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SEMI-SLANT SUBMANIFOLDS OF A LOCALLY PRODUCT MANIFOLD

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Abstract. In the present paper, we define and study the slant, bi-slant and semi-slant submanifolds of a locally product manifold. We give some characterization theorems for slant submanifolds and semi-slant submanifolds. Moreover, we obtain integrability conditions for the distributions which are involved in the definition of semi-slant submanifolds. We also get some results about mixed totally geodesic submanifolds.

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1. Introduction

Let $(\widetilde{M}, g, F)$ be a C^∞ differentiable almost product Riemannian manifold, where g is a Riemannian metric and F is a non-trivial tensor field of type (1,1). Moreover, g and F satisfy the following conditions:

$$F^2 = I, (F \neq \pm I), \quad g(FX, FY) = g(X, Y), \quad X, Y \in T\widetilde{M}, \tag{1.1}$$

where I is the identity map and $T\widetilde{M}$ is the Lie algebra of vector fields on $\widetilde{M}$.

We denote by $\widetilde{\nabla}$ the Levi–Civita connection on $\widetilde{M}$ with respect to g and furthermore we assume that $\widetilde{M}$ is a locally product manifold, that is

$$\widetilde{\nabla} F = 0, \quad X \in T\widetilde{M}. \tag{1.2}$$

Locally product manifolds are a class of important manifolds introduced by S. Tachibana [1] in the early 60s. After that, many authors discussed this class of manifolds. Adati [2] defined and studied invariant, anti-invariant and non-invariant submanifolds of locally product manifolds, while Bejancu [3] studied semi-invariant submanifolds which correspond to CR-submanifolds of a Kaehlerian manifold [4]. Recently, Liu and Shao [5] have defined and studied skew semi-invariant submanifolds and many related interesting results have been obtained.

Since B. Y. Chen introduced the theory of slant immersions in complex geometry (see [6]), the differential geometry of slant submanifolds has shown an increasing development. Recently, N. Papaghiuc has introduced in [7] a class of submanifolds in an almost Hermitian manifold, called semi-slant submanifolds.

The purpose of this paper is to define and study three new classes of submanifolds of a locally product manifold, i.e., slant submanifolds, bi-slant submanifolds and semi-slant submanifolds. We will focus our attention mainly on

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semi-slant submanifolds which contain semi-invariant submanifolds as a special case.

In Section 2, we review the basic formulas for locally product manifolds and submanifolds in locally product manifolds. Slant immersions are introduced in Section 3. In Section 4, we define slant distributions and introduce a more general class of submanifolds, that is bi-slant submanifolds. We also give a sufficient and necessary condition for a distribution to be slant. In Section 5, we define semi-slant immersions and obtain a useful characterization of semi-slant submanifolds in locally product manifolds.

2. Preliminaries

Let M be a Riemannian manifold isometrically immersed in $\widetilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. Denote by ∇ the Levi–Civita connection of M. Then the Gauss–Weigarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.2}$$

for any X, $Y \in TM$ and any $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle, σ is the second fundamental form of M, and A_V is the Weingarten endomorphism associated with V. The second fundamental form σ and the shape operator A are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V). \tag{2.3}$$

For any $X \in TM$, we write

$$FX = TX + CX, \tag{2.4}$$

where TX is the tangential component of FX and CX is the normal component of FX.

Similarly, for any $V \in T^{\perp}M$, we have

$$FV = tV + nV, \tag{2.5}$$

where tV (resp. nV) is the tangential component (resp. normal component) of FV. From $F(\widetilde{\nabla}_X Y)=\widetilde{\nabla}_X FY$ and (2.1), (2.2), (2.5) we have

$$\begin{aligned} &T\nabla_X Y + C\nabla_X Y + t\sigma(X,Y) + n\sigma(X,Y) \\ &= \nabla_X TY + \sigma(X,TY) - A_{CY}X + \nabla^{\perp}_X CY, \end{aligned} \tag{2.6}$$

for $X, Y \in TM$. Comparing the tangential and normal components in (2.6) we obtain

$$\begin{aligned} T\nabla_X Y &= \nabla_X TY - t\sigma(X,Y) - A_{CY}X, \\ C\nabla_X Y &= \nabla^{\perp}_X CY - n\sigma(X,Y) + \sigma(X,TY), \end{aligned} \tag{2.7}$$

for $X, Y \in TM$.

