**Georgian Mathematical Journal**
Volume 12 (2005), Number 2, 321–330

# ARCWISE CONNECTED CONTINUA AND WHITNEY MAPS

IVAN LONČAR

**Abstract.** Let $X$ be a non-metric continuum, and $C(X)$ be the hyperspace of subcontinua of $X$. It is known that there is no Whitney map on the hyperspace $2^X$ for non-metric Hausdorff compact spaces $X$. On the other hand, there exist non-metric continua which admit and ones which do not admit a Whitney map for $C(X)$. In particular, a locally connected or a rim-metrizable continuum $X$ admits a Whitney map for $C(X)$ if and only if it is metrizable. In this paper we investigate the properties of continua $X$ which admit a Whitney map for $C(X)$ or for $C^2(X)$.

**2000 Mathematics Subject Classification**: Primary 54F15, 54B20; Secondary 54B35.
**Key words and phrases:** Arcwise connected continuum, hyperspace, inverse system, property of Kelley, smoothness, Whitney map.

## 1. Introduction

Introduction contains some basic definitions, results and notation.

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space $X$ is denoted by $w(X)$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space $X$ is *arcwise connected* provided that for every two distinct points $x, y \in X$ there exists a generalized arc $xy$ with end points $x$ and $y$.

For a compact space $X$ we denote by $2^X$ the hyperspace of all nonempty closed subsets of $X$ equipped with the Vietoris topology. $C(X)$ and $X(n)$, where $n$ is a positive integer, stand for the sets of all connected members of $2^X$ and of all nonempty subsets consisting of at most $n$ points, respectively, both considered as subspaces of $2^X$. The hyperspace $C(C(X))$ is denoted by $C^2(X)$.

For a mapping $f : X \to Y$ define $2^f : 2^X \to 2^Y$ by $2^f(K) = f(K)$ for $K \in 2^X$. By [17, 5.10] $2^f$ is continuous, $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

We will use the notion of inverse system as in [7, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$. If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then an element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \le b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the *limit of an inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [7, p. 135].

ISSN 1072-947X / \$8.00 / 

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim \mathbf{X} \to X_a$ for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}, C^2(\mathbf{X}) = \{C^2(X_a), C^2(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}| X_b(n), A\}$ form inverse systems.

**Lemma 1.1** ([10, Lemma 2]). *Let $X = \lim \mathbf{X}$. Then $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X}), C^2(X) = \lim C^2(\mathbf{X})$ and $X(n) = \lim \mathbf{X}(n)$.*

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *$\sigma$-directed* if for each sequence $a_1, a_2, \ldots, a_k, \ldots$ of the members of $A$ there is $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

In the next we will use the following expanding theorem of non-metric compact spaces into a $\sigma$-directed inverse system of compact metric spaces.

**Theorem 1.2.** *Let $X$ be a compact Hausdorff space such that $w(X) \geq \aleph_1$. Then there exists a $\sigma$-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces $X_a$ and surjective bonding mappings $p_{ab}$ such that $X$ is homeomorphic to $\lim \mathbf{X}$. Moreover, if $X$ is a Hausdorff continuum, then each coordinate space $X_a$ can be chosen as a metric continuum.*

*Proof.* In [13, Theorem 1.8] it is proved that for a compact Hausdorff space with $w(X) \geq \aleph_1$ there exists a $\sigma$-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces $X_a$ such that $X$ is homeomorphic to $\lim \mathbf{X}$. From the proof of [13, p. 397, Theorem 1.8] it follows that the bonding mappings $p_{ab}$ are surjective. Now, if $X$ is a Hausdorff continuum, then it is clear that each $X_a = p_a(X)$ is a metric continuum. □

The following result [23, p. 173, Problem 23C] will be used.

