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MAPPING PROPERTIES OF INTEGRAL OPERATORS OF LEVY TYPE

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Abstract. We study the boundedness and compactness of a special class of integral operators defined on generalized Sobolev spaces.

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1. Introduction and the Main Results

Let us consider the integral operator S,

$$Su(x) = \int\limits_{\mathbb{R}^n\setminus\{0\}} \Big[u(x+z) - u(x) - \sum_{j=1}^{n} z_j \frac{\partial u}{\partial x_j}(x)\Big] K(x,z)\, d\mu(z), \quad x \in \mathbb{R}^n,$$

where the kernel K is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies the following growth conditions:

- (H1): $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$,
- (H2): for fixed $0 < \theta_0 < 1$, we suppose that $H(\theta_0) < +\infty$, where

$$H(\theta_0) = \sup_{x,z\in\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \big|K(x+h,z) - K(x,z)\big|^2 \frac{dh}{|h|^{n+2\theta_0}}.$$

Furthermore, the Radon measure $d\mu(z)$ on $\mathbb{R}^n \setminus \{0\}$ satisfies the moment condition:

- (H3): $\displaystyle\int\limits_{0<|z|\le 1} |z|^\alpha\, d\mu(z) + \int\limits_{|z|\ge 1} |z|\, d\mu(z) < +\infty$, where $1 \le \alpha \le 2$ holds.

Our first mapping result deals with the generalized Sobolev spaces $H^s_p(\mathbb{R}^n)$ of functions defined on $\mathbb{R}^n$.

Theorem 1. *Suppose that* (H1), (H2) *and* (H3) *hold. Then*

$$S:\, H^{\theta+\alpha}_p(\mathbb{R}^n) \to H^\theta_p(\mathbb{R}^n)$$

is a bounded operator for all $1 < p < \infty$ *and all* $0 \le \theta \le \theta_0$.

In our second result we consider the case of bounded connected domains Ω in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

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Theorem 2. *Let $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $K(x,z) = 0$ if $x + z \notin \overline{\Omega}$. Suppose that* (H2) *and* (H3) *are satisfied with $0 < \theta_0 < 1$ and $1 \le \alpha \le 2$, respectively. Then*

$$S:\ H_p^{\alpha+\theta}(\Omega) \to H_p^\theta(\Omega)$$

is bounded for $0 \le \theta \le \theta_0$ and all $1 < p < +\infty$. Moreover, if $\frac{n}{p} + 1 < \alpha$, then

$$S:\ H_p^{\alpha+\theta}(\Omega) \to H_p^\theta(\Omega)$$

is compact for all $0 \le \theta < \theta_0$.

Note that the condition $K(x,z) = 0$ for $x + z \notin \overline{\Omega}$ implies that our integral operator can be interpreted as a mapping acting on functions u which are defined in $\overline{\Omega}$.

Operators of such a type play an important role in the theory of Waldenfels operators $W = P + S$, where P is a second-order elliptic partial differential operator of the type

$$Pu(x) = \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j}(x) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x), \quad x \in \Omega,$$

see [1] and [4]. In particular, the Levy operator S occurs as a compact perturbation.

The paper is organized as follows. First we recall some properties of spaces of type H_p^s. In Section 3, we prove Theorem 1 and then Theorem 2 in Section 4.

2. Function Spaces

In general, all functions, distributions, etc. are defined on the Euclidean space $\mathbb{R}^n$. As usual $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of test functions, $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ is its dual. For $f \in \mathcal{S}'$, $\hat{f}$ denotes the Fourier transform of f.

To define the Bessel potential space $H_p^s = H_p^s(\mathbb{R}^n)$, we make use of the Fourier-analytic approach.

Throughout the paper let ψ in $\mathcal{S}(\mathbb{R}^n)$ be fixed so that $\hat{\psi}$ be supported by the set $\{\xi \in \mathbb{R}^n :\ \frac{1}{2} \le |\xi| \le 2\}$ and

$$\sum_{j \in \mathbb{Z}} \hat{\psi}(2^j \xi) = 1 \quad \text{for} \quad \xi \neq 0\,. \tag{1}$$

Define φ by

$$\hat{\varphi}(\xi) = 1 - \sum_{j \ge 1} \hat{\psi}(2^{-j}\xi)\,, \tag{2}$$

and denote now by Δ_j ($j \in \mathbb{N}$) the convolution operator with symbol $\hat{\psi}(2^{-j}\xi)$.

