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BOUNDARY REGULARITY FOR CAPILLARY SURFACES

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Abstract. For solutions of capillarity problems with the boundary contact angle being bounded away from 0 and π and the mean curvature being bounded from above and below, we show the Lipschitz continuity of a solution up to the boundary locally in any neighborhood in which the solution is bounded and $\partial\Omega$ is C^2; the Lipschitz norm is determined completely by the upper bound of $|\cos\theta|$, together with the lower and upper bounds of H, the upper bound of the absolute value of the principal curvatures of $\partial\Omega$ and the dimension n.

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0. Introduction

Given a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, let $H(x, u(x))$ be a given Lipschitz continuous function in $\Omega \times \mathbb{R}$. A solution of the capillarity problem can be regarded at as a solution of the equation of surfaces of the prescribed mean curvature

$$\operatorname{div} Tu = H(x, u) \quad \text{in} \quad \Omega, \tag{0.1}$$

subject to the "contact angle" boundary condition

$$Tu \cdot \nu = \cos\theta, \tag{0.2}$$

where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}}\ , \tag{0.3}$$

$Du = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_n)$ and ν is the outward pointing unit normal to $\partial\Omega$. Thus, geometrically, we are looking for a function u over $\overline{\Omega}$ whose graph has the prescribed mean curvature H and which meets the boundary cylinder in the prescribed angle θ. $H = H(x, t)$ is assumed to be a given locally Lipschitz function on $\overline{\Omega} \times \mathbb{R}$ satisfying the structural condition

$$\frac{\partial H}{\partial t}(x, t) \geq 0, \quad \text{for} \ \ x \in \Omega, \quad t \in \mathbb{R}. \tag{0.4}$$

As (0.1) is the Euler equation of the functional

$$I(v) = \int\limits_{\Omega} \sqrt{1 + |Dv|^2}\, dx + \int\limits_{\Omega} \int\limits_{0}^{v} H(x, t)\ dt dx,$$

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it corresponds to the following variational problem for the capillarity problem:

$$I(v) + \int_{\partial\Omega} \cos\theta v \, d\mathcal{H}_{n-1} \to \min, \quad \text{for all} \quad v \in BV(\Omega), \tag{0.5}$$

with $\mathcal{H}_{n-1}$ being the $(n-1)$-dimensional Hausdorff measure.

We are interested in regularity near the boundary $\partial\Omega$ for solutions $u \in C^2(\Omega)$. In this work our main interest is in *the case where* $|\cos\theta|$ *is bounded away from* 0 *and* 1 and *the mean curvature* H *is bounded from above and below.* We shall show that in such a case the solution is Lipschitz continuous up to the boundary locally in any neighborhood in which u is bounded and $\partial\Omega$ is C^2; the Lipschitz norm is determined completely by the upper and lower bounds of $|\cos\theta|$, together with the lower and upper bounds of H, the upper bound of the absolute value of the principal curvatures of $\partial\Omega$ and n.

Spruck and Simon treat in [12] the case where Ω is C^4, θ in (0.2) is $C^{1,\alpha}$ on $\partial\Omega$ for some $0 < \alpha < 1$, and $H(x,t)$ is strictly monotone in t:

$$\inf_{x\in\bar{\Omega};t\in\mathbb{R}} \frac{\partial H}{\partial t}(x,t) > 0. \tag{0.6}$$

In case $0 < \theta < \pi$, the existence of a $C^2(\bar{\Omega})$ solution of (0.1) and (0.2) is established in [12]. In case θ is allowed to take 0 and/or π, setting

$$\begin{aligned} S_1^+ =& \{x : x \in \partial\Omega, \theta \equiv 0 \ \text{ in some neighborhood of } x\} \\ S_1^- =& \{x : x \in \partial\Omega, \theta \equiv \pi \ \text{ in some neighborhood of } x\} \\ S_2 =& \{x : x \in \partial\Omega, \ 0 < \theta < \pi\} \end{aligned}$$

a function $u \in C^2(\Omega \cup S_2)$ is shown to exist in [12], which satisfies (0.1) in Ω and satisfies (0.2) on S_2; furthermore, u is Hölder continuous at each point of $S_1^+ \cup S_1^-$, has a restriction to $\partial\Omega$ which is Lipschitz continuous at each point of $S_1^+ \cup S_1^-$ in the sense that

$$\lim_{\varepsilon^+\to 0} \int_{U\cap\Omega_\varepsilon} |Tu \cdot \nu \pm 1| \, dx = 0 \ \text{ for each } \ U \subset \Omega \ \text{ with } \ \overline{U} \cap \partial\Omega \subset S_1^{\mp},$$

assuming that Tu is extended to some boundary strip Ω_ε with width ε so that it is constant along the normals to $\partial\Omega$. This result is obtained first by establishing estimates of tangential derivatives under the condition that $|\cos\theta| \le \gamma < 1$ for some positive constant γ; in case θ is constant in a neighborhood of the point under consideration, the estimates of tangential derivatives are independent of γ. This proves the Lipschitz continuity of the trace of u on $\partial\Omega$, which and the result in [10] yield the Hölder continuity of u. Estimates for the tangential derivatives are obtained by performing the transformation of the coordinates near the boundary analogously to that in [11], with a subsequent differentiation of (0.1), (0.2) and substituting (0.6) into the resultant identities. The disadvantage of their proofs is that H has to be assumed to satisfy the strict inequality (0.6) rather than the less restrictive condition (0.4).

In contrast, the following estimates for the boundary oscillation of u are established in [8, Main Theorem III].

Theorem 1. *Let $u \in C(\overline{\Omega})$ be a bounded solution of (0.1) and (0.2) in Ω in the sense that*

$$\int\limits_{\Omega} \frac{Du}{\sqrt{1+|Du|^2}} \cdot D\eta \, dx + \int\limits_{\Omega} H \cdot \eta \, dx \\ - \int\limits_{\partial\Omega} \beta \cdot \eta \, d\mathcal{H}_{n-1} = 0 \quad \text{for all} \quad \eta \in H^{1,1}(\Omega), \tag{0.7}$$

and with $\beta = \cos\theta$. Suppose that for two positive constant $\tilde{\tilde{\beta}}$, $\tilde{\beta} \le 1$ and a ball $B_R(x_0)$ intersecting the interior of Ω, the function $\cos\theta$ is continuous on $\partial\Omega \cap B_R(x_0)$ and there holds

$$0 < \tilde{\tilde{\beta}} \le |\cos\theta| \le \tilde{\beta} < 1, \tag{0.8}$$

for all $x \in \partial\Omega \cap B_R(x_0)$, and such that

$$\hat{H}^{\pm}(x) = H(x, \pm \inf_{\partial\Omega} u) \in L^p(\Omega), \qquad \hat{\hat{H}}^{\pm}(x) = H(x, \pm \sup_{\partial\Omega} u) \in L^p(\Omega) \tag{0.9}$$

and

$$H(x,0) \in L^1(\Omega). \tag{0.10}$$

Suppose $\partial\Omega$ is piecewise Lipschitz continuous with possible outward and/or inward cusps. Then the trace of u on $\partial\Omega$ is Lipschitz continuous locally in $\partial\Omega \cap B_R(x_0)$ if $\partial\Omega \cap B_R(x_0)$ is either C^2 or is the graph of a Lipschitz continuous function with Lipschitz constant L such that $\tilde{\beta}\sqrt{1+L^2} < 1$. The Lipschitz constant $L_0(\tilde{\beta}, \tilde{\tilde{\beta}})$ of the trace of u on $\partial\Omega \cap B_R(x_0)$ depends only on H, n, together with the constants $\tilde{\beta}$, $\tilde{\tilde{\beta}}$ and $\tilde{\mathcal{K}}_{\partial\Omega \cap B_R(x_0)}$, where for a set A, we set $\tilde{\mathcal{K}}_{\partial\Omega\cap A} = \mathcal{K}_{\partial\Omega\cap A}$ in case $\partial\Omega \cap A$ is C^2 and $\tilde{\mathcal{K}}_{\partial\Omega\cap A} = \sqrt{1+L^2}$ in case $\partial\Omega \cap A$ is Lipschitz continuous with Lipschitz constant L; here $\mathcal{K}_{\partial\Omega\cap A}$ is an upper bound for the absolute value of the principal curvatures of $\partial\Omega \cap A$ in case $\partial\Omega \cap A$ is C^2.

We notice that (0.9) and (0.10) hold in particular if $|H(x,t)|$ is bounded in $\bar{\Omega} \times \mathbb{R}$.

The following global estimates for u are also established in [8, Main Theorem IV].

Theorem 2. *Suppose that $\partial\Omega$ is Lipschitz continuous without outward cusps. Suppose that (0.10) holds and*

$$H_{t_0} \in L^p(\Omega), \quad \text{for some} \;\; p > n \;\; \text{and} \;\; t_0 \in \mathbb{R}.$$

If $u \in C(\overline{\Omega})$ is a solution to (0.1) and (0.2) in Ω such that (0.2) is fulfilled in the sense of (0.7) and if $\cos\theta(x) = \beta(x)$ satisfies the condition (0.8) for all $x \in \partial\Omega$

and is piecewise continuous on $\partial\Omega$, then

$$\sup_\Omega u - \inf_\Omega u$$

can be estimated in terms of t_0, n, $\|H_{t_0}\|_{L^p(\Omega)}$, $\int_\Omega H(x,0)dx$, the constant $\tilde{\beta}$ and the constant $\tilde{\mathcal{K}_{\partial\Omega}}$ which depends only on the geometry of Ω; here $\tilde{\mathcal{K}_{\partial\Omega}} = \max_{i\in\mathbb{I}} \tilde{\mathcal{K}_{\partial\Omega\cap A_i}}$, in which $\{\Omega\cap A_i\}_{i\in\mathbb{I}}$ is a covering of $\partial\Omega$ such that $\partial\Omega\cap A_i$, $i\in\mathbb{I}$, is either C^2 or is a Lipschitz function with Lipschitz constant L, $\tilde{\beta}\cdot\sqrt{1+L^2}<1$.

The results in the previous two theorems are established by modifying the approach taken in [2], [3] and [4], which is based on the minimizing property (0.5) u satisfies and the iteration technique used in [13].