**Theorem 1.3.** *The following are all equivalent, for a locally compact metric space $X$ :*

a) *$X$ is separable,*

b) *$X = \bigcup_{n=1}^{\infty} K_n$, where $K_n$ is compact and $K_n \subset Int K_{n+1}$ for each $n \in \mathbb{N}$,*

c) *The one point compactification $X^*$* [23, p. 136] *of $X$ is metrizable.*

A function $F : X \to 2^Y$ is *upper semi-continuous at a point $p \in X$* provided that for every open set $V \subset Y$ such that $F(p) \subset V$ there is an open set $U \subset X$ such that $p \in U$ and satisfying $F(x) \subset V$ for all $x \in U$. The function $F$ is said to be *upper semi-continuous* if it is upper semi-continuous at each of its points.

We say that a function $F : X \to 2^Y$ is *lower semi-continuous at a point $p \in X$* provided that for every open set $G \subset Y$ such that $F(p) \cap G \neq \varnothing$ there exists an open set $U \subset X$ such that $p \in U$ and $F(x) \cap G \neq \varnothing$ for every $x \in U$. The function $F$ is said to be *lower semi-continuous* if it is lower semi-continuous at each of its points.

## 2. A Whitney Map and Hereditarily Irreducible Mappings

The notion of an irreducible mapping was introduced by Whyburn [22, p. 162]. If $X$ is a continuum, a surjection $f : X \to Y$ is *irreducible* provided that no proper subcontinuum of $X$ maps onto all of $Y$ under $f$.

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [18, p. 204, (1.212.3)] provided that for any given subcontinuum $Z$ of $X$, no proper subcontinuum of $Z$ maps onto $f(Z)$.

A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [7, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subset of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

Every hereditarily irreducible mapping is light. If $f : X \to Y$ is monotone and hereditarily irreducible, then $f$ is one-to-one.

Let $\Lambda$ be a subspace of $2^X$. By a *Whitney map* for $\Lambda$ [18, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $g(A) < g(B)$, and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If $X$ is a metric continuum, then there exists a Whitney map for $2^X$ and $C(X)$ [18, pp. 24-26], [9, p. 106]. If $X$ is a metric continuum, then so is $C(X)$. Hence, there exists a Whitney map for $C^2(X) = C(C(X))$. On the other hand, if $X$ is non-metric, then it admits no Whitney map for $2^X$ [2, p. 305]. It is known that there exist non-metric continua which admit and ones which do not admit a Whitney map for $C(X)$ [2, p. 307]. Moreover, if $X$ is a non-metric locally connected or a rim-metrizable continuum, then $X$ admits no Whitney map for $C(X)$ [12, Theorem 8 and 11].

The following external characterization of non-metric continua which admit a Whitney map is proved in [13, p. 399, Theorem 2.3] for continua, but the proof given in [13, p. 399, Theorem 2.3] can be applied without essential changes to compact spaces.

**Theorem 2.1.** *Let $X$ be a compact space. Then $X$ admits a Whitney map for $C(X)$ if and only if for each $\sigma$-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact spaces $X_a$ which admit Whitney maps for $C(X_a)$ and $X = \lim \mathbf{X}$, there exists a subset $B$ of $A$ cofinal in $A$ and such that, for each $b \in B$, the projection $p_b : X \to X_b$ is hereditarily irreducible. Moreover, for the necessity the condition that each space $X_a$ admits a Whitney map for $C(X_a)$ is not required.*

We say that a continuum $X$ admits a Whitney map for $C^2(X)$ if there is a mapping $h : C^2(X) \to [0, +\infty)$ such that

c) if $\mathcal{A}$,$\mathcal{B} \in C^2(X)$ are such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then $h(\mathcal{A}) < h(\mathcal{B})$ and

d) $h(\mathcal{D}) = 0$ for each $\mathcal{D} \in (X(1))(1)$.

