad x \in G, \tag{1.14}$$

for some unitary character χ of S.

5.4. Let G be a unimodular group containing a normal commutative group A with no elements of order 2 and let K be a nonnormal compact subgroup of G. Then (G, K) is a Gel'fand pair [9]. Let Φ be a finite subgroup of $\mathrm{Aut}(G)$. Then the bounded $\Phi \propto K$-spherical functions associated to the Gel'fand pair $(\Phi \propto G, \Phi \propto K)$ are solutions of the functional equation (1.6) and have the form

$$f(x) = \frac{1}{|\Phi|} \Sigma_{\varphi \in \Phi} \int_K \chi(k\varphi(s)k^{-1})\, d\omega_K(k), \quad x \in G, \tag{1.15}$$

where $x = ts$, $t \in K$, $s \in A$ and χ is a unitary character of A.

5.5. Let G be a locally compact group group and let K be a compact subgroup of G such that (G, K) is a Gel'fand pair. Let $\sigma : G \longrightarrow G$ be a continuous automorphism of order 2 such that $\sigma(K) \subset K$. Then $(\{I, \sigma\} \propto G, \{I, \sigma\} \propto K)$ is a Gel'fand pair and the associated bounded $\{I, \sigma\} \propto K$-spherical functions are solutions of the functional equation

$$\int_K f(xky)\, d\omega_K(k) + \int_K f(xk\sigma(y))\, d\omega_K(k) = 2f(x)f(y), \quad x, y \in G, \tag{1.16}$$

and have the form

$$f = \frac{\omega + \omega \circ \sigma}{2} \tag{1.17}$$

for some bounded K-spherical function ω on G (see [1] and [10]).

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