We define the covariant derivatives of T and C as follows

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.8}$$

$$(\nabla_X C)Y = \nabla^{\perp}_X CY - C\nabla_X Y, \tag{2.9}$$

for all $X, Y \in TM$.

Using (2.7) we have

$$(\nabla_X T)Y = t\sigma(X, Y) + A_{CY}X, \tag{2.10}$$

$$(\nabla_X C)Y = n\sigma(X, Y) - \sigma(X, TY). \tag{2.11}$$

Let D_1 and D_2 be two distributions defined on a manifold M. We say that D_1 is parallel to D_2, if for all $X \in D_2$ and $Y \in D_1$ we have $\nabla_X Y \in D_1$. D_1 is called parallel if for $X \in TM$ and $Y \in D_1$, we have $\nabla_X Y \in D_1$. It is easy to verify that D_1 is parallel if and only if the orthogonal complementary distribution of D_1 is also parallel.

Let M be a submanifold of $\widetilde{M}$. A distribution D on M is said to be totally geodesic if for all $X, Y \in D$ we have $\sigma(X, Y) = 0$. In this case we also say that M is D-totally geodesic. For two distributions D_1 and D_2 defined on M, we say that M is $D_1 - D_2$ mixed totally geodesic if for all $X \in D_1$ and $Y \in D_2$ we have $\sigma(X, Y) = 0$.

The submanifold M is said to be invariant if C is identically zero, that is, $FX \in TM$ for any $X \in TM$. On the other hand, M is said to be an anti-invariant submanifold if T is identically zero, that is, $FX \in T^{\perp}M$ for any $X \in TM$. M is called a semi-invariant submanifold if there exists two orthogonal distributions D_1 and D_2 on M, such that:

(a) $TM = D_1 \oplus D_2$,

(b) the distribution D_1 is invariant, i.e., $FD_1 = D_1$,

(c) the distribution D_2 is anti-invariant, i.e., $FD_2 = T^{\perp}M$.

3. Slant Immersions

Let M be a Riemannian manifold, isometrically immersed in a locally product manifold $(\widetilde{M}, g, F)$. For each nonzero vector X tangent to M at x, we denote by $\theta(X)$ the angle between FX and T_xM.

Definition 3.1. M is said to be slant if the angle $\theta(X)$ is constant, which is independent of the choice of $x \in M$ and $X \in TM$. The angle θ of a slant immersion is called the slant angle of the immersion.

Invariant and anti-invariant immersions are slant immersions with slant angles $\theta = 0$ and $\theta = \pi/2$, respectively.

The following theorems are useful characterization of slant submanifolds in a locally product manifold.

Theorem 3.1. *Let M be a submanifold of a locally product manifold $\widetilde{M}$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that $T^2 = \lambda I$. Furthermore, in this case, if θ is the slant angle of M, it satisfies $\lambda = \cos^2\theta$.*

Proof. Suppose that M is a slant submanifold. Then for any $X \in TM$ we have

$$g(T^2X, X) = g(TX, TX) = \cos^2\theta\ g(FX, FX) = g(\cos^2\theta X, X),$$

by using $\frac{|TX|}{|FX|} = \cos\theta$, where θ is the slant angle. Furthermore, for any $Y \in TM$, we have,

$$\begin{aligned} g(\cos^2\theta(X+Y), X+Y) &= g(T^2(X+Y), X+Y) \\ &= g(T^2X, X) + g(T^2Y, Y) + 2g(T^2X, Y). \end{aligned} \tag{3.1}$$

On the other hand,

$$g(\cos^2\theta(X+Y), X+Y) = g(\cos^2\theta X, X) + g(\cos^2\theta Y, Y) + 2g(\cos^2\theta X, Y), \tag{3.2}$$

By comparing (3.1) and (3.2) we have

$$g(T^2X, Y) = g(\cos^2\theta X, Y)$$

for any $X, Y \in TM$. Let $\lambda = \cos^2\theta$, then $\lambda \in [0, 1]$ and $T^2 = \lambda I$.