**Theorem 2.2.** *If a continuum $X$ admits a Whitney map for $C^2(X)$, then $X$ admits a Whitney map for $C(X)$.*

*Proof.* Since $X$ admits a Whitney map for $C^2(X)$ there is a mapping $h : C^2(X) \to [0, +\infty)$ such that: c) if $\mathcal{A}$,$\mathcal{B} \in C^2(X)$ are such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then $h(\mathcal{A}) < h(\mathcal{B})$ and d) $h(\{\{x\}\}) = 0$ for each $x \in X$. Given $A \in C(X)$ it follows that $A(1) \in C^2(X)$. Thus we can consider the function

$\mu : C(X) \to [0, +\infty)$ defined by $\mu(A) = h(A(1))$, for any $A \in C(X)$. Let $A, B \in C(X)$ be such that $A \subset B$ and $A \neq B$. Then $A(1), B(1) \in C^2(X)$ are such that $A(1) \subset B(1)$ and $A(1) \neq B(1)$, By c) we have $h(A(1)) < h(B(1))$. This implies that $\mu(A) < \mu(B)$. Now, let $x \in X$. By d) $\mu(\{x\}) = h(\{x\}(1)) = h(\{\{x\}\}) = 0$. Using the Vietoris topology on both $C(X)$ and $C^2(X)$ it can be shown that $\mu$ is continuous. Thus, $X$ admits a Whitney map for $C(X)$. □

We say that a continuum $C(X)$ admits a Whitney map for $C^2(X)$ if there is a mapping $f : C^2(X) \to [0, +\infty)$ such that

e) if $\mathcal{A}$,$\mathcal{B} \in C^2(X)$ are such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then $f(\mathcal{A}) < f(\mathcal{B})$ and

d) $f(\mathcal{D}) = 0$ for each $\mathcal{D} \in C(X)(1)$.

**Theorem 2.3.** *Let $X$ be a continuum. If $C(X)$ admits a Whitney map for $C^2(X)$, then $X$ admits a Whitney for $C(X)$.*

*Proof.* Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a $\sigma$-directed inverse system such that each coordinate space $X_a$ is a continuum which admits a Whitney map for $C(X_a)$ and $X = \lim \mathbf{X}$. Then $C(\mathbf{X})$=$\{C(X_a),\ C(p_{ab}),\ A\}$ is a $\sigma$-directed inverse system such that $C(X)$ is homeomorphic to $\lim C(\mathbf{X})$ (Lemma 1.1). Since $C(X)$ admits a Whitney map for $C^2(X)$, we can apply the necessity of Theorem 2.1 to $C(X)$ in place of $X$ to conclude that there is a subset $B$ of $A$ cofinal in $A$ such that the projections $C(p_b) : C(X) \to C(X_b)$ are hereditarily irreducible. Now, the restriction $C(p_b)|X(1)$ is again hereditarily irreducible since $X(1)$ is a subcontinuum of $C(X)$. Let us observe that $(C(p_b)|X(1))(X(1)) \subset X_b(1)$ and that both $X(1)$ and $X_b(1)$ are homeomorphic to $X$ and $X_b$, respectively. Thus $C(p_b)|X(1) = p_b$, which means that the projections $p_b : X \to X_b$ are hereditarily irreducible, for each $b \in B$. Finally, from Theorem 2.1 it follows that $X$ admits a Whitney map for $C(X)$. □

## 3. The Metrizability of $C(X) \setminus X(1)$ if $X$ is Arcwise Connected

Now we will prove the metrizability of $C(X) \setminus X(1)$ if $X$ is an arcwise connected continuum which admits a Whitney map for $C(X)$.