Below we give the result, in which we let the set $A_r(\hat{x})$, for some small positive number r, be chosen as follows. Namely, setting

$$\partial^*\Omega_t = \{x : x\in\Omega,\ \mathrm{dist}(x,\partial\Omega)=t\} \quad \text{for} \quad t>0,$$

we let the boundary $\partial(\Omega\cap A_r(\hat{x}))$ be made up of three parts, namely

$$\partial(\Omega\cap A_r(\hat{x})) = \big(\partial\Omega\cap\overline{A_r(\hat{x})}\big)\cup\big(\partial^*A_r(\hat{x})\big)\cup\big(\partial^{**}A_r(\hat{x})\big),$$

such that

$$\partial^*A_r(\hat{x}) = \partial A_r(\hat{x})\cap\partial^*\Omega_r,$$
$$\partial^{**}A_r(\hat{x}) = (\partial A_r(\hat{x})\cap\Omega)\setminus\partial^*\Omega_r,$$

and

$$Dd\cdot\nu_{\Omega\cap A_r(\hat{x})}\Big|_{\partial^{**}A_r(\hat{x})} = 0,$$

where we let $\nu_{\Omega\cap A_r(\hat{x})}$ be the unit outward normal to $\partial(\Omega\cap A_r(\hat{x}))$; furthermore,

$$\mathrm{diam}(\partial\Omega\cap\overline{A_r(\hat{x})})\le r \quad \text{and} \quad \mathrm{diam}(\partial^*A_r(\hat{x}))\le r,$$
$$|\partial\Omega\cap\overline{A_r(\hat{x})}|\ge\left(\frac{r}{2}\right)^{n-1} \quad \text{and} \quad |\partial^*A_r(\hat{x})|\ge\left(\frac{r}{2}\right)^{n-1}.$$

Proposition 1. *Let $u\in C(\overline{\Omega})$ be a solution to* (0.1) *and* (0.2) *in Ω such that* (0.2) *be fulfilled in the sense of* (0.7). *Suppose that for a constant H_*,*

$$|H(x,t)|\le H_* \quad \text{for} \quad (x,t)\in\overline{\Omega}\times\mathbb{R}. \tag{0.11}$$

Suppose that for a positive constant $\tilde{\beta}<1$ and a point $x_0\in\partial\Omega$, the function $\cos\theta$ is continuous in $\partial\Omega\cap A_{\delta_0}(x_0)$ and there holds

$$0<\tilde{\tilde{\beta}}<\cos\theta\le\tilde{\beta} \quad \text{or} \quad 0>-\tilde{\tilde{\beta}}>\cos\theta>-\tilde{\beta} \quad \text{for} \quad x\in\partial\Omega\cap\overline{A_{\delta_0}(x_0)}. \tag{0.12}$$

Suppose that $\partial\Omega\cap\overline{A_{\delta_0}(x_0)}$ is of the class C^2 and

$$\lim_{x_k\to x_0}|Du|\ge\frac{2}{\tilde{\beta}}$$

for each *subsequence of points* x_k *approaching* x_0. *Then, setting* $R = (\delta_0/4)^{1+\varepsilon_0}$, ε_0 *being a positive number which can be arbitrarily small, there hold, for* δ_0 *sufficiently small, respectively,*

$$u(x) - \inf_{\Omega\cap A_R(x_0)} u \le \mathfrak{C}\cdot\hat{\mathfrak{C}}\cdot(\delta_0)^{1-n\varepsilon_0} + 2(\delta_0)^{1+\varepsilon_0} + \mathfrak{C}\cdot\hat{\mathfrak{C}}\cdot(\delta_0)^{1-n\varepsilon_0}\cdot\big(\sup_{\Omega\cap A_{\delta_0}(x_0)} u - \inf_{\Omega\cap A_{\delta_0}(x_0)} u\big) \tag{0.13}$$

if the first case in (0.12) *holds, and*

$$\sup_{\Omega\cap A_R(x_0)} u - u(x) \le \mathfrak{C}\cdot\hat{\mathfrak{C}}\cdot(\delta_0)^{1-n\varepsilon_0} + 2(\delta_0)^{1+\varepsilon_0} + \mathfrak{C}\cdot\hat{\mathfrak{C}}\cdot(\delta_0)^{1-n\varepsilon_0}\cdot\big(\sup_{\Omega\cap A_{\delta_0}(x_0)} u - \inf_{\Omega\cap A_{\delta_0}(x_0)} u\big), \tag{0.14}$$

$$\mathfrak{C} = 2^{n+7}\cdot(n+1)\cdot k_{(n+1)}\cdot[1+(3/\hat{\beta})], \tag{0.15}$$

if the second case in (0.12) *holds, where* $k_{(n+1)}$ *is the isoperimetric constant in* $\mathbb{R}^{n+1}$ *and* $\hat{\mathfrak{C}}$ *is a constant determined by* $\tilde{\beta}$, $\tilde{\tilde{\beta}}$, H_*, $\mathcal{K}_{\partial\Omega\cap\overline{A_{\delta_0}(x_0)}}$ *and* n.

From the interior regularity of u and (0.12), we obtain in Appendix 3

$$\tilde{\tilde{\beta}}/2 < \cos\theta \le (1+\tilde{\beta})/2 \quad \text{or} \quad -\tilde{\tilde{\beta}}/2 > \cos\theta > -(1+\tilde{\beta})/2 \quad \text{for } x \in \partial^* A_{\delta_0}(x_0). \tag{0.16}$$

Using this and the interior regularity of u, we obtain

Theorem 3. *Let* $u \in C(\overline{\Omega})$ *be a solution to* (0.1) *and* (0.2) *in* Ω *such that* (0.2) *be fulfilled in the sense of* (0.7) *and such that* H *and* $\cos\theta$ *satisfy respectively* (0.11) *and* (0.12). *Suppose that* $\partial\Omega\cap\overline{A_{\delta_0}(x_0)}$ *is of the class* C^2. *If*

$$|Du(x)| \ge \frac{5}{\tilde{\tilde{\beta}}}, \quad \text{for a point } x \in \partial^* A_{\delta_0}(x_0)$$

then, for $x \in \partial^* A$, *there holds*

$$u(x) - \inf_{\Omega\cap A_R(x_0)} u \le \mathfrak{C}\cdot\hat{\mathfrak{C}}_*\cdot(\delta_0)^{1-n\varepsilon_0} + 2(\delta_0)^{1+\varepsilon_0} + \mathfrak{C}\cdot\hat{\mathfrak{C}}_*\cdot(\delta_0)^{1-n\varepsilon_0}\cdot\big(\sup_{\Omega\cap A_{\delta_0}(x_0)} u - \inf_{\Omega\cap A_{\delta_0}(x_0)} u\big) \tag{0.17}$$

if the first case in (0.12) *holds, and*

$$\sup_{\Omega\cap A_R(x_0)} u - u(x) \le \mathfrak{C}\cdot\hat{\mathfrak{C}}_*\cdot(\delta_0)^{1-n\varepsilon_0} + 2(\delta_0)^{1+\varepsilon_0} + \mathfrak{C}\cdot\hat{\mathfrak{C}}_*\cdot(\delta_0)^{1-n\varepsilon_0}\cdot\big(\sup_{\Omega\cap A_{\delta_0}(x_0)} u - \inf_{\Omega\cap A_{\delta_0}(x_0)} u\big) \tag{0.18}$$

if the second case in (0.12) *holds, where* $\hat{\mathfrak{C}}_*$ *is a constant determined by* $\tilde{\beta}$, $\tilde{\tilde{\beta}}$, H_*, $\mathcal{K}_{\partial\Omega\cap\overline{A_{\delta_0}(x_0)}}$ *and* n.

Letting $\varepsilon_0 \to 0$ in Theorem 3 and combining the latter theorem with Theorem 2, we obtain

Main Theorem. *Let $u \in C(\overline{\Omega})$ be a solution to* (0.1) *and* (0.2) *in Ω such that* (0.2) *is fulfilled in the sense of* (0.7). *Let $|H(x,t)|$ be bounded by the constant H_* in $\overline{\Omega} \times \mathbb{R}$. Suppose that for a positive constant $\tilde{\beta} < 1$, the inequalities* (0.12) *hold for x_0 and $A_{\delta_0}(x_0)$, δ_0 being sufficiently small and $x_0 \in \partial\Omega$. Suppose that δ_0 is so small that $\partial\Omega \cap \overline{A_{\delta_0}(x_0)}$ is of class C^2. Then u is Lipschitz continuous in $\Omega \cap A_{\delta_0}(x_0)$ up to the boundary; the Lipschitz norm of u in $\overline{\Omega \cap A_{\delta_0}(x_0)}$ is either less than $\frac{5}{\tilde{\beta}}$ or is determined by H_*, n, $\tilde{\beta}$, $\tilde{\tilde{\beta}}$, $\mathcal{K}_{\partial\Omega\cap\overline{A_r(\hat{x})}}$, and $|\Omega|$.*

Here and in the following, we denote by $|\cdot|$ either an n-dimensional or an $(n+1)$-dimensional Hausdorff measure.

Proposition 1 is based on the reasoning in Giusti [6, pp. 312–313] which leads to estimates for the oscillation of u in terms of the L^1-norm of u, under the conditions (1.1) or (1.3) indicated below, which says that the subgraph of u or the complement of the subgraph of u includes a large portion of a sufficiently small cylinder-type region around $u(x_0)$, $x_0 \in \partial\Omega$. This reasoning is given in Subsection 1. In Subsection 2, we formulate a result which is essentially Theorem 3.2 in Giusti [6] and which assures us of the fulfillment of (1.1) or (1.3) for capillarity surfaces. This suggests us that we should estimate the L^1-norm of u by writing (0.1) and (0.2) in weak a form in which the assumed boundedness of $|u|$ allows us to take the test function to be $\big(u(x) - \inf_{\Omega} u\big)$ or $\big(\sup_{\Omega} u - u(x)\big)$. The resultant inequalities (1.7) and (1.8) suggest us to restrict our consideration to a small region $\Omega \cap A$ of the type indicated in the beginning of Subsection 4 which is analogous to that of $A_R(x_0)$, for which the resultant boundary integrals are treated in Subsection 3.2. To proceed with obtaining L^1-estimates of $\big(u(x) - \inf_{\Omega\cap A} u\big)$ and $\big(\sup_{\Omega\cap A} u - u(x)\big)$ in $\Omega \cap A$, we shall appeal to the modified Sobolev inequality given in Proposition A.1 in Appendix I, for which we have to estimate $\int_{\partial^{**}A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1}$ and $\int_{\partial^{**}A} \big(u(x) - \inf_{\Omega\cap A} u\big)\, d\mathcal{H}_{n-1}$ with an application of the condition (0.11).