Conversely, suppose that there exists a constant $\lambda \in [0, 1]$ such that $T^2 = \lambda I$. Then for any $X \in TM$, we have

$$\cos\theta(X) = \frac{g(FX, TX)}{|FX||TX|} = \frac{g(X, T^2X)}{|FX||TX|} = \lambda\frac{g(X, F^2X)}{|FX||TX|}. \tag{3.3}$$

On the other hand, since $\cos\theta(X) = \frac{|TX|}{|FX|}$ and by using (3.3), we obtain $\cos^2\theta(X) = \lambda$. Hence $\theta(X)$ is a constant and so M is slant. □

Corollary 3.1. *Let M be a slant submanifold of a locally product manifold $\widetilde{M}$ with slant angle θ. Then for any $X, Y \in TM$ we have*

$$g(TX, TY) = \cos^2\theta\ g(X, Y), \qquad g(CX, CY) = \sin^2\theta\ g(X, Y). \tag{3.4}$$

Since for any submanifold of a locally product manifold we have $tC + T^2 = I$, (i.e., $CT + nC = 0$), by using (1.1), (2.4), (2.5), we can obtain the following result.

Corollary 3.2. *Let M be a submanifold of a locally product manifold $\widetilde{M}$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that $tC = \lambda I$. Furthermore, if θ is the slant angle of M, it satisfies $\lambda = \sin^2\theta$.*

4. Slant Distributions and Bi-Slant Submanifolds

From now on, let M be a Riemannian manifold, isometrically immersed in locally product manifold $(\widetilde{M}, g, F)$.

Definition 4.1. We call a differentiable distribution ν on M a slant distribution if for each $x \in M$ and each nonzero vector $X \in \nu_x$, the angle θ_ν between FX and ν_x is a constant which is independent of the choice of $x \in M$ and $X \in \nu_x$. In this case, the constant angle θ_ν is called the slant angle of the distribution ν.

Next, we will give a sufficient and necessary condition for a distribution to be slant.

Theorem 4.1. *Let ν be a distribution on M. Then ν is slant if and only if there exists a constant $\lambda \in [0,1]$ such that $(P_1T)^2X = \lambda X$ for any $X \in \nu$, where P_1 denotes the orthogonal projection on ν. Furthermore in this case, $\lambda = \cos^2\theta_\nu$.*

Proof. Suppose that there exists a constant $\lambda \in [0,1]$ such that $(P_1T)^2X = \lambda X$ for any $X \in \nu$, and P_2 is the orthogonal projection on $\nu^\perp$. For any $X \in \nu$, we have

$$\begin{aligned} FX &= TX + CX = P_1TX + P_2TX + CX, \\ \cos\theta_\nu &= \frac{g(FX, P_1TX)}{|FX||P_1TX|} = \frac{g(X, FP_1TX)}{|FX||P_1TX|} = \frac{g(X,(P_1T)^2X)}{|FX||P_1TX|} \\ &= \lambda\frac{g(X, F^2X)}{|FX||P_1TX|} = \lambda\frac{|FX|}{|P_1TX|}. \end{aligned}$$

On the other hand, $\cos\theta_\nu = \frac{|P_1TX|}{|FX|}$; then $\cos^2\theta_\nu = \lambda$, θ_ν is a constant, i.e., ν is slant. Conversely, since $|P_1TX| = \cos\theta_\nu|FX|$, we have

$$g(X,(P_1T)^2X) = \cos\theta_\nu|FX||P_1TX| = \cos^2\theta_\nu|FX|^2 = g(X, \cos^2\theta_\nu X),$$

which means $(P_1T)^2X = \cos^2\theta_\nu X$. □

Definition 4.2. We say M is a bi-slant submanifold of $\widetilde{M}$ if there exist two orthogonal distributions D_1 and D_2 on M such that:
(a) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2$.
(b) For any $i = 1, 2$, D_i is a slant distribution with slant angle θ_i.