**Theorem 3.1.** *If an arcwise connected continuum $X$ admits a Whitney map for $C(X)$, then $C(X) \setminus X(1)$ is metrizable and $w(C(X) \setminus X(1)) \leq \aleph_0$.*

*Proof.* Assume that a non-metric arcwise connected continuum $X$ admits a Whitney map for $C(X)$. From Theorem 1.2 it follows that there exists a $\sigma$-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua and surjective bonding mappings such that $X$ is homeomorphic to $\lim \mathbf{X}$. Consider the inverse system $C(\mathbf{X}) = \{C(X_a),\ C(p_{ab}),\ A\}$ whose limit is $C(X)$ (Lemma 1.1). From Theorem 2.1 it follows that there exists a subset $B$ cofinal in $A$ such that the projections $p_b$ are hereditarily irreducible. By [18, p. 204, (1.212.3)] the hereditarily irreducibility of $p_b$ implies that $C(p_b)$ is light for every $b \in B$. Since $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$, we may assume that $B = A$. Let $Y_a = C(p_a)(C(X))$. Furthermore, $C(p_a)^{-1}(X_a(1)) = X(1)$ since from the hereditary irreducibility of $p_a$ it follows that no non-degenerate subcontinuum of $X$ maps under $p_a$ onto a point. We infer that $C(p_a)^{-1}[Y_a \setminus X_a(1)] = C(X) \setminus X(1)$. Let us prove that the restriction $C(p_a)|[C(X) \setminus X(1)]$ is one-to-one. Suppose that $C(p_a)|[C(X) \setminus X\ (1)]$ is not one-to-one. Then there exists a continuum $C_a$ in $X_a$ and two non-degenerate and distinct continua $C, D$ in $X$ such that $p_a(C) = p_a(D) = C_a$. It is impossible that $C \subset D$ or $D \subset C$ since $p_a$ is hereditarily irreducible. Otherwise, if $C \cap D \neq \varnothing$, then for the continuum $Y = C \cup D$ we have that $C$ and $D$ are proper subcontinua of $Y$ and $p_a(Y) = p_a(C) = p_a(D) = C_a$, which is impossible since $p_a$ is hereditarily irreducible. We infer that $C \cap D = \varnothing$. There exists a generalized arc $E$ with end points in $C$ and $D$, respectively. Moreover, since $C$ and $D$ are non-degenerate, we may assume that $E \cap C \neq C$ and $E \cap D \neq D$. Now $p_a(E \cup D) = p_a(E)$, which is impossible since $p_a$ is hereditarily irreducible. It follows that the restriction $P_a = C(p_a)|[C(X) \setminus X(1)]$ is one-to-one and closed [7, p. 95, Proposition 2.1.4]. Hence, $P_a$ is a homeomorphism and $C(X) \setminus X(1)$ is metrizable. Moreover, $w(C(X) \setminus X(1)) \leq \aleph_0$ since $Y_a$ as a compact metrizable space is separable and, consequently, second-countable [7, p. 320]. □

It is known that if $X$ is a continuum, then $C(X)$ is arcwise connected [16, p. 1209, Theorem]. Hence, we have the following corollary.

**Corollary 3.2.** *If $X$ is a continuum which admits a Whitney map for the hyperspace $C^2(X)$, then $C^2(X) \setminus C(X)(1)$ is metrizable and $w(C^2(X) \setminus C(X)(1)) \leq \aleph_0$.*

We close this section with the following result.

**Theorem 3.3.** *If an arcwise connected continuum $X$ admits a Whitney map for the hyperspace $C(X)$, then $C(X) \setminus X(1)$ admits a Whitney map for both $C(C(X) \setminus X(1))$ and $(C(X) \setminus X(1))(1)$.*

*Proof.* By Theorem 3.1 the space $C(X) \setminus X(1)$ is metrizable and $w(C(X) \setminus X(1)) \leq \aleph_0$. This means that $C(X) \setminus X(1)$ is separable. Now we will use the theorem due to T. Watanabe [21, Theorem 1] which states that if $Z$ is a separable metric space, then $Z$ admits a Whitney map $\mu : 2^Z \to \mathbb{R}$. This means that there exists a Whitney map $\upsilon$ for $2^{C(X) \setminus X(1)}$. The restrictions of $\upsilon$ to $C(C(X) \setminus X(1))$ and $(C(X) \setminus X(1))(1)$ are Whitney maps as well. □

## 4. Smoothness and Whitney Maps

There are many definitions of smoothness in the literature. The following concept of smoothness is due to Maćkowiak [15] for metric continua and to Rakowski [19] for Hausdorff continua. We call this concept the *MR-smoothness.*

4.1. **MR-smoothness.** We say that a pointed continuum $(X, p)$ is *MR-smooth* provided that $X$ is smoth at $p$, i.e., for each subcontinuum $L$ of $X$ which contains $p$ and for each open set $V$ which contains $L$ there exists an open connected set $U$ such that $L \subset U \subset V$ [3, p. 103].