1. Proof of Proposition 1

1. Oscillation of u in terms of the L^1-norm of u. We modify the approach taken by Giusti [6, pp. 312–313].

Let u be a function with the subgraph

$$U = \{(x,t) \in \Omega \times \mathbb{R},\, t < u(x)\},$$

and set for points $\hat{z} = (\hat{x}, \hat{t}) \in \Omega \times \mathbb{R}$ and for $r > 0$,

$$U_r(\hat{z}) = C_r(\hat{z}) \cap U \quad \text{and} \quad U'_r(\hat{z}) = C_r(\hat{z}) \setminus U,$$

where

$$C_r(\hat{z}) = \{(x,t) : x \in A_r(\hat{x}), |t - \hat{t}| < r\},$$

with $A_r(\hat{x})$ being chosen as indicated before Theorem 3.

We make the assumption that there exist positive constants R_- and α_* such that

$$|U_r(\hat{z})| > \alpha_* r^{n+1} \quad \text{for every} \quad r \le R_-, \qquad \text{whenever} \ |U_r(\hat{z})| > 0 \ \text{for every} \ r > 0. \tag{1.1}$$

Suppose $x_0 \in \mathbb{R}^n$ such that $A_R(x_0) \cap \Omega$ is nonempty. Let us set

$$M_R = \sup_{\Omega \cap A_R(x_0)} u, \quad \text{and} \quad m_R = \inf_{\Omega \cap A_R(x_0)} u.$$

We shall establish below the fulfilment of the following inequality *under the assumption that* (1.1) *holds*; namely, for $R \le R_-$:

$$u(x_0) - m_R \le \frac{2^{n+2}}{\alpha_* R^n} \int_{\Omega \cap A_R(x_0)} (u(x) - m_R)\, dx + 2R. \tag{1.2}$$

Indeed, let

$$z_j = (x_0, m_R + 2jR), \quad \text{for} \ \ j \in \mathbb{N}.$$

Then

$$z_j \in U, \quad \text{for} \ \ j \le j_* = \left[\frac{u(x_0) - m_R}{2R}\right],$$

where $[s]$ denotes the largest integer less than s for $s > 0$. Under the assumption (1.1), we have

$$|U_{R/2}(z_j)| \ge \alpha_* \left(\frac{R}{2}\right)^{n+1}, \quad \text{for} \ \ 1 \le j \le j_*,$$

and therefore

$$\int_{\Omega \cap A_R(x_0)} (u(x) - m_R)\, dx \ge \sum_{j=1}^{j_*} |U_{R/2}(z_j)| \ge j_* \cdot \alpha_* \cdot \left(\frac{R}{2}\right)^{n+1}.$$

Hence

$$\begin{aligned} M_R =& u(x_0) + (M_R - u(x_0)) \le 2(j_* + 1)R + m_R + (M_R - u(x_0)) \\ \le& \frac{2^{n+2}}{\alpha_* R^n} \int_{\Omega \cap A_R(x_0)} (u(x) - m_R)\, dx + 2R + m_R + (M_R - u(x_0)), \end{aligned}$$

which is (1.2).

Assume now, instead of (1.1), that there exist positive constants R_+ and α_* such that

$$|U_r'(\hat{z})| > \alpha_* r^{n+1} \quad \text{for every} \quad r \le R_+, \qquad \text{whenever} \ |U_r'(\hat{z})| > 0 \ \text{for every} \ r > 0. \tag{1.3}$$

Under the assumption (1.3), we shall analogously obtain, for $R \leq R_+$, the inequality

$$M_R - u(x_0) \leq \frac{2^{n+2}}{\alpha_* R^n} \int\limits_{\Omega \cap A_R(x_0)} (M_R - u(x))\, dx + 2R. \tag{1.4}$$

Indeed, let

$$z_j^+ = (x_0, M_R - 2jR), \quad \text{for} \quad j \in \mathbb{N}.$$

Then

$$z_j^+ \in U' = (\Omega \times \mathbb{R}) \setminus U, \quad \text{for} \quad j \leq j_*^+ = \left[\frac{M_R - u(x_0)}{2R} \right].$$

By assumption, we have

$$|U_{R/2}(z_j^+)| \geq \alpha_* \left(\frac{R}{2} \right)^{n+1}, \quad \text{for} \quad 1 \leq j \leq j_*^+,$$

and therefore

$$\int\limits_{\Omega \cap A_R(x_0)} (M_R - u(x_0))\, dx \geq \sum_{j=1}^{j_*} |U'_{R/2}(z_j^+)| \geq j_*^+ \cdot \alpha_* \cdot \left(\frac{R}{2} \right)^{n+1},$$

which yields

$$\begin{aligned} -m_R =& u(x_0) + (u(x_0) - m_R) \leq 2(j_*^+ + 1)R - M_R + (u(x_0) - m_R) \\ \leq& \frac{2^{n+2}}{\alpha_* R^n} \int\limits_{\Omega \cap A_R(x_0)} (M_R - u(x))\, dx + 2R - M_R + (u(x_0) - m_R); \end{aligned}$$

this is (1.4).

2. (1.1) or (1.3) for capillary surfaces. The above consideration suggests that we should apply the estimates in Giusti [6, Theorem 3.2]. Indeed, below we appeal to estimates in Proposition 1, which are essentially obtained in Giusti [6, Theorem 3.2] and which can be proved by the argument given in [7, Appendix] without any essential modification.

Proposition 2. *Let u be a solution to* (0.5) *with subgraph U. Suppose that $\partial\Omega \cap \overline{A_{R_0}(x_0)}$ is of the class C^2 whose principal curvatures are bounded in the absolute value by $\mathcal{K}_{\partial\Omega \cap \overline{A_{R_0}(x_0)}}$. If there exists a constant $\hat{\gamma}$, $0 \leq \hat{\gamma} < 1$, such that*

$$\beta(x) \geq -\hat{\gamma}, \quad \text{for all} \quad x \in \partial\Omega \cap \overline{A_{R_0}(x_0)},$$

and if

$$|U_r(\hat{z})| > 0, \quad \text{for all} \quad r > 0,$$

then there exist positive constants R_- and α_* determined completely by n, $\inf\limits_{\Omega \times \mathbb{R}} H$, $\hat{\gamma}$, $\mathcal{K}_{\partial\Omega \cap \overline{A_{R_0}(x_0)}}$, R_0 and the largest possible radius R_Ω of the inscribed disks in Ω such that

$$|U_r(\hat{z})| > \alpha_* r^{n+1}, \quad \text{for every} \quad r \leq R_-.$$

In particular, we can take

$$\alpha_* = \frac{1-\hat{\gamma}}{16(n+1)k_{(n+1)}}, \tag{1.5}$$

with $k_{(n+1)}$ *being the isoperimetric constant in* $\mathbb{R}^{n+1}$*, and*

$$R_- = \begin{cases} \min\left(\dfrac{C_{\hat{\gamma}}}{C_{\varepsilon,\Omega} k_{(n+1)}}, R_0\right), & \text{if } \inf\limits_{\Omega\times\mathbb{R}} H \geq 0, \\ \min\left(\dfrac{C^-_{\hat{\gamma}}}{2^{1/(n+1)}\cdot C_{\varepsilon,\Omega}\cdot k_{(n+1)}}, \tilde{R}_-\right), R_0, & \text{if } \inf\limits_{\Omega\times\mathbb{R}} H < 0, \end{cases}$$

in which we set

$$C_{\hat{\gamma}} = \min\left(\frac{1}{2}, \frac{1-\hat{\gamma}}{3\hat{\gamma}+1}\right),$$

$$C_{\varepsilon,\Omega} = \frac{2}{\varepsilon} + 2(n-1)\mathcal{K}_{\partial\Omega}, \quad \text{with} \quad \varepsilon \leq \min\left(\frac{1}{2\mathcal{K}_{\partial\Omega}}, R_\Omega\right),$$

$$\tilde{R}_- = \left(\frac{1-\hat{\gamma}}{4nk_{(n+1)}\cdot|\inf_{\Omega\times\mathbb{R}} H|}\right)^{n+1},$$

and

$$C^-_{\hat{\gamma}} = \min\left(\frac{1}{2}, \frac{1-\hat{\gamma}-2nk_{(n+1)}\cdot|\inf_{\Omega\times\mathbb{R}} H|\cdot(\tilde{R}_-)^n}{3\hat{\gamma}+1}\right).$$

If there exists a constant $\hat{\gamma}$*,* $0 \leq \hat{\gamma} < 1$*, such that*

$$\beta(x) \leq \hat{\gamma}, \quad \text{for all} \quad x \in \partial\Omega \cap \overline{A_{R_0}(x_0)},$$

and if

$$|U_r'(\hat{z})| > 0, \quad \text{for all} \quad r > 0,$$

then there exists a positive constant R_+ *determined completely by* n*,* $\sup\limits_{\Omega\times\mathbb{R}} H$*,* $\hat{\gamma}$*,* $\mathcal{K}_{\partial\Omega\cap\overline{A_{R_0}(x_0)}}$*,* R_0 *and* R_Ω *such that*

$$|U_r'(\hat{z})| > \alpha_* r^{n+1}, \quad \text{for every} \quad r \leq R_+,$$

for the same constant α_* *as above. In particular, we can take*

$$R_+ = \begin{cases} \min\left(\dfrac{C_{\hat{\gamma}}}{C_{\varepsilon,\Omega} k_{(n+1)}}, R_0\right), & \text{if } \sup\limits_{\Omega\times\mathbb{R}} H \leq 0, \\ \min\left(\dfrac{C^+_{\hat{\gamma}}}{2^{1/(n+1)}\cdot C_{\varepsilon,\Omega}\cdot k_{(n+1)}}, \tilde{R}_+, R_0\right), & \text{if } \sup\limits_{\Omega\times\mathbb{R}} H > 0, \end{cases}$$

with

$$\tilde{R}_+ = \left(\frac{1-\hat{\gamma}}{4nk_{(n+1)}\cdot|\sup_{\Omega\times\mathbb{R}} H|}\right)^{n+1},$$

and

$$C^+_{\hat{\gamma}} = \min\left(\frac{1}{2}, \frac{1-\hat{\gamma}-2nk_{(n+1)}\cdot|\sup_{\Omega\times\mathbb{R}} H|\cdot(\tilde{R}_+)^n}{3\hat{\gamma}+1}\right).$$

3. L^1-norm of $|Du|$ in terms of the L^1-norm of $\big(u - \inf_\Omega u\big)$ or $\big(\sup_\Omega u - u\big)$.

An initial stage. Assume that there exists a nonnegative constant H_* such that

$$|H(x,t)| \le H_* \quad \text{for} \quad x \in \Omega \quad \text{and} \quad t \in \mathbb{R}.$$