Given a bi-slant submanifold M, we can write, for any $X \in TM$,

$$X = P_1X + P_2X, \tag{4.1}$$

where P_i denotes the component of X in D_i for any $i = 1, 2$. In particular, if $X \in D_i$, then we obtain $X_i = P_iX$. If we define $T_i = P_i \circ T$, then we have

$$FX = T_1X + T_2X + CX \tag{4.2}$$

for any $X \in TM$.

Proposition 4.1. *Let M be a bi-slant submanifold with angles $\theta_1 = \theta_2 = \theta$. If $g(FX, Y) = 0$, for any $X \in D_1$, $Y \in D_2$, then M is slant with angle θ.*

Proof. For all $X \in D_1, Y \in D_2$, since $g(FX, Y) = 0$, we have

$$g(TX, Y) = g(FX, Y) = 0;$$

then $TX \in D_1$. Similarly, we can obtain $TY \in D_2$. For any $X \in TM = D_1 \oplus D_2$, there must be $X_1 \in D_1$, $X_2 \in D_2$ such that $X = X_1 + X_2$, and $\cos^2\theta_1 = \frac{|TX_1|^2}{|FX_1|^2}$, $\cos^2\theta_2 = \frac{|TX_2|^2}{|FX_2|^2}$. Since $\theta_1 = \theta_2 = \theta$, we have

$$\frac{g(TX, TX)}{g(FX, FX)} = \frac{g(TX_1, TX_1) + g(TX_2, TX_2)}{g(FX_1, FX_1) + g(FX_2, FX_2)}.$$

Hence M is slant with angle θ. □

Lemma 4.1. *Suppose that there exist two orthogonal distributions D_1 and D_2 on M, such that $TM = D_1 \oplus D_2$. Then D_1 is invariant if and only if it is slant with angle $\theta_1 = 0$. Moreover, in this case, $TX = T_2X$ for any $X \in D_2$.*

Proof. It is clear that if D_1 is invariant, then it is slant with zero angle. The converse is easy to prove. Since $\cos\theta = 1 = \frac{|T_1X|}{|FX|}$, we have

$$|T_1X| = |FX| = \sqrt{|T_1X|^2 + |T_2X|^2 + |CX|^2},$$

thus $|T_2X| = |CX| = 0$ for any $X \in TM$. Consequently, $FX = T_1X \in D_1$ and we know that D_1 is invariant. On the other hand, if D_1 is invariant, then we have

$$g(TX, Y) = g(X, FY) = 0$$

for any $X \in D_2$ and $Y \in D_1$. Thus $T_1X = 0$ and the result holds. $\square$

5. Semi-Slant Submanifolds

Definition 5.1. M is called a semi-slant submanifold of $\widetilde{M}$ if there exist two orthogonal distributions D_1 and D_2 on M such that:

(a) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2$.
(b) The distribution D_1 is invariant distribution, i.e., $F(D_1) = D_1$.
(c) The distribution D_2 is slant with angle $\theta \neq 0$.