For a given pointed continuum $(X, p)$ consider a function $\delta_{(X,p)} : X \to C^2(X)$ defined by

$$\delta_{(X,p)}(x) = \{K \in C(X) : p, x \in K\}.$$

The following two theorems have been proved in the metric case [5, Propositions 1 and 2] respectively, but they remain valid for Hausdorff continua [3, p. 103].

**Theorem 4.1.** *The function $\delta_{(X,p)}$ is upper semi-continuous.*

**Theorem 4.2** ([3, Theorem 8.1])**.** *The function $\delta_{(X,p)}$ is continuous if and only if the pointed continuum $(X, p)$ is MR-smooth.*

**Lemma 4.3.** *If $(X, p)$ is a pointed arcwise connected MR-smooth continuum, then $\delta_{(X,p)} : X \to C^2(X)$ is an embedding and $\delta_{(X,p)}(X) \subset C^2(X) \setminus C(X)(1)$.*

*Proof.* According to Theorem 4.2 the function $\delta_{(X,p)} : X \to C^2(X)$ is continuous since $(X, p)$ is MR-smooth. Moreover, if $x \neq y$, then the generalized arcs $px$ and $py$ are distinct, whence $\{K \in C(X) : p, x \in K\} \neq \{K \in C(X) : p, y \in K\}$. This means then $\delta_{(X,p)}$ is one-to-one. Thus it is an embedding. Let us prove that $\delta_{(X,p)}(X) \subset C^2(X) \setminus C(X)(1)$. If $X$ is not a generalized arc, then $\delta_{(X,p)}(x) = \{K \in C(X) : p, x \in K\}$ is a non-degenerate continuum in $C(X)$ which contains a generalized arc $px$ and $X$. Hence, $\delta_{(X,p)}(x) \in C^2(X) \setminus C(X)(1)$. If $X$ is a generalized arc, then $X$ is MR-smooth at each of its points. We may assume that $p$ is not an end point. This implies that $\delta_{(X,p)}(x)$ is a non-degenerate continuum in $C(X)$ which contains a generalized arc $px$ and $X$. □

**Theorem 4.4.** *If $(X, p)$ is a pointed arcwise connected MR-smooth continuum which admits a Whitney map for $C^2(X)$, then $X$ is metrizable.*

*Proof.* From Corollary 3.2 it follows that $C^2(X) \setminus C(X)(1)$ is metrizable and $w(C^2(X) \setminus C(X)(1)) \leq \aleph_0$ since $C(X)$ is arcwise connected. Using Lemma 4.3 we infer that $\delta_{(X,p)}(X) \subset C^2(X) \setminus C(X)(1)$. Hence $X$ is metrizable and separable. □

**Corollary 4.5.** *An MR-smooth arcwise connected pointed continuum $(X, p)$ admits a Whitney map for $C^2(X)$ if and only if it is metrizable.*

An *arboroid* is a hereditarily unicoherent continuum which is arcwise connected. A metrizable arboroid is a *dendroid*. If $X$ is an arboroid and $x, y \in X$, then there exists a unique generalized arc $xy$ in $X$ with end points $x$ and $y$.

**Corollary 4.6.** *An MR-smooth pointed arboroid $(X, p)$ admits a Whitney map for $C^2(X)$ if and only if it is metrizable.*

*Proof.* Apply Theorem 4.4. □

4.2. **Arc-smoothness.** The notion of arc-smoothness was introduced by Fugate, Gordh and Lum in [8]. We will use the generalization of this notion from [11].