Consider the identity (0.7). Assuming $|u|$ is bounded up to the boundary, [8, Theorem 1] assures us of that $u \in H^{1,1}(\Omega)$ and thus we are allowed to set in (0.7)

$$\eta(x) = u(x) - \inf_\Omega u \ge 0,$$

and obtain

$$\int_\Omega \frac{|Du|^2}{1+|Du|^2}\,dx - H_* \cdot \int_\Omega \big(u(x) - \inf_\Omega u\big)\,dx \le \int_{\partial\Omega} \beta \cdot \big(u(x) - \inf_\Omega u\big)\,d\mathcal{H}_{n-1}. \quad (1.6)$$

Since

$$\frac{|Du|^2}{\sqrt{1+|Du|^2}} = \sqrt{1+|Du|^2} - \frac{1}{\sqrt{1+|Du|^2}},$$

the last inequality yields

$$\int_\Omega \sqrt{1+|Du|^2}\,dx \le |\Omega| + H_* \cdot \int_\Omega \big(u(x) - \inf_\Omega u\big)\,dx + \int_{\partial\Omega\cap\{x:\beta(x)>0\}} \beta(x) \cdot \big(u(x) - \inf_\Omega u\big)\,d\mathcal{H}_{n-1}. \quad (1.7)$$

Analogously, we are allowed to set in (0.7)

$$\eta(x) = u(x) - \sup_\Omega u \le 0,$$

and obtain

$$\int_\Omega \sqrt{1+|Du|^2}\,dx \le |\Omega| + H_* \cdot \int_\Omega \big(\sup_\Omega u - u(x)\big)\,dx - \int_{\partial\Omega\cap\{x:\beta(x)<0\}} \beta(x) \cdot \big(\sup_\Omega u - u(x)\big)\,d\mathcal{H}_{n-1}. \quad (1.8)$$

3.1. Restricting to small domains.

This consideration suggests that we should restrict our consideration to a small region $\Omega \cap A_0$ of the type indicated below. Namely, setting

$$\partial^*\Omega_t = \{x : x \in \Omega,\ \mathrm{dist}(x, \partial\Omega) = t\} \quad \text{for} \quad t > 0,$$

we first let the boundary $\partial(\Omega \cap A_0)$ be made up of three parts, namely

$$\partial(\Omega \cap A_0) = \big(\partial\Omega \cap \overline{A_0}\big) \cup \big(\partial^* A_0\big) \cup \big(\partial^{**} A_0\big),$$

such that

$$\partial^* A_0 = \partial A_0 \cap \partial^*\Omega_{\delta_0}, \quad (1.9)$$

for some small positive number δ_0,

$$\partial^{**} A_0 = (\partial A_0 \cap \Omega) \setminus \partial^* \Omega_{\delta_0},$$

and

$$Dd \cdot \nu_{\Omega\cap A_0}\Big|_{\partial^{**}A_0} = 0, \tag{1.10}$$

where we let $\nu_{\Omega\cap A_0}$ be the unit outward normal to $\partial(\Omega \cap A_0)$; furthermore,

$$\operatorname{diam}(\partial\Omega \cap \overline{A_0}) \le (\delta_0)^{1+\varepsilon_0} \quad \text{and} \quad \operatorname{diam}(\partial^* A_0) \le (\delta_0)^{1+\varepsilon_0}, \tag{1.11}$$

$$|\partial\Omega \cap \overline{A_0}| \ge \left(\frac{\delta_0}{2}\right)^{(1+\varepsilon_0)(n-1)} \quad \text{and} \quad |\partial^* A_0| \ge \left(\frac{\delta_0}{2}\right)^{(1+\varepsilon_0)(n-1)}, \tag{1.12}$$

for some small positive constant $\varepsilon_0 < 1$. We choose δ_0 sufficiently small so that each component of $\partial^{**}A_0$ is entirely included in either $\partial^{**}_+(\Omega\cap A_0)$ or $\partial^{**}_-(\Omega\cap A_0)$, where

$$\partial^{**}_-(\Omega \cap A_0) = \big(\partial^{**}A_0\big) \cap \{x : \beta_{\Omega\cap A_0} < 0\}$$
$$\partial^{**}_+(\Omega \cap A_0) = \big(\partial^{**}A_0\big) \cap \{x : \beta_{\Omega\cap A_0} > 0\}.$$

Next, we let the region $\Omega \cap A$ be as follows. Namely,

Case 1. If

$$\beta_{\Omega\cap A_0}\Big|_{\partial^{**}A_0} \le -\hat{\beta} \quad \text{or} \quad \beta_{\Omega\cap A_0}\Big|_{\partial^{**}A_0} \ge \hat{\beta} \tag{1.13}$$

for some positive constant $\hat{\beta}$, then we let $A = A_0$.

Case 2. Suppose both sets $\partial^{**}_-(\Omega \cap A_0)$ and $\partial^{**}_-(\Omega \cap A_0)$ are nonempty and there hold

$$\beta_{\Omega\cap A_0}\Big|_{\partial^{**}_-(\Omega\cap A_0)} \le -\hat{\beta} \quad \text{and} \quad \beta_{\Omega\cap A_0}\Big|_{\partial^{**}_+(\Omega\cap A_0)} \ge \hat{\beta}. \tag{1.14}$$

For

$$\beta(x) > \tilde{\tilde{\beta}} > 0 \quad \text{for all} \quad x \in \partial\Omega \cap \overline{A_0} \tag{1.15}$$

we set

$$E_{+-} = \{x : x \in \overline{\Omega_{\delta_0}},\, u(x) = \inf_{\partial\Omega\cap\overline{A_0}} u\},$$
$$E_{-+} = \{x : x \in \overline{\Omega_{\delta_0}},\, u(x) = \sup_{\partial^* A_0} u\},$$

and let A_{11} be the region enclosed by $\partial^*\Omega_{\delta_0}$ and E_{+-}, together with the components of $\overline{\partial^{**}A_0}$ passing through $E_{+-} \cap \partial\Omega$; let A_{12} be the region enclosed by $\partial^*\Omega_{\delta_0}$ and E_{-+}, together with the components of $\overline{\partial^{**}A_0}$ passing through $E_{-+} \cap \partial\Omega$; for

$$\beta(x) < -\tilde{\tilde{\beta}} < 0 \quad \text{for all} \quad x \in \partial\Omega \cap \overline{A_0} \tag{1.16}$$

we set

$$E_{++} = \{x : x \in \overline{\Omega_{\delta_0}},\, u(x) = \sup_{\partial\Omega\cap\overline{A_0}} u\},$$
$$E_{--} = \{x : x \in \overline{\Omega_{\delta_0}},\, u(x) = \inf_{\partial^* A_0} u\},$$

and let A_{11} be the region enclosed by $\partial^*\Omega_{\delta_0}$ and E_{++}, together with components of $\overline{\partial^{**}A_0}$ passing through $E_{++}\cap\partial\Omega$, and let A_{12} be the region enclosed by $\partial^*\Omega_{\delta_0}$ and E_{--}, together with components of $\overline{\partial^{**}A_0}$ passing through $E_{--}\cap\partial\Omega$. We then denote

$$A = A_0 \cup A_{11} \cup A_{12}.$$

Furthermore, we let δ_0 be so small that in the case of (1.15) or (1.16), there hold respectively

$$-(1+\tilde{\beta})/2 < \beta_{\Omega\cap A}\Big|_{\partial^* A} < -\tilde{\tilde{\beta}}/2, \quad \text{or} \quad (1+\tilde{\beta})/2 > \beta_{\Omega\cap A}\Big|_{\partial^* A} > \tilde{\tilde{\beta}}/2, \tag{1.17}$$

where we set

$$\partial^* A = A \cap \partial^*\Omega_{\delta_0}.$$

We shall prove in Appendix 5 the following.

Proposition 3. *Suppose that*

$$\lim_{x_k \to x_0} |Du| \geq \frac{2}{\tilde{\tilde{\beta}}}, \tag{1.18}$$

for each *sequence of points x_k approaching x_0. Then we have*

$$|E_{\pm\pm}| \leq 2|\partial^{**}A| \cdot \left(\sqrt{1 - \{(1+\tilde{\beta})/2 \Big/ \sqrt{1+[(1+\tilde{\beta})/2]^2}\}}\right)^{-1}, \tag{1.19}$$

and

$$|A_1| \leq 2|\partial^{**}A|^2 \cdot \frac{(1+\tilde{\beta})/2/\sqrt{1+[(1+\tilde{\beta})/2]^2}}{\sqrt{1-\{(1+\tilde{\beta})/2/\sqrt{1+[(1+\tilde{\beta})/2]^2}\}}}. \tag{1.20}$$

For a domain $\Omega\cap A$, we may, without loss of generality, assume that

$$\int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A}(x) u(x)\, d\mathcal{H}_{n-1} = 0, \tag{1.21}$$

where

$$\beta_{\Omega\cap A} = \frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu_{\Omega\cap A}.$$

Applying to $\Omega\cap A$ the reasoning leading to (1.7) and (1.8), we obtain from (1.21)

$$\int_{\Omega\cap A} |Du|\, dx \leq |\Omega\cap A| + H_* \cdot \int_{\Omega\cap A} \big(u(x) - \inf_{\Omega\cap A} u\big)\, dx + \int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A} \cdot \big(u(x) - \inf_{\Omega\cap A} u\big)\, d\mathcal{H}_{n-1} \tag{1.22}$$

and

$$\int_{\Omega\cap A} |Du|\,dx \le |\Omega\cap A| + H_* \cdot \int_{\Omega\cap A} \big(\sup_{\Omega\cap A} u - u(x)\big)\,dx + \int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A}\cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\,d\mathcal{H}_{n-1}. \quad (1.23)$$

Adding (1.22) and (1.23), we obtain

$$2\int_{\Omega\cap A} |Du|\,dx \le |\Omega\cap A| + H_*\cdot\int_{\Omega\cap A}\big(u(x)-\inf_{\Omega\cap A} u\big)\,dx + H_*\cdot\int_{\Omega\cap A}\big(\sup_{\Omega\cap A} u - u(x)\big)\,dx + 2\int_{\partial(\Omega\cap A)}\beta(x)u(x)\,d\mathcal{H}_{n-1} - \big(\sup_{\Omega\cap A} u - \inf_{\Omega\cap A} u\big)\cdot\int_{\partial(\Omega\cap A)}\beta(x)\,d\mathcal{H}_{n-1}.$$