For $s \in \mathbb{R}$ and $1 < p < +\infty$, the generalized Sobolev space or the Bessel potential space $H_p^s = H_p^s(\mathbb{R}^n)$ is a subspace of $\mathcal{S}'(\mathbb{R}^n)$ given by the norm

$$\|g\|_{H_p^s} = \|\varphi * g\|_p + \Big\| \Big[\sum_{j \ge 1} 4^{sj} |\Delta_j g|^2 \Big]^{\frac{1}{2}} \Big\|_p < +\infty,$$

and for $1 \le q \le \infty$, the Besov space $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$ is a subspace of $\mathcal{S}'(\mathbb{R}^n)$ with the norm

$$\|g\|_{B^s_{p,q}} = \|\varphi * g\|_p + \Big[\sum_{j\ge 1} 2^{sjq}\|\Delta_j g\|_p^q\Big]^{\frac{1}{q}} < +\infty.$$

Note that if $s < s'$ holds, then $H^s_p \subset B^s_{p,\infty}$ and $B^{s'}_{p,\infty} \subset H^s_p$.

Now let Ω be a bounded connected domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and its closure be given by $\overline{\Omega} = \Omega \cup \partial\Omega$. To define function spaces on Ω we use restriction arguments. Let $\mathcal{D}'(\Omega)$ be the space of distributions on Ω. Then we put

$$H^s_p(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : \quad \exists\, g \in H^s_p(\mathbb{R}^n) \quad \text{such that} \quad g|_\Omega = f \Big\}$$

and

$$\| f\|_{H^s_p(\Omega)} = \inf_{g|_\Omega = f} \ \| g\|_{H^s_p}.$$

In the same way we define the Besov space

$$B^s_{p,q}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : \quad \exists\, g \in B^s_{p,q}(\mathbb{R}^n) \quad \text{such that} \quad g|_\Omega = f \Big\}$$

and

$$\| f\|_{B^s_{p;q}(\Omega)} = \inf_{g|_\Omega = f} \ \| g\|_{B^s_{p,q}},$$

see for example [5] and [3].

For the proof of Theorem 1 we make use of the following characterization of generalized Sobolev spaces H^s_p (*Strichartz' norm*) which can be found, for example, in [5, p. 194].

Lemma 1. *Let* $0 < s < 1$ *and* $1 < p < +\infty$. *Then*

$$N^s_p(g) = \|g\|_p + \|L_s(g)\|_p$$

defines an equivalent norm on H^s_p, *where*

$$L_s(g)(x) = \begin{cases} \Big(\displaystyle\int_{\mathbb{R}^n} \big|g(x+h) - g(x)\big|^2 \frac{dh}{|h|^{n+2s}} \Big)^{1/2} & \text{if } p \ge 2, \\ \Big(\displaystyle\int_0^\infty \Big[\int_{|h|\le 1} \big|g(x+th) - g(x)\big| dh \Big]^2 \frac{dt}{t^{1+2s}} \Big)^{1/2} & \text{if } 1 < p \le 2. \end{cases}$$

The second Lemma which will be used in our proof is the following.

Lemma 2. *Let* $0 \le s \le 1$, $0 \le \gamma \le 1$, $1 < p < +\infty$ *and set* $T_h(f)(x) = f(x+h) - f(x)$. *Then there exists* $C > 0$ *such that*

$$\big\|T_h(f)\big\|_{H^\gamma_p} \le C|h|^s \, \big\|f\big\|_{H^{s+\gamma}_p}$$

for all $f \in H^{s+\gamma}_p$.

Proof. We are not able to give a reference for this result. For the reader's convenience we prove the assertion.

1) The case $\gamma = 0$:

for $s = 0$ the lemma is trivial and for $s = 1$ it is known, see [6, pp. 45–46], for example;

for $0 < s < 1$ we recall that

$$\left[\int_{\mathbb{R}^n} \left| f(x+h) - f(x) \right|^p dx \right]^{\frac{1}{p}} \leq C|h|^s \, \|f\|_{B^s_{p,\infty}}.$$

Since $H^s_p \subset B^s_{p,\infty}$ the lemma follows.

2) The case $\gamma = 1$:

Note that

$$\left\| T_h(f) \right\|_{H^1_p} \leq C \left[\sum_{j=1}^n \left\| \frac{\partial T_h(f)}{\partial x_j} \right\|_p + \left\| T_h(f) \right\|_p \right]$$

and it follows from **1)** that

$$\left\| T_h(f) \right\|_p + \left\| \frac{\partial T_h(f)}{\partial x_j} \right\|_p \leq C|h|^s \left[\left\| \frac{\partial f}{\partial x_j} \right\|_{H^s_p} + \left\| f \right\|_p \right],$$

and hence

$$\left\| T_h(f) \right\|_{H^1_p} \leq C|h|^s \left\| f \right\|_{H^{s+1}_p}.$$

3) From **1)** and **2)** we obtain that T_h is a bounded linear operator from H^s_p into L^p and from H^{s+1}_p into H^1_p, respectively.