An *arc-structure on a continuum $X$* [11, p. 172] is a function $A : X \times X \to C(X)$ such that for $x \neq y$ in $X$, the set $A(x, y)$ is a generalized arc from $x$ to $y$

and such that the following metric-like conditions are satisfied for all $x, y$ and $z$ in $X$ :

(a) $A(x,x) = \{x\}$,

(b) $A(x,y) = A(y,x)$, and

(c) $A(x,z) \subseteq A(x,y) \cup A(y,z)$ with equality prevailing whenever $y$ belongs to $A(x,z)$.

The pair $(X, A)$ is *arc-smooth at point $p$ in $X$* if the induced function $A_p : X \to C(X)$ defined by $A_p(x) = A(p,x)$ is continuous. The pair $(X, A)$ is *arc-smooth* if there exists a point in $X$ at which $(X, A)$ is arc-smooth.

*Remark.* In [14] the set $A_p(X)$ is denoted by $\mathcal{D}(X,p)$ and it is proved that if $X$ is smooth at $p$, then $\mathcal{D}(X,p)$ is arcwise connected [14, Theorem 4.8]. Moreover, if the continuum $X$ is arcwise connected and smooth at a point $p$, then there exists a homeomorphism $h : X \to \mathcal{D}(X,p)$ [14, Theorem 8.2]. Namely, $h$ is defined by $h(x) = A(p,x)$.

If a continuum $X$ is arc-smooth at the point $p$, then $A_p : X \to C(X)$ is one-to-one. Thus, we have the following lemma.

**Lemma 4.7.** *Let $X$ be a continuum with an arc-structure $A$. If $(X, A)$ is arc-smooth at a point $p \in X$, then $A_p(X) \subset C(X)$ is homeomorphic to $X$.*

Now we are ready to prove the following theorem.

**Theorem 4.8.** *If $X$ is an arc-smooth continuum, then $X$ admits a Whitney map for $C(X)$ if and only if $X$ is metrizable.*

*Proof.* It is known that if $X$ is metrizable, then $X$ admits a Whitney map for $C(X)$. Suppose that $X$ is non-metrizable and there exists a Whitney map for $C(X)$. Let $X$ be arc-smooth at a point $p$. By Lemma 4.7 $X$ is homeomorphic to $A_p(X) \subset C(X)$. It is clear that $A_p(X) \setminus \{\{p\}\} \subset C(X) \setminus X(1)$. We infer that $A_p(X) \setminus \{\{p\}\}$ is metrizable since $C(X) \setminus X(1)$ is metrizable (Theorem 3.1). Hence $X \setminus \{p\}$ is metrizable since it is homeomorphic to $A_p(X) \setminus \{\{p\}\}$ under the homeomorphism $h(x) = A(p,x)$. Moreover, from Theorem 3.1 it follows that $X$ is separable since $w(C(X) \setminus X(1)) \leq \aleph_0$ and $A_p(X) \setminus \{\{p\}\} \subset C(X) \setminus X(1)$. Furthermore, $X$ is the one point compactification of $X \setminus \{p\}$. Finally, from Theorem 1.3 it follows that $X$ is metrizable, a contradiction. □

Formerly the smoothness was defined for fans [1, p. 7] and for metric arboroids, i.e., for dendroids [4, p. 298, Definition].

An arboroid $X$ is said to be *smooth* if there exists a point $p \in X$, called an *initial point* of $X$, such that for every convergent net $\{a_d : d \in D\}$ of points $a_d$ of $X$ the condition $\lim\{a_d : d \in D\} = a$ implies that the net of arcs $\{pa_d : d \in D\}$ is convergent and $\mathrm{Lim}\{pa_d : d \in D\} = pa$.