In this case, we call θ the slant angle of submanifold M. By virtue of Lemma 4.1, we can see that the invariant distribution of a semi-slant submanifold is slant with zero angle. Thus it is obvious that semi-slant submanifolds are particular cases of bi-slant submanifolds. Furthermore, it is clear that if $\theta = \pi/2$, then a semi-slant submanifold is a semi-invariant submanifold. On the other hand, if we denote the dimension of D_i by d_i, for $i = 1, 2$, then we have the following cases:

(a) If $d_2 = 0$, then M is an invariant submanifold.
(b) If $d_1 = 0$ and $\theta = \pi/2$, then M is an anti-invariant submanifold.
(c) If $d_1 = 0$ and $\theta \neq \pi/2$, then M is a proper slant submanifold with slant angle θ.
(d) If $d_1d_2 \neq 0$ and $\theta \neq \pi/2$, then M is a proper semi-slant submanifold.

Given a semi-slant submanifold M, we denote by P_i the projection on the distribution D_i for $i = 1, 2$. We also put $T_i = P_iT$. Hence we obtain

$$FX = FP_1X + TP_2X + CP_2X \tag{5.1}$$

for any $X \in TM$. By a direct calculation, we can prove that for any $X \in TM$,

$$FP_1X = TP_1X, \quad CP_1X = 0, \tag{5.2}$$

$$TP_2X \in D_2. \tag{5.3}$$

In particular, (5.1) and (5.2) imply for any $X \in TM$,

$$TX = FP_1X + TP_2X = TP_1X + TP_2X.$$

Then from (5.2) and (5.3) we obtain

$$g(TX, TP_2Y) = \cos^2\theta g(X, P_2Y), \quad g(CX, CP_2Y) = \sin^2\theta g(X, P_2Y)$$

for any $X, Y \in TM$.

We are going to characterize semi-slant submanifolds by the following theorem.

Theorem 5.1. *Let M be a submanifold of a locally product manifold $\widetilde{M}$. Then M is semi-slant if and only if there exists a constant $\lambda \in [0,1)$ such that $\mathcal{D}=\{X \in TX | T^2X = \lambda X\}$ is a distribution. Furthermore, in this case, $\lambda = \cos^2\theta$, where θ denotes the slant angle of M.*

Proof. Let M be semi-slant and $TM = D_1 \oplus D_2$, where D_1 is invariant and D_2 is slant. We put $\lambda = \cos^2\theta$, where θ denotes the slant angle of M. For any $X \in \mathcal{D}$, if $X \in D_1$, then $X = F^2X = T^2X = \lambda X$, which means that $\lambda = 1$, but this contradicts that $\lambda \in [0,1)$. So $X \notin D_1$ and $\mathcal{D} \subseteq D_2$. On the other hand, since D_2 is a slant distribution, it follows from Theorem 4.1 and Lemma 4.1 that $T^2X = (P_2T)^2X = \lambda X$, which means that $D_2 \subseteq \mathcal{D}$. Thus $\mathcal{D}= D_2$ is a distribution.

Conversely, we can consider the orthogonal direct decomposition $TM = \mathcal{D} \oplus \mathcal{D}^\perp$. It is obvious that $T\mathcal{D} \subseteq \mathcal{D}$, from which we have $g(FX,Y) = g(X,FY) = g(X,TY) = 0$ for any $X \in \mathcal{D}^\perp$ and $Y \in \mathcal{D}$. Hence $\mathcal{D}^\perp$ is an invariant distribution. Finally, Lemma 4.1 and Theorem 4.1 imply that $\mathcal{D}$ is a slant distribution, with slant angle θ satisfying $\lambda = \cos^2\theta$. □

Remark. The result above is also valid for almost product Riemannian manifolds, since they do not deal with the Levi–Civita connection and from now on, we are going to deal with semi-slant submanifolds of a locally product manifold. Our goal is to study the integrability.

At first, we will prove the following lemma.