Then by interpolation arguments we obtain that T_h is bounded from $H^{s+\gamma}_p$ into H^γ_p for all $0 \leq \gamma \leq 1$ and the norm of T_h is bounded by $C|h|^s$. □

Lemma 3. *Let $0 < \theta_0 < 1$. There exists $C_1 = C_1(\theta_0) > 0$ such that*

$$\sup_{x,h \in \mathbb{R}^n} \left[\frac{|g(x+h) - g(x)|}{|h|^{\theta_0}} \right] \leq C_1 \left[\sup_{x \in \mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left| g(x+h) - g(x) \right|^2 \frac{dh}{|h|^{n+2\theta_0}} \right]^{\frac{1}{2}} + \|g\|_\infty \right]$$

for all $g \in L^\infty(\mathbb{R}^n)$. Moreover, for every $0 \leq \theta < \theta_0$, there exists a finite constant $C_2 = C_2(\theta_0) > 0$ such that

$$\int_{\mathbb{R}^n} \left| g(x+h) - g(x) \right|^2 \frac{dh}{|h|^{n+2\theta}}]^{\frac{1}{2}} \leq C_1 \left[\sup_{x,h \in \mathbb{R}^n} \frac{|g(x+h) - g(x)|}{|h|^{\theta_0}} + \|g\|_\infty \right]$$

for all $g \in L^\infty(\mathbb{R}^n)$.

In particular, if $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then $H(\theta_0) < +\infty$ implies $H(\theta) < +\infty$ for all $0 \leq \theta < \theta_0$ and

$$\left| g(x+h) - g(x) \right| \leq C|h|^{\theta_0}$$

for all $x, h \in \mathbb{R}^n$.

Proof. Indeed, we have

$$\|g\|_{B^{\theta_0}_{\infty,\infty}} = \|g\|_\infty + \sup_{x,h\in\mathbb{R}^n} \frac{|g(x+h)-g(x)|}{|h|^{\theta_0}}$$

and

$$\|g\|_{B^{\theta_0}_{\infty,2}} = \|g\|_\infty + \sup_{x\in\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \big|g(x+h)-g(x)\big|^2 \frac{dh}{|h|^{n+2\theta_0}}\right]^{\frac{1}{2}},$$

see [5] or [3] for example. The embeddings $B^{\theta_0}_{\infty,2} \subset B^{\theta}_{\infty,2} \subset B^{\theta}_{\infty,\infty}$ finish the proof. □

3. Proof of Theorem 1

To prove Theorem 1 we show first the boundedness of S from $H^\alpha_p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ (the case $\theta = 0$) and after that we prove that there exists a positive constant C such that

$$\big\|L_\theta(Su)\big\|_p \le C\ \|u\|_{H^{\alpha+\theta}_p}$$

holds for all $u \in H^{\alpha+\theta}_p(\mathbb{R}^n)$ and $0 < \theta \le \theta_0$. This implies

$$\|Su\|_{H^\theta_p} \le C\|u\|_{H^{\theta+\alpha}_p}.$$

Indeed, $H(\theta_0) < +\infty$ implies $H(\theta) < +\infty$ by Lemma 3.

3.1. **L^p-boundedness.** To prove the L^p-boundedness, we write $Su(x) = S_1u(x) + S_2u(x)$, where

$$S_1u(x) = \int_{|z|\ge 1} \left[u(x+z) - u(x) - \sum_{j=1}^n z_j \frac{\partial u}{\partial x_j}(x)\right] K(x,z)\, d\mu(z),$$

$$S_2u(x) = \int_{|z|\le 1} \left[u(x+z) - u(x) - \sum_{j=1}^n z_j \frac{\partial u}{\partial x_j}(x)\right] K(x,z)\, d\mu(z).$$

First observe that

$$|S_1u(x)| \le \|K\|_\infty \big[g_1(x) + g_2(x)\big],$$

where

$$g_1(x) = \int_{|z|\ge 1} \big|u(x+z) - u(x)\big|\, d\mu(z)$$

and

$$g_2(x) = \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) \int_{|z|\ge 1} |z_j|\, d\mu(z).$$