This and (1.21) yield

$$2\int_{\Omega\cap A} |Du|\,dx \le |\Omega\cap A| + H_*\cdot\int_{\Omega\cap A}\big(u(x)-\inf_{\Omega\cap A} u\big)\,dx + H_*\cdot\int_{\Omega\cap A}\big(\sup_{\Omega\cap A} u - u(x)\big)\,dx - \big(\sup_{\Omega\cap A} u - \inf_{\Omega\cap A} u\big)\cdot\int_{\partial(\Omega\cap A)}\beta(x)\,d\mathcal{H}_{n-1}. \quad (1.24)$$

3.2. Boundary integral in (1.24). Taking $\eta = 1$ in the identity (0.7) with the domain of integration Ω replaced by $\Omega\cap A$, we obtain

$$\int_{\Omega\cap A} H\,dx = \int_{\Omega\cap A} \operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}}\,dx = \int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A}(x)\,d\mathcal{H}_{n-1}.$$

Hence

$$\Big|\int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A}(x)\,d\mathcal{H}_{n-1}\Big| \le H_*\cdot|\Omega\cap A|. \quad (1.25)$$

3.3. Inserting (1.25) into (1.24), we obtain

$$2\int_{\Omega\cap A} |Du|\,dx \le |\Omega\cap A| + H_*\cdot\int_{\Omega\cap A}\big(u(x)-\inf_{\Omega\cap A} u\big)\,dx + H_*\cdot\int_{\Omega\cap A}\big(\sup_{\Omega\cap A} u - u(x)\big)\,dx + \big(\sup_{\Omega\cap A} u - \inf_{\Omega\cap A} u\big)\cdot H_*\cdot|\Omega\cap A|. \quad (1.26)$$

4. Estimating the L^1-norm of $\big(u(x)-\inf_{\Omega\cap A} u\big)$ and $\big(\sup_{\Omega\cap A} u - u(x)\big)$ in $\Omega\cap A$.

4.1. L^1-norm of $\big(\sup_{\Omega\cap A} u - u(x)\big)$ and $\big(u(x)-\inf_{\Omega\cap A} u\big)$ in $\Omega\cap A$. The case where $\beta(x) > 0$ for all $x\in\partial\Omega\cap\bar{A}$. By the modified Sobolev inequality (A.8), we have

$$\|\sup_{\Omega\cap A} u - u(x)\|_{L^{n*}(\Omega\cap A)} \le \frac{\omega_n}{n}\cdot\Big[\int_{\Omega\cap A}|Du|\,dx + \int_{\partial(\Omega\cap A)}\big(\sup_{\Omega\cap A} u - u(x)\big)\,d\mathcal{H}_{n-1}\Big]. \quad (1.27)$$

Assume that (1.15) and (1.17) hold. By (A.2) and (A.7), we have

$$\int_{\partial^* A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} \le \int_{\Omega\cap A} |Du|\, dx + \Big[2(n-1)\mathcal{K}_{\partial\Omega} + 2(\delta_0)^{-1}\Big] \cdot \int_{\Omega\cap A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx, \quad (1.28)$$

and

$$\int_{\partial\Omega\cap\overline{A}} \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} \le \int_{\Omega\cap A} |Du|\, dx + \Big[2(n-1)\mathcal{K}_{\partial\Omega} + 2(\delta_0)^{-1}\Big] \cdot \int_{\Omega\cap A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx. \quad (1.29)$$

Consider, rather than (1.6), the identity

$$\int_{\Omega\cap A} \frac{Du}{\sqrt{1+|Du|^2}} \cdot D\eta\, dx + \int_{\Omega\cap A} H \cdot \eta\, dx - \int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A} \cdot \eta\, d\mathcal{H}_{n-1} = 0, \quad (1.30)$$

for all $\eta \in H^{1,1}(\Omega\cap A)$. By setting $\eta(x) = \big(u(x) - \sup_{\Omega\cap A} u\big)$ and $\eta(x) = \big(\sup_{\Omega\cap A} u - u(x)\big)$ in (1.30) we have

$$\int_{\Omega\cap A} \frac{|Du|^2}{\sqrt{1+|Du|^2}} dx - \int_{\Omega\cap A} H \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx + \int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} = 0,$$

and

$$\int_{\Omega\cap A} \frac{|Du|^2}{\sqrt{1+|Du|^2}} dx + \int_{\Omega\cap A} H \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx - \int_{\partial(\Omega\cap A)} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} = 0.$$

These yield

$$\begin{aligned} &- \int_{\partial^{**}_{-}(\Omega\cap A)} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} - \int_{\partial^* A} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} \\ &\le \int_{\Omega\cap A} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\, dx - \int_{\Omega\cap A} H \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx \\ &\quad + \int_{\partial\Omega\cap\overline{A}} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} \end{aligned}$$

$$+ \int\limits_{\partial^{**}_{+}(\Omega\cap A)} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1}, \tag{1.31}$$

and

$$\begin{aligned}
&\int\limits_{\partial^{**}_{+}(\Omega\cap A)} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} + \int\limits_{\partial\Omega\cap\overline{A}} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} \\
&\le \int\limits_{\Omega\cap A} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\, dx + \int\limits_{\Omega\cap A} H \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx \\
&\quad - \int\limits_{\partial^{*} A} \beta_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} \\
&\quad - \int\limits_{\partial^{**}_{-}\Omega\cap A} \beta_{(\Omega\cap A)} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1},
\end{aligned} \tag{1.32}$$

where we set

$$\begin{aligned}
\partial^{**}_{-}(\Omega\cap A) &= \big(\partial^{**} A\big) \cap \{x : \beta_{\Omega\cap A} < 0\} \\
\partial^{**}_{+}(\Omega\cap A) &= \big(\partial^{**} A\big) \cap \{x : \beta_{\Omega\cap A} > 0\}.
\end{aligned}$$

4.1.1. If (1.13) holds for some positive constant $\hat{\beta}$, then we have $A = A_0$ and we obtain from (1.31) or (1.32)

$$\begin{aligned}
\int\limits_{\partial^{**} A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} &\le (\hat{\beta})^{-1} \cdot \Big[\int\limits_{\Omega\cap A} |Du|\, dx + H_* \int\limits_{\Omega\cap A} \cdot \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx\Big] \\
&\quad + (\hat{\beta})^{-1} \cdot \int\limits_{\partial\Omega\cap\overline{A}} \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1}.
\end{aligned}$$

Inserting (1.29) into this, we obtain

$$\begin{aligned}
\int\limits_{\partial^{**} A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, d\mathcal{H}_{n-1} &\le 2(\hat{\beta})^{-1} \cdot \int\limits_{\Omega\cap A} |Du|\, dx \\
&\quad + (\hat{\beta})^{-1} \big[H_* + 2(n-1)\mathcal{K}_{\partial\Omega} + 2(\delta_0)^{-1}\big] \cdot \int\limits_{\Omega\cap A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx\Big].
\end{aligned} \tag{1.33}$$

Inserting (1.33), (1.28) and (1.29) into (1.27), we obtain

$$\begin{aligned}
\|\sup_{\Omega\cap A} u - u(x)\|_{L^{n*}(\Omega\cap A)} &\le \frac{\omega_n}{n} \cdot \big[1 + 2(\hat{\beta})^{-1}\big] \cdot \int\limits_{\Omega\cap A} |Du|\, dx \\
&\quad + \hat{C}_1 \cdot \int\limits_{\Omega\cap A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx,
\end{aligned}$$

where

$$\hat{C}_1 = 2(\hat{\beta})^{-1} \cdot H_* + \big[1 + 2(\hat{\beta})^{-1}\big] \cdot \big[2(n-1)\mathcal{K}_{\partial\Omega} + 2(\delta_0)^{-1}\big]. \tag{1.34}$$

Hence, by Hölder's inequality and the fact that $A = A_0$, we obtain

$$\int\limits_{\Omega\cap A} |Du|\,dx \geq [1 + 2(\hat{\beta})^{-1}]^{-1} \cdot [n/\omega_n - \hat{C}_1 \cdot |\Omega \cap A_0|^{1/n}] \times \|\sup_{\Omega\cap A} u - u(x)\|_{L^{n*}(\Omega\cap A)}. \tag{1.35.1}$$

Analogously, we can establish in the case of (1.13) that

$$\int\limits_{\Omega\cap A} |Du|\,dx \geq [1 + 2(\hat{\beta})^{-1}]^{-1} \cdot [n/\omega_n - \hat{C}_1 \cdot |\Omega \cap A_0|^{1/n}] \times \|u(x) - \inf_{\Omega\cap A} u\|_{L^{n*}(\Omega\cap A}. \tag{1.35.2}$$

4.1.2. Suppose both sets $\partial^{**}_{-}(\Omega\cap A_0)$ and $\partial^{**}_{-}(\Omega\cap A_0)$ are nonempty and (1.14) holds. We notice that by our choice of A_0 and A indicated above in (1.14) and (1.15), we have

$$\sup_{\Omega\cap A} u = \sup_{\partial\Omega\cap\overline{A_0}} u \quad \text{and} \quad \inf_{\Omega\cap A} u = \inf_{\partial^* A_0} u.$$

Thus

$$\int\limits_{\partial(\Omega\cap A)} \big(\sup_{\Omega\cap A} u - u(x)\big)\,d\mathcal{H}_{n-1} = \int\limits_{\partial(\Omega\cap A)} (\inf_{\partial\Omega\cap\overline{A_0}} u - u(x))\,d\mathcal{H}_{n-1} + \int\limits_{\partial(\Omega\cap A)} (\sup_{\partial\Omega\cap\overline{A_0}} u - \inf_{\partial\Omega\cap\overline{A_0}} u)\,d\mathcal{H}_{n-1}.$$

By Theorem 1, we obtain

$$\Big|\int\limits_{\partial(\Omega\cap A)} \beta_{\Omega\cap A} \cdot (\sup_{\partial\Omega\cap\overline{A_0}} u - \inf_{\partial\Omega\cap\overline{A_0}} u)\,d\mathcal{H}_{n-1}\Big| \leq L_0(\tilde{\beta}, \tilde{\tilde{\beta}}) \cdot \operatorname{diam}(\partial\Omega \cap \overline{A_0}) \cdot |\partial(\Omega\cap A)|, \tag{1.36}$$

where $L_0(\tilde{\beta}, \tilde{\tilde{\beta}})$ is the Lipschitz norm of the trace of u on the boundary, which depends only on $\tilde{\beta}$, $\tilde{\tilde{\beta}}$, H, n and $\mathcal{K}_{\partial\Omega\cap\overline{A}}$.