Lemma 5.1. *Let M be a semi-slant submanifold of a locally product manifold $\widetilde{M}$. Then for any $X, Y \in TM$, we have:*

$$P_1(\nabla_X FP_1Y) + P_1(\nabla_X TP_2Y) = FP_1(\nabla_X Y) + P_1A_{CP_2Y}X, \tag{5.4}$$

$$P_2(\nabla_X FP_1Y) + P_2(\nabla_X TP_2Y) = FP_2(\nabla_X Y) + P_2A_{CP_2Y}X + t\sigma(X,Y), \tag{5.5}$$

$$\sigma(FP_1Y, X) + \sigma(TP_2Y, X) + \nabla^\perp_X CP_2Y = CP_2\nabla_X Y + n\sigma(X,Y). \tag{5.6}$$

Proof. Since $\overline{\nabla}_X FY = F\overline{\nabla}_X Y$ for any $X, Y \in TM$, by using Gauss–Weigarten formulas we obtain:

$$\begin{aligned}
&\nabla_X FP_1Y + \sigma(FP_1Y, X) + \nabla_X TP_2Y + \sigma(TP_2Y, X) - A_{CP_2Y}X + \nabla^\perp_X CP_2Y \\
&\quad = P_1(\nabla_X FP_1Y) + P_2(\nabla_X FP_1Y) + P_1(\nabla_X TP_2Y) + P_2(\nabla_X TP_2Y) - P_1A_{CP_2Y}X \\
&\qquad - P_2A_{CP_2Y}X + \sigma(FP_1Y, X) + \sigma(TP_2Y, X) + \nabla^\perp_X CP_2Y \\
&\quad = FP_1\nabla_X Y + TP_2\nabla_X Y + CP_2\nabla_X Y + t\sigma(X,Y) + n\sigma(X,Y).
\end{aligned}$$

By comparing the components of D_1, D_2 and $T^\perp M$, respectively, we can obtain the above results. □

Proposition 5.1. *Let M be a semi-slant submanifold of a locally product manifold $\widetilde{M}$. Then we have:*

(a) *The distribution* D_1 *is integrable if and only if*

$$\sigma(X, FY) = \sigma(FX, Y) \tag{5.7}$$

for any $X, Y \in D_1$.

(b) *The distribution* D_2 *is integrable if and only if*

$$P_1(\nabla_X TY - \nabla_Y TX) = P_1(A_{CY}X - A_{CX}Y) \tag{5.8}$$

for any $X, Y \in D_2$.

Proof. Let D_1 be integrable; by using (5.6) we see that

$$\sigma(X, FY) - \sigma(FX, Y) = CP_2[X, Y] \tag{5.9}$$

for any $X, Y \in D_1$. Hence if D_1 is integrable, then (5.7) holds directly from (5.9). Conversely, let $X, Y \in D_1$, by using (5.7) and (5.9) it follows that $CP_2[X, Y] = 0$. So we can easily deduce that $P_2[X, Y]$ must vanish, since D_2 is a slant distribution with nonzero slant angle. Therefore, $[X, Y] \in D_1$ and statement (a) holds. As to statement (b), we first compute

$$FP_1[X, Y] = P_1(\nabla_X TY - \nabla_Y TX) - P_1(A_{CY}X - A_{CX}Y)$$

for any $X, Y \in D_2$, by virtue of (5.4). Hence (5.8) holds if and only if $FP_1[X, Y] = 0$, i.e., $P_1[X, Y] = 0$, i.e., D_2 is integrable. □

We can also obtain from the above that when M is a semi-invariant submanifold, (b) becomes $P_1(A_{FY}X\text{-}A_{FX}Y) = 0$ for any $X, Y \in D_2$, which is consistent to the results in [5].

Lemma 5.2. *A semi-slant submanifold* M *of a locally product manifold* $\widetilde{M}$ *is* $D_i - D_j$, $i \neq j$, *mixed totally geodesic if and only if* $A_N X \in D_i$ *for any* $X \in D_i$, $N \in T^{\perp}M$, $i = 1, 2$.