**Lemma 4.9** ([8, p. 647]). *An arboroid is smooth if and only if it is arc-smooth.*

**Theorem 4.10.** *If a smooth arboroid $X$ admits a Whitney map for $C(X)$, then $X$ is metrizable.*

## 5. The Property of Kelley

We say that a continuum $X$ has the *property of Kelley at a point* $p \in X$ if for every subcontinuum $K \subset X$ containing $p$ and for every open neighborhood $\mathcal{U}$ of $K$ in the hyperspace $C(X)$, there exists a neighborhood $U$ of $p$ in $X$ such that if $q \in U$ then there is a continuum $L \in C(X)$ with $q \in L \in \mathcal{U}$. A continuum $X$ has the *property of Kelley* if it has the property of Kelley at each of its points.

For a given continuum $X$ we define the function $\alpha_X : X \to C^2(X)$ by

$$\alpha_X(x) = \{A \in C(X) : x \in A\}$$

for each point $x \in X$ [3, p. 91].

**Lemma 5.1.** *The function $\alpha_X$ is upper semi-continuous.*

*Proof.* See [20, p. 292, (2.1) Theorem]. □

**Theorem 5.2** ([3, Theorem 3.1])**.** *The function $\alpha_X$ is continuous if and only if $X$ has the property of Kelley.*

Hence we have the following lemma.

**Lemma 5.3.** *If a continuum $X$ has the property of Kelley, then the function $\alpha_X : X \to C^2(X) \setminus C(X)(1)$ is an embedding.*

*Proof.* Let us note that

$$\alpha_X(X) \subset C^2(X) \setminus C(X)(1).$$

The rest follows from Theorem 5.2. □

Now we are ready to prove the following theorem.

**Theorem 5.4.** *If a continuum $X$ with the property of Kelley admits a Whitney map for $C^2(X)$, then it is metrizable.*

*Proof.* By Corollary 3.2 the set $C^2(X) \setminus C(X\ )(1)$ is metrizable and

$$w(C^2(X) \setminus C(X)(1)) \leq \aleph_0.$$

Using Lemma 5.3 we see that $\alpha_X(X) \subset C^2(X) \setminus C(X)(1)$ is metrizable. Moreover, $X$ is homeomorphic to $\alpha_X(X)$. Hence $X$ is metrizable. □

**Problem 1.** Is it true that a continuum $X$ with the property of Kelley is metrizable if it admits a Whitney map for $C(X)$?

We say that a continuum $X$ is *hereditarily indecomposable* if no subcontinuum of $X$ can be written as the union of two proper subcontinua [9, p. 61].

**Lemma 5.5** ([6, p. 211, Proposition 2.7])**.** *Hereditarily indecomposable continua have the property of Kelley.*

From Theorem 5.4 we obtain the following result.

**Theorem 5.6.** *If a hereditarily indecomposable continuum $X$ admits a Whitney map for $C^2(X)$, then $X$ is metrizable.*

## 6. Concluding Remarks

It is known [7, p. 171, Corollary 3.1.20] that if a compact space $X$ is a countable union of its subspaces $X_n, n \in \mathbb{N}$, such that $w(X_n) \leq \aleph_0$, then $w(X) \leq \aleph_0$. Using this fact and the theorems proved in the previous sections we obtain the following results.

**Theorem 6.1.** *If a continuum $X$ is the countable union of its arcwise connected MR-smooth continua and if $X$ admits a Whitney map for $C^2(X)$, then $X$ is metrizable.*

*Proof.* Apply Theorem 4.4. □

**Theorem 6.2.** *If a continuum $X$ is the countable union of its arc-smooth continua and if $X$ admits a Whitney map for $C(X)$, then $X$ is metrizable.*

*Proof.* Apply Theorem 4.8. □

Finally, applying Theorem 5.4 we obtain the following theorem.

**Theorem 6.3.** *If a continuum $X$ is the countable union of its subcontinua with the property of Kelley and if $X$ admits a Whitney map for $C^2(X)$, then $X$ is metrizable.*


## Acknowledgement

The author is very grateful to the referee for his/her valuable suggestions.