To treat the first integral on the right-hand side of (1.32), we set

$$(\Omega\cap A)_{+-} = \{x : x \in \Omega\cap A, u(x) \leq \inf_{\partial\Omega\cap\overline{A_0}} u\},$$

and

$$\partial^{**}_{+-}(\Omega\cap A) = \partial((\Omega\cap A)_{+-}) \cap \partial^{**} A$$

to obtain

$$\int\limits_{(\partial\Omega\cap\overline{A_0})\cup\partial^{**}_{+}(\Omega\cap A)} (\inf_{\partial\Omega\cap\overline{A}} u - u(x))\,d\mathcal{H}_{n-1} \leq \int\limits_{\partial^{**}_{+-}(\Omega\cap A)} (\inf_{\partial\Omega\cap\overline{A}} u - u(x))\,d\mathcal{H}_{n-1} = 0, \tag{1.37}$$

since $\partial^{**}_{+-}(\Omega \cap A) \subset E_{+-}$. Furthermore, we have

$$\int\limits_{\partial\Omega\cap(\overline{A}\setminus\overline{A_0})} (\inf_{\partial\Omega\cap\overline{A_0}} u - u(x))\, d\mathcal{H}_{n-1} \le L_0(\tilde{\beta}, \tilde{\tilde{\beta}}) \cdot \operatorname{diam}(\partial\Omega \cap \overline{A} \setminus \overline{A_0})) \cdot |\partial\Omega \cap (\overline{A} \setminus \overline{A_0})|, \quad (1.38)$$

and

$$\int\limits_{\partial^* A\cup\partial^{**}_{-} A} (\inf_{\partial\Omega\cap\overline{A_0}} u - u(x))\, d\mathcal{H}_{n-1} \le \int\limits_{\partial(\cap(\Omega\cap A)_{+-})} (\inf_{\partial\Omega\cap\overline{A_0}} u - u(x))\, d\mathcal{H}_{n-1}, \quad (1.39)$$

and

$$\begin{aligned} &- \int\limits_{\partial(\cap(\Omega\cap A)_{+-})} \beta_{\Omega\cap A}(x) \cdot (\inf_{\partial\Omega\cap\overline{A_0}} u - u(x))\, d\mathcal{H}_{n-1} \\ &\le \int\limits_{(\Omega\cap A)_{+-}} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\, dx + H_* \int\limits_{(\Omega\cap A)_{+-}} (\inf_{\partial\Omega\cap\overline{A}} u - u(x))\, dx \\ &\le \int\limits_{(\Omega\cap A)_{+-}} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\, dx + H_* \int\limits_{(\Omega\cap A)_{+-}} (\sup_{\Omega\cap A} u - u(x))\, dx. \end{aligned} \quad (1.40)$$

From (1.39) and (1.40), we obtain

$$\int\limits_{\partial^* A\cup\partial^{**}_{-} A} (\inf_{\partial\Omega\cap\overline{A_0}} u - u(x))\, d\mathcal{H}_{n-1} \le (\hat{\hat{\beta}})^{-1} \cdot \int\limits_{(\Omega\cap A)_{+-}} |Du|\, dx + (\hat{\hat{\beta}})^{-1} \cdot H_* \cdot \int\limits_{(\Omega\cap A)_{+-}} (\inf_{\partial\Omega\cap\overline{A_0}} u - u(x)\, dx, \quad (1.41)$$

if we have

$$|\beta_{\Omega\cap A}|\Big|_{\partial^{**}_{-} A} \ge \hat{\hat{\beta}}_*, \quad (1.42.1)$$

and we set

$$\hat{\hat{\beta}} = \min(\hat{\hat{\beta}}_*, \tilde{\tilde{\beta}}/2). \quad (1.42.2)$$

Inserting (1.36), (1.37), (1.38) and (1.41) into (1.27) and using Hölder's inequality, we obtain the inequality

$$\begin{aligned} &(1 + (\hat{\hat{\beta}})^{-1}) \cdot \int\limits_{\Omega\cap A} |Du|\, dx + 2L_0(\tilde{\beta}, \tilde{\tilde{\beta}}) \cdot \operatorname{diam}(\partial\Omega \cap \overline{A}) \cdot |\partial(\Omega \cap A)| \\ &\ge \frac{n}{\omega_n}\left(1 - (\hat{\hat{\beta}})^{-1} \cdot \frac{\omega_n}{n} \cdot H_* \cdot |\Omega \cap A|^{1/n}\right) \cdot \|\sup_{\Omega\cap A} u - u(x)\|_{L^{n*}(\Omega\cap A)} \end{aligned} \quad (1.43)$$

if (1.42.1) holds and $\hat{\hat{\beta}}$ is given by (1.42.2).

4.1.3. Analogously, we can use (1.17) and Theorem 1 to establish

$$(1+(\hat{\hat{\beta}})^{-1})\cdot\int\limits_{\Omega\cap A}|Du|\,dx+2L_0((1+\tilde{\beta})/2,\tilde{\tilde{\beta}}/2)\cdot\operatorname{diam}(\partial^*A)\cdot|\partial(\Omega\cap A)|$$
$$\geq\frac{n}{\omega_n}\cdot\left(1-(\hat{\hat{\beta}})^{-1}\cdot\frac{\omega_n}{n}\cdot H_*\cdot|\Omega\cap A|^{1/n}\right)\cdot\|u(x)-\inf_{\Omega\cap A}u\|_{L^{n*}(\Omega\cap A},\qquad(1.44)$$

where the constant $L_0((1+\tilde{\beta})/2,\tilde{\tilde{\beta}}/2)$ is given in Theorem 1.

4.1.4. Inserting (1.35.1), (1.35.2), (1.43) and (1.44) into (1.26) and applying Hölder's inequality, we obtain

$$\int\limits_{\Omega\cap A}\big(\sup_{\Omega\cap A}u-u(x)\big)\,dx+\int\limits_{\Omega\cap A}\big(u(x)-\inf_{\Omega\cap A}u\big)\,dx$$
$$\leq\big[\hat{C}_2+(1+(\hat{\hat{\beta}})^{-1})\cdot\hat{C}_3\big]\cdot|\Omega\cap A|^{1+1/n}+\big[\hat{C}_2+(\hat{\hat{\beta}})^{-1}\cdot\hat{C}_3\big]$$
$$\times|\Omega\cap A|^{1/n}\cdot\int\limits_{\Omega\cap A}\big(u(x)-\inf_{\Omega\cap A}u\big)\,dx$$
$$+\big[\hat{C}_2+(\hat{\hat{\beta}})^{-1}\cdot\hat{C}_3\big]\cdot|\Omega\cap A|^{1/n}\cdot\int\limits_{\Omega\cap A}\big(\sup_{\Omega\cap A}u-u(x)\big)\,dx$$
$$+\hat{C}_3\cdot L_0(\tilde{\beta},\tilde{\tilde{\beta}})\cdot\operatorname{diam}(\partial\Omega\cap\overline{A})\cdot|\partial(\Omega\cap A)|\cdot|\Omega\cap A|^{1/n},$$
$$+\hat{C}_3\cdot L_0((1+\tilde{\beta})/2,\tilde{\tilde{\beta}}/2)\cdot\operatorname{diam}(\partial^*A)\cdot|\partial(\Omega\cap A)|\cdot|\Omega\cap A|^{1/n},$$

where

$$\hat{C}_2=\frac{2\omega_n}{n}[1+3(\hat{\beta})^{-1}+(\hat{\hat{\beta}})^{-1}]\cdot[n/\omega_n-\hat{C}_1\cdot|\Omega\cap A_0|^{1/n}]^{-1}$$
$$+\frac{\omega_n}{n}\big(\sup_{\Omega\cap A}u-\inf_{\Omega\cap A}u\big)\cdot H_*,\qquad(1.45)$$
$$\hat{C}_3=\frac{\omega_n}{n}\cdot\left(1-(\hat{\hat{\beta}})^{-1}\cdot(\omega_n/n)\cdot H_*\cdot|\Omega\cap A|^{1/n}\right)^{-1}.\qquad(1.46)$$

Hence

$$\int\limits_{\Omega\cap A}\big(\sup_{\Omega\cap A}u-u(x)\big)\,dx+\int\limits_{\Omega\cap A}\big(u(x)-\inf_{\Omega\cap A}u\big)\,dx$$
$$\leq\big[\hat{C}_2+(\hat{\hat{\beta}})^{-1}\cdot\hat{C}_3\big]\cdot\hat{C}_4\cdot|\Omega\cap A|^{1+1/n}$$
$$+\hat{C}_3\cdot\hat{C}_4\cdot L_0(\tilde{\beta},\tilde{\tilde{\beta}})\cdot\operatorname{diam}(\partial\Omega\cap\overline{A})\cdot|\partial(\Omega\cap A)|\cdot|\Omega\cap A|^{1/n}$$
$$+\hat{C}_3\cdot\hat{C}_4\cdot L_0((1+\tilde{\beta})/2,\tilde{\tilde{\beta}}/2)\cdot\operatorname{diam}(\partial^*A)\cdot|\partial(\Omega\cap A)|\cdot|\Omega\cap A|^{1/n},$$

where

$$\hat{C}_4=\left(1-\big[\hat{C}_2+(\hat{\hat{\beta}})^{-1}\cdot\hat{C}_3\big]\cdot H_*\cdot|\Omega\cap A|^{1/n}\right)^{-1},\qquad(1.47)$$

This yields

$$\int\limits_{\Omega\cap A} \big(\sup_{\Omega\cap A} u - u(x)\big)\, dx \le [\hat{C}_2 + (\hat{\hat{\beta}})^{-1} \cdot \hat{C}_3] \cdot \hat{C}_4 \cdot |\Omega \cap A|^{1+1/n}$$

$$+ \hat{C}_3 \cdot \hat{C}_4 \cdot L_0(\tilde{\beta}, \tilde{\tilde{\beta}}) \cdot \operatorname{diam}(\partial\Omega \cap \overline{A}) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n}$$

$$+ \hat{C}_3 \cdot \hat{C}_4 \cdot L_0((1+\tilde{\beta})/2, \tilde{\tilde{\beta}}/2) \cdot \operatorname{diam}(\partial^* A) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n}, \quad (1.48)$$

and

$$\int\limits_{\Omega\cap A} \big(u(x) - \inf_{\Omega\cap A} u\big)\, dx \le \big[\hat{C}_2 + (\hat{\hat{\beta}})^{-1} \cdot \hat{C}_3\big] \cdot \hat{C}_4 \cdot |\Omega \cap A|^{1+1/n}$$

$$+ \hat{C}_3 \cdot \hat{C}_4 \cdot L_0(\tilde{\beta}, \tilde{\tilde{\beta}}) \cdot \operatorname{diam}(\partial\Omega \cap \overline{A}) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n}$$

$$+ \hat{C}_3 \cdot \hat{C}_4 \cdot L_0((1+\tilde{\beta})/2, \tilde{\tilde{\beta}}/2) \cdot \operatorname{diam}(\partial^* A) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n}, \quad (1.49)$$

By (1.9), (1.11) and (1.12), we have

$$|\Omega \cap A_0| \le (\delta_0)^{(n-1)\cdot(1+\varepsilon_0)+1} \qquad (1.50)$$

which yields

$$|\Omega \cap A_0|^{\frac{1}{n}} \le (\delta_0)^{(1-\frac{1}{n})\cdot(1+\varepsilon_0)+\frac{1}{n}} = (\delta_0)^{1+\varepsilon_0(1-\frac{1}{n})}. \qquad (1.51)$$

In view of (1.34), (1.45) and (1.47), we see that

$$\hat{C}_1 \cdot |\Omega \cap A_0|^{1/n} \le \mathfrak{C}_1 \cdot (\delta_0)^{\varepsilon_0(1-\frac{1}{n})} \le \frac{1}{2} \quad \text{for } \delta_0 \text{ sufficiently small,} \qquad (1.52)$$

where $\mathfrak{C}_1$ is a constant depending only on H_*, $\hat{\beta}$ and $\mathcal{K}_{\partial\Omega}$.