Hence

$$\|g_1\|_p \le \int_{|z|\ge 1} \left[\int_{\mathbb{R}^n} \big|u(x+z)-u(x)\big|^p\, dx\right]^{\frac{1}{p}} d\mu(z)$$

and

$$\|g_2\|_p \le \sum_{j=1}^n \Big\|\frac{\partial u}{\partial x_j}\Big\|_p \Big[\sum_{j=1}^n \int\limits_{|z|\ge 1} |z_j|\, d\mu(z)\Big].$$

Combining now (H3), Lemma 2 with $s = 1$ and $\gamma = 0$, we obtain

$$\|S_1 u\|_p \le C\|K\|_\infty \|u\|_{H_p^1} \int\limits_{|z|\ge 1} |z|\, d\mu(z).$$

Next we give an estimate for $\|S_2 u\|_p$. We write

$$u(x+z) - u(x) - \sum_{j=1}^n z_j \frac{\partial u}{\partial x_j}(x) = \sum_{j=1}^n z_j \int_0^1 \Big[\frac{\partial u}{\partial x_j}(x+tz) - \frac{\partial u}{\partial x_j}(x)\Big] dt,$$

and it follows that

$$|S_2 u(x)| \le \|K\|_\infty \sum_{j=1}^n \int_0^1 \int\limits_{|z|\le 1} |z| \Big|\frac{\partial u}{\partial x_j}(x+tz) - \frac{\partial u}{\partial x_j}(x)\Big| d\mu(z)\, dt.$$

Hence

$$\|S_2 u\|_p \le C \sum_{j=1}^n \|K\|_\infty \int_0^1 \int\limits_{|z|\le 1} |z| \Big[\int\limits_{\mathbb{R}^n} \Big|\frac{\partial u}{\partial x_j}(x+tz) - \frac{\partial u}{\partial x_j}(x)\Big|^p dx\Big]^{\frac{1}{p}} d\mu(z)\, dt.$$

Applying now Lemma 2 with $s = \alpha - 1$ and $\gamma = 0$, we obtain

$$\Big[\int\limits_{\mathbb{R}^n} \Big|\frac{\partial u}{\partial x_j}(x+tz) - \frac{\partial u}{\partial x_j}(x)\Big|^p dx\Big]^{\frac{1}{p}} \le C|tz|^{\alpha-1} \Big\|\frac{\partial u}{\partial x_j}\Big\|_{H_p^{\alpha-1}} \le C|tz|^{\alpha-1}\|u\|_{H_p^\alpha}.$$

Thus

$$\|S_2 u\|_p \le C\|K\|_\infty \|u\|_{H_p^\alpha} \int\limits_{|z|\le 1} |z|^\alpha d\mu(z).$$

3.2. **Estimation of $\|L_\theta(Su)\|_p$.** In the following, we consider the case where $p \ge 2$ holds. The other case can be handled similarly. We are to prove that $L_\theta(Su) \in L^p$, where

$$L_\theta(Su)(x) = \Big[\int \frac{\big|Su(x+h) - Su(x)\big|^2}{|h|^{n+2\theta}}\, dh\Big]^{\frac{1}{2}}.$$

As above we put $Su = S_1 u + S_2 u$, where

$$S_1 u(x) = \int\limits_{|z|\ge 1} \Big[u(x+z) - u(x) - \sum_{j=1}^n z_j \frac{\partial u}{\partial x_j}(x)\Big] K(x,z)\, d\mu(z),$$

$$S_2 u(x) = \int\limits_{|z|\le 1} \Big[u(x+z) - u(x) - \sum_{j=1}^n z_j \frac{\partial u}{\partial x_j}(x)\Big] K(x,z)\, d\mu(z).$$

A) Estimation of $L_\theta(S_1u)$. First we write $S_1u(x+h)-S_1u(x)$ in the following form:

$$\begin{aligned}
&S_1u(x+h)-S_1u(x)\\
&=\int\limits_{|z|\geq 1}\big[u(x+h+z)-u(x+z)+u(x)-u(x+h)\big]K(x+h,z)d\mu(z)\\
&\quad+\int\limits_{|z|\geq 1}\big[u(x+z)-u(x)\big]\big[K(x+h,z)-K(x,z)\big]d\mu(z)\\
&\quad+\sum_{j=1}^{n}\Big[\frac{\partial u}{\partial x_j}(x)-\frac{\partial u}{\partial x_j}(x+h)\Big]\int\limits_{|z|\geq 1} z_jK(x+h,z)d\mu(z)\\
&\quad+\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}(x)\int\limits_{|z|\geq 1} z_j\big[K(x,z)-K(x+h,z)\big]d\mu(z)