Proof. If M is $D_1 - D_2$ mixed totally geodesic, then for any $X \in D_1$, $Y \in D_2$, $N \in T^{\perp}M$,

$$g(A_N X, Y) = g(\sigma(X, Y), N) = 0,$$

which implies that $A_N X \in D_1$. Conversely, suppose $A_N X \in D_1$ for any $X \in D_1$, $N \in T^{\perp}M$ and let $\{N_1, \cdots, N_{m-n}\}$ be a local orthogonal basis of $T^{\perp}M$, where $n = \dim M$, $m = \dim \widetilde{M}$; we have

$$0 = g(A_{N_\alpha} X, Y) = g(\sigma(X, Y), N_\alpha),$$

$\alpha = 1, \ldots, m - n$, for any $X \in D_1$, $Y \in D_2$. So $\sigma(X, Y) = 0$ and M is $D_1 - D_2$ mixed totally geodesic.

In the same way we can also prove that M is $D_2 - D_1$ mixed totally geodesic if and only if $A_N X \in D_2$ for any $X \in D_2$, $N \in T^{\perp}M$. □

Proposition 5.2. *Any invariant submanifold* M *of a locally product manifold* $\widetilde{M}$ *is a locally product manifold.*

Proof. Suppose M is an invariant submanifold of a locally product manifold $\widetilde{M}$; then $C \equiv 0$ and we can obtain $(\nabla_X F)Y = t\sigma(X,Y)$, by using (2.7), for any $X, Y \in TM$. On the other hand,

$$g(Z, t\sigma(X,Y)) = g(Z, F\sigma(X,Y)) = g(FZ, \sigma(X,Y)) = 0.$$

So we have $t\sigma(X,Y)$=0, which implies $(\nabla_X F)Y$=0 for any $X, Y \in TM$. This is equivalent to saying that M is a locally product manifold. □

Proposition 5.3. *Let M be a semi-slant submanifold of a locally product manifold $\widetilde{M}$. If $\nabla C \equiv 0$, then M is $D_1 - D_2$ mixed totally geodesic. Furthermore, if $X, Y \in D_2$, then either $\sigma(X,Y) = 0$ or $\sigma(X,Y)$ is an eigenvector of n^2 with eigenvalue $\cos^2\theta$. If $X, Y \in D_1$, then either $\sigma(X,Y) = 0$ or $\sigma(X,Y)$ is an eigenvector of n^2 with eigenvalue 1.*

Proof. If $(\nabla_X C)Y \equiv 0$ for any $X, Y \in TM$, then from (2.11) we have $f\sigma(X,Y) = \sigma(X,TY)$. In particular, if $Y \in D_2$, then by Lemma 4.1

$$\begin{aligned} f^2\sigma(X,Y) &= f\sigma(X,TY) = f\sigma(X,T_2Y) = \sigma(X,TT_2Y) \\ &= \sigma(X,T_2^2Y) = \cos^2\theta\sigma(X,Y), \end{aligned}$$

where θ is the slant angle of D_2. Furthermore, if $X \in D_1$, then by Lemma 4.1

$$\begin{aligned} f^2\sigma(X,Y) &= f^2\sigma(Y,X) = f\sigma(Y,TX) = \sigma(Y,T^2X) = \sigma(Y,F^2X) \\ &= \sigma(Y,X) = \sigma(X,Y). \end{aligned}$$

Since $\theta \neq 0$, we have $\sigma(X,Y)$=0 by virtue of (5.13). Hence M is $D_1 - D_2$ mixed totally geodesic. If $X, Y \in D_i$, $i = 1, 2$, then from (5.13) we can obtain the result easily. □

Proposition 5.4. *Let M be a mixed totally geodesic semi-slant submanifold of a locally product manifold $\widetilde{M}$. If the distribution D_1 is integrable, then FA_NX=A_NFX for any $X \in D_1$, $N \in T^\perp M$.*

Proof. From (2.3) and (5.7) we have

$$g(FA_NX,Y) = g(A_NX,FY) = g(\sigma(X,FY),N) = g(\sigma(FX,Y),N)$$

for any $X, Y \in D_1$ and $N \in T^\perp M$, thus we get FA_NX=A_NFX by virtue of Lemma 5.2. □

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