If (1.18) holds for each sequence of points x_k approaching x_0, then we obtain from (1.19) and (1.50)

$$|\Omega \cap A| \le \tilde{\mathfrak{C}}_0 (\delta_0)^n, \qquad (1.53)$$

where $\tilde{\mathfrak{C}}_0$ is determined by $\tilde{\beta}$ and n. Hence, in view of (1.45),

$$\hat{C}_2 \cdot |\Omega \cap A|^{\frac{1}{n}} \le \mathfrak{C}_2 \cdot (\delta_0) \le 1/[2H_*(\sup_{\Omega\cap A} u - \inf_{\Omega\cap A} u)] \qquad (1.54)$$

for δ_0 sufficiently small, and, by (1.47),

$$\hat{C}_3 \le 2, \quad \hat{C}_4 \le 2, \quad \text{for } \delta_0 \text{ sufficiently small,} \qquad (1.55)$$

where $\mathfrak{C}_2$ is a constant depending only on H_*, $\tilde{\beta}$, $\hat{\beta}$ and n.

We remark here that (1.49) will not be used in the next sections.

4.2. L^1-norm of $\big(\sup_{\Omega\cap A} u - u(x)\big)$ and $\big(u(x) - \inf_{\Omega\cap A} u\big)$ in $\Omega \cap A$. The Case where $\beta(x) < 0$ for all $x \in \partial\Omega \cap \overline{A}$. Assume (1.16) holds. The case where $\beta(x) < 0$ can be treated in a similar way as the case $\beta(x) \ge 0$: repeating the corresponding reasoning, in case (1.13) holds, we obtain (1.35.1) and (1.35.2), in case (1.14) holds, we obtain (1.43) and (1.44). Inserting (1.35.1), (1.35.2), (1.44) and (1.45) into (1.26), we obtain (1.48) and (1.49).

We also remark here that in this setting, (1.48) will not used in the nect sections.

5. Oscillation of u near the boundary.

5.1. The case where $1 > \beta(x_0) > 0$. Consider a point $x_0 \in \partial\Omega$ such that for δ_0 sufficiently small such there holds

$$1 > \tilde{\beta} > \beta(x) > \tilde{\tilde{\beta}} > 0, \quad \text{for} \quad x \in \partial\Omega \cap B_{\delta_0}(x_0). \tag{1.56}$$

We assume, without loss of generality, δ_0 to be sufficiently small that $\delta_0 \leq R_-$, R_- being given in Proposition 1.

Choose a boundary strip A adjacent to $\partial\Omega \cap B_{\tilde{\delta}_0}(x_0)$, $\tilde{\delta}_0 = (\delta_0)^{1+\varepsilon_0}$, to be with of width δ_0 and of the type indicated in the beginning of Subsection 3.1 such that (1.17) hold. We obtain from (1.56) that (1.13) or (1.14) hold with

$$\hat{\beta} = \frac{\tau}{\sqrt{1+\tau^2}} \cdot \sqrt{1 - [(1+\tilde{\beta})/2)]^2}, \tag{1.57}$$

where

$$\tau = \sqrt{(\tilde{\tilde{\beta}}/2)^2 \Big/ (1 - (\tilde{\tilde{\beta}}/2)^2)} \tag{1.58}$$

and (1.42.1) holds with

$$\hat{\hat{\beta}}_* = \tilde{\tilde{\beta}}/2, \tag{1.59}$$

for δ_0 sufficiently small. We shall establish this for sufficiently small δ_0 in Appendix 4.

Let us set

$$R = (\delta_0/4)^{1+\varepsilon_0}, \tag{1.60}$$

and choose a boundary strip $A_R(x_0)$ adjacent to $\partial\Omega \cap B_R(x_0)$ to be of width R and of the type indicated above in Theorem 2. From inserting (1.46) into (1.2) with the value $\hat{\beta}$ given in (1.57) and setting $\hat{\gamma} = 0$ in (1.5), we obtain

$$\begin{aligned}
u(x_0) - \inf_{A_R(x_0)} u \leq \frac{2^{n+2}}{\alpha_* R^n} \cdot \Big\{ & \big[\hat{C}_2 + (\hat{\hat{\beta}})^{-1} \cdot \hat{C}_3\big] \cdot \hat{C}_4 \cdot |\Omega \cap A|^{1+\frac{1}{n}} \\
& + \hat{C}_3 \cdot \hat{C}_4 \cdot L_0(\tilde{\beta}, \tilde{\tilde{\beta}}) \cdot \operatorname{diam}(\partial\Omega \cap \overline{A}) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n} \\
& + \hat{C}_3 \cdot \hat{C}_4 \cdot L_0((1+\tilde{\beta})/2, \tilde{\tilde{\beta}}/2) \cdot \operatorname{diam}(\partial^* A) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n} \Big\} + 2R \\
\leq 2^{n+6} \cdot (n+1) \cdot k_{(n+1)} \cdot \Big\{ & \big[\hat{C}_2 + (\hat{\hat{\beta}})^{-1} \cdot \hat{C}_3\big] \cdot \hat{C}_3 \cdot \frac{|\Omega \cap A|^{1+\frac{1}{n}}}{R^n} \\
& + \hat{C}_3 \cdot \hat{C}_4 \cdot L_0(\tilde{\beta}, \tilde{\tilde{\beta}}) \cdot \operatorname{diam}(\partial\Omega \cap \overline{A}) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n} \Big/ R^n \\
& + \hat{C}_3 \cdot \hat{C}_4 \cdot L_0((1+\tilde{\beta})/2, \tilde{\tilde{\beta}}/2) \cdot \operatorname{diam}(\partial^* A) \cdot |\partial(\Omega \cap A)| \cdot |\Omega \cap A|^{1/n} \Big/ R^n \Big\} \\
& + 2R.
\end{aligned} \tag{1.61}$$

If (1.18) holds for *each* sequence of points x_k approaching x_0, then we obtain from (1.20)

$$\operatorname{diam}(\partial\Omega\cap\overline{A})\cdot|\partial(\Omega\cap A)|\cdot|\Omega\cap A|^{1/n}\le\tilde{\mathfrak{C}}_0^*\cdot(\delta_0)^{1+\frac{1}{n}},$$
$$\operatorname{diam}(\partial^*A)\cdot|\partial(\Omega\cap A)|\cdot|\Omega\cap A|^{1/n}\le\tilde{\mathfrak{C}}_0^*\cdot(\delta_0)^{1+\frac{1}{n}},$$

where $\tilde{\mathfrak{C}}_0^*$ is determined by $\tilde{\beta}$ and n. Inserting these, together with (1.17), (1.42.1), (1.53) into (1.61) and then using (1.45), (1.54), (1.55), (1.17), (1.42.1), (1.42.2), (1.57), (1.58), (1.59), we arrive at (0.13) with $\hat{\mathfrak{C}}$ determined by $\tilde{\beta}$, $\tilde{\tilde{\beta}}$, H_* and $\mathcal{K}_{\partial\Omega\cap\overline{A_{\delta_0}(x_0)}}$.

5.2. The case where $-1<\beta(x_0)<0$. Consider a point $x_0\in\partial\Omega$ such that for δ_0 sufficiently small there holds

$$-1<-\tilde{\beta}<\beta(x)<-\tilde{\tilde{\beta}}<0,\quad\text{for }\ x\in\partial\Omega\cap B_{\delta_0}(x_0).\tag{1.62}$$

Choose δ_0 to be sufficiently small such that $\delta_0\le R_+$, R_+ being given in Proposition 1. Choose a boundary strip A_0 adjacent to $\partial\Omega\cap B_{\tilde{\delta}_0}(x_0)$, $\tilde{\delta}_0=(\delta_0)^{1+\varepsilon_0}$, to be of width δ_0 and of the type as before. For τ given in (1.58), we obtain from (1.62) that (1.13) or (1.14) holds with $\hat{\hat{\beta}}$ given in (1.57), and (1.42.1) holds with $\hat{\hat{\beta}}$ given in (1.59), which will be established for sufficiently small δ_0 in Appendix 4.

From inserting (1.49) into (1.4) with the value of $\hat{\hat{\beta}}$ given in (1.57) and setting $\hat{\gamma}=0$ in (1.5), we obtain, analogously to **5.1**, the estimate (0.14) with the same $\hat{\mathfrak{C}}$ as in (0.13).

Appendix 1. Boundary Integrals along a Piecewise C^2 Boundary

The proof of the following can be modified from that of [6, Lemma 1.1] in an obvious way.