## References


1. J. J. Charatonik, On fans. *Dissertationes Math.* (*Rozprawy Mat.*) **54**(1967), 39 pp.
2. J. J. Charatonik and W. J. Charatonik, Whitney maps – a non-metric case. *Colloq. Math.* **83**(2000), No. 2, 305–307.
3. J. J. Charatonik and W. J. Charatonik, Connections between classes of spaces and of mappings. *Proceedings of the* 2000 *Topology and Dynamics Conference* (*San Antonio,* TX). *Topology Proc.* **25**(2000), *Spring,* 87–111.
4. J. J. Charatonik and C. Eberhart, On smooth dendroids. *Fund. Math.* **67**(1970), 297–322.
5. W. J. Charatonik, Inverse limits of smooth continua. *Comment. Math. Univ. Carolin.* **23**(1982), No. 1, 183–191.
6. W. J. Charatonik, A homogeneous continuum without the property of Kelley. *Topology Appl.* **96**(1999), No. 3, 209–216.
7. R. Engelking, General topology. Translated from the Polish by the author. *Monografie Matematyczne, Tom* 60. [*Mathematical Monographs, Vol.* 60]. *PWN—Polish Scientific Publishers, Warsaw,* 1977.
8. J. B. Fugate, G. R. Gordh, Jr., and Lewis Lum, On arc-smoth continua. *Topology Proc.* **2**(1977), 645–656.
9. A. Illanes and S. B. Nadler, Jr., Hyperspaces. Fundamentals and recent advances. *Monographs and Textbooks in Pure and Applied Mathematics,* 216. *Marcel Dekker, Inc., New York,* 1999.
10. Y. Kodama, S. Spiez, and T. Watanabe, On shape of hyperspaces. *Fund. Math.* **100**(1978), No. 1, 59–67.

11. A. Koyama, Weakly arc-smooth continua. I. *Glas. Mat. Ser. III* **22(42)**(1987), No. 1, 171–185.
12. I. Lončar, A note on a Whitney map for continua. *Math. Commun.* **6**(2001), No. 1, 1–9.
13. I. Lončar, A fan $X$ admits a Whitney map for $C(X)$ iff it is metrizable. *Glas. Mat. Ser. III* **38(58)**(2003), 395–411.
14. L. Lum, Weakly smooth continua. *Trans. Amer. Math. Soc.* **214**(1975), 153–167.
15. T. Maćkowiak, On smooth continua. *Fund. Math.* **85**(1974), No. 1, 79–95.
16. M. M. McWaters, Arcs, semigroups, and hyperspaces. *Canad. J. Math.* **20**(1968), 1207–1210.
17. E. Michael, Topologies on spaces of subsets. *Trans. Amer. Math. Soc.* **71**(1951), 152–182.
18. S. B. Nadler, Jr., Hyperspaces of sets. A text with research questions. *Monographs and Textbooks in Pure and Applied Mathematics,* 49. *Marcel Dekker, Inc., New York–Basel,* 1978.
19. Z. M. Rakowski, Smooth Hausdorf continua. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **25**(1977), No. 6, 563–566.
20. R. W. Wardle, On a property of J. L. Kelley. *Houston J. Math.* **3**(1977), No. 2, 291–299.
21. T. Watanabe, A Whitney map and the Lindelöf property. *Houston J. Math.* **15**(1989), No. 1, 147–151.
22. G. T. Whyburn, Analytic topology. *American Mathematical Society Colloquium Publications,* XXVIII. *American Mathematical Society, Providence, R.I.,* 1963.
23. S. Willard, General topology. *Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.,* 1970.

(Received 4.12.2003; revised 3.02.2005)

Author' address:

Faculty of Organization and Informatics Varaždin
(University of Zagreb, Croatia)
Pavlinska 2, HR-42000 Varaždin
Croatia
E-mails: ivan.loncar1@vz.tel.hr
ivan.loncar@foi.hr