Lemma A.1. *Let E be a Caccioppoli set in $\mathbb{R}^n$ and Γ be a subset of ∂E which is a C^2 manifold and $d(x)=\operatorname{dist}(x,\partial E)$ for $x\in E$. Let*

$$E_{\Gamma,t}=\{x:x\in E:\ \operatorname{dist}(x,\Gamma)\le t\},\quad\text{for }\ t>0.\tag{A.1}$$

Let ε_Γ be so small that the function $d(x)$ be of the class C^2 in $E_{\Gamma,\varepsilon_\Gamma}$, and consider, for $0<\varepsilon'<\varepsilon_\Gamma$, a domain $E^*_{\Gamma,\varepsilon_\Gamma}$,

$$E_{\Gamma,\varepsilon'}\subseteq E^*_{\Gamma,\varepsilon_\Gamma}\subseteq E_{\Gamma,\varepsilon_\Gamma},$$

such that a portion of its boundary $\partial^*E^*_{\Gamma,\varepsilon_\Gamma}\subset E_{\Gamma,t}\setminus E_{\Gamma,\varepsilon'}$, and on the remaining portion of its boundary in Ω, we have

$$Dd\cdot\nu\Big|_{(\partial E^*_{\Gamma,\varepsilon_\Gamma}\cap\Omega)\setminus\partial^*E^*_{\Gamma,\varepsilon_\Gamma}}=0,$$

ν being the unit outward normal to $\partial E^*_{\Gamma,\varepsilon_\Gamma}$. Then, there exists a constant $C_{\Gamma,\varepsilon'}$ depending only on Γ and ε' such that the inequality

$$\int\limits_{\Gamma} w\, d\mathcal{H}_{n-1} \le \int\limits_{E^*_{\Gamma,\varepsilon_\Gamma}} |Dw|\, dx + C_{\Gamma,\varepsilon'} \cdot \int\limits_{E^*_{\Gamma,\varepsilon_\Gamma}} |w|\, dx, \tag{A.2}$$

holds for all $w \in BV(E_{\Gamma,\varepsilon_\Gamma})$. In fact, let $\eta_{\varepsilon'}$ be a C^∞ function with

$$\begin{cases} o \le \eta_{\varepsilon'} \le 1, \\ \eta_{\varepsilon'} = 1 \quad \text{on } \Gamma, \\ \eta_{\varepsilon'} = 0 \quad \text{in } E \setminus E_{\Gamma,\varepsilon'}, \end{cases} \tag{A.3}$$

then we can take

$$C_{\Gamma,\varepsilon'} = \sup_{E^*_{\Gamma,\varepsilon_\Gamma}} |\mathrm{div}(\eta_{\varepsilon'} Dd)|. \tag{A.4}$$

In order to apply Lemma A.1, we have to estimate the value of $C_{\Gamma,\varepsilon'}$ in (A.4). For this, we formulate the following result which is well known and can be found, e.g., in [5, pp. 420–422].

Lemma A.2. *Let $\Gamma \subseteq \partial E$ be of the class C^2 whose principal curvatures are bounded in the absolute value by $\mathcal{K}_\Gamma$. Then $d(x) = \mathrm{dist}(x,\Gamma)$ is of the class C^2 in $E_{\Gamma,\varepsilon_\Gamma}$, for $\varepsilon_\Gamma \le \frac{1}{\mathcal{K}_\Gamma}$, where $E_{\Gamma,\varepsilon_\Gamma}$ is given in* (A.1).

Furthermore, for points $\bar{x}$ in $E_{\Gamma,\varepsilon_\Gamma}$, $\varepsilon_\Gamma \le \frac{1}{\mathcal{K}_\Gamma}$, define $\bar{y} = \bar{y}(\bar{x})$ to be the (*unique*) *point of Γ nearest to $\bar{x}$. Consider the special coordinate frame in which the x_n-axis is oriented along the inward normal to Γ at $\bar{y}$ and the coordinates $x_1, \cdots, x_{n-1}$ lie along the principal directions of Γ at the point $\bar{y}$. In this special coordinates, we have at $\bar{x}$,*

$$Dd = (0, \cdots, 0, 1) \tag{A.5}$$

and

$$D^2 d = \text{diagonal}\left[\frac{-k_1}{1-k_1 d}, \cdots, \frac{-k_{n-1}}{1-k_{n-1} d}, 0\right] \tag{A.6}$$

where $k_1, \cdots, k_{n-1}$ are the principal curvatures of Γ at $\bar{y}$.

Inserting (A.5) and (A.6) into (A.3) and (A.4), we obtain the following.

Lemma A.3. *Let $\Gamma \subseteq \partial E$ be of the class C^2 whose principal curvatures are bounded in the absolute value by $\mathcal{K}_\Gamma$. Then, for $\varepsilon_\Gamma \le \frac{1}{\mathcal{K}_\Gamma}$ and for each δ, $0 < \delta \le 1$, we can take in* (A.4)

$$C_{\Gamma,\varepsilon'} \le |D\eta_{\varepsilon'}| + 2(n-1)\mathcal{K}_\Gamma \le \left(\frac{1+\delta}{\varepsilon'}\right) + 2(n-1)\mathcal{K}_\Gamma. \tag{A.7}$$

Appendix 2. Modified Sobolev Inequality

The following result is a special case of the so-called Friedrichs inequality and can be found, e.g., in [9, Theorem 6.5.7].

Proposition A.1. *Suppose E is a Caccioppoli set with piecewise Lipschitz continuous boundary. Then, for any $f \in BV(\Omega)$, the inequality*

$$\|f\|_{L^{n*}(E)} \le \frac{n}{\omega_n}\Big(\int_E |Df|\,dx + \int_{\partial E} |f|\,d\mathcal{H}_{n-1}\Big) \tag{A.8}$$

is valid, where ω_n is the Lebesgue measure of the n-dimensional unit ball.

Appendix 3. A Proof of (0.16)

Since u is assumed to be bounded up to the boundary, [8, Theorem 1] implies $u \in H^{1,1}(\Omega)$. By this and the fact that the restriction $u\Big|_{\Omega\setminus\Omega_\varepsilon}$ is a minimizing function of the functional

$$J(v) = \int_\Omega \sqrt{1+|Du|^2}\,dx + \int_{\partial\Omega_\varepsilon\cap\Omega} (Tu\cdot\nu_{\Omega_\varepsilon})u\,d\mathcal{H}_{n-1},$$

with ν_{Ω_ε} being the unit outward normal to Ω_ε, we are allowed to set $\eta = 1$ in the identities

$$\int_\Omega \frac{Du}{\sqrt{1+|Du|^2}}\cdot D\eta\,dx + \int_\Omega H\cdot\eta\,dx = \int_{\partial\Omega} \beta\cdot\eta\,d\mathcal{H}_{n-1} \tag{A.9}$$

and

$$\int_{\Omega\setminus\Omega_\varepsilon} \frac{Du}{\sqrt{1+|Du|^2}}\cdot D\eta\,dx + \int_{\Omega\setminus\Omega_\varepsilon} H\cdot\eta\,dx = \int_{\partial\Omega_\varepsilon\cap\Omega} (Tu\cdot\nu_{\Omega_\varepsilon})\cdot\eta\,d\mathcal{H}_{n-1}; \tag{A.10}$$

(see [5, (7.6)]); here $\Omega_\varepsilon = \{x : x\in\Omega,\ \mathrm{dist}(x,\partial\Omega)\le\varepsilon\}$ and ε is sufficiently small. Subtracting (A.10) (with $\eta = u\Big|_{\Omega\setminus\Omega_\varepsilon}$) from (A.9) (with $\eta = u$), we obtain

$$\int_{\Omega_\varepsilon} H\,dx = \int_{\partial\Omega} \beta\cdot u\,d\mathcal{H}_{n-1} - \int_{\partial\Omega_\varepsilon} (Tu\cdot\nu_{\Omega\setminus\Omega_\varepsilon})\,d\mathcal{H}_{n-1}.$$

The left-hand side of the last identity approaches zero as $\varepsilon\to 0$, hence the same is true for the right-hand side of the last identity. Hence, a subsequence can be extracted from the sequence $\{Tu\cdot\nu_{\Omega\setminus\Omega_\varepsilon}\}(x+Dd)$, with $d(x) = \mathrm{dist}(x,\partial\Omega)$, which approaches $\beta(x)$ for almost every $x\in\partial\Omega$ as $\varepsilon\to 0$. This, together with the interior regularity of u, yields under the assumption (0.12) that if δ_0 is sufficiently small, then

$$\frac{1+\tilde\beta}{2} \ge |Tu\cdot\nu_{\Omega_{\delta_0}}(x)| \ge \frac{\tilde{\tilde\beta}}{2}, \tag{A.11}$$

for $x\in\partial^*\Omega_\varepsilon$, $\varepsilon\le\delta_0$, which is sufficiently close to $A_{\delta_0}(x_0)$. In particular, (0.16) is proved.

Appendix 4. Proof of (1.13), (1.14) with the Value of $\hat{\beta}$ Given by (1.57). A Proof of (1.59).

We shall prove in Appendix 5 that points of $\partial^* A$ are close to $\partial^* A_{\delta_0}(x_0)$ for small δ_0. The second inequality in (A.11) thus yields

$$\frac{|Du|}{\sqrt{1+|Du|^2}}\Big|_{\partial^* A} \geq \frac{\tilde{\tilde{\beta}}}{2},$$

from which we obtain

$$|Du|\Big|_{\partial^* A} \geq \tau,$$

with τ given in (1.58). This and the first inequality in (A.11) yield (1.57).

Since $E_{\pm\pm}$ are level sets of u, the normal of $E_{\pm\pm}$ lies toward or opposite to the direction of that of $\frac{Du}{\|Du\|}$, and thus

$$|\beta_{\Omega\cap A}|\Big|_{E_{\pm\pm}} = \frac{Du}{\sqrt{1+|Du|^2}}\cdot\frac{Du}{|Du|} = \frac{|Du|}{\sqrt{1+|Du|^2}},$$

from which by (1.61) it follows the second inequality in (A.11).

Appendix 5. Proof of Proposition 2

To prove Proposition 2, we assume without loss of generality that (1.15) holds. To show the inequality (1.19) for E_{+-}, we observe that the unit normal of E_{+-} is $\pm\frac{Du}{|Du|}$ and thus it suffices to show that for points x in a sufficiently small neighborhood of x_0

$$\Big|\frac{Du}{|Du|}(x)\cdot\nu_\Omega(x_0)\Big| \leq \frac{(1+\tilde{\beta})/2}{\sqrt{1+[(1+\tilde{\beta})/2]^2}},$$

if (1.18) holds for each sequence of points x_k approaching x_0. This follows from the second inequality in (A.11) and the inequality

$$\frac{\sqrt{1+|Du|^2}}{|Du|}(x) = \sqrt{1+\frac{1}{|Du|^2}} \leq \sqrt{1+[(1+\tilde{\beta})/2]^2}$$

for points x in a sufficiently small neighborhood of x_0, which is obtained from the assumption that (1.18) holds for each sequence of points x_k approaching x_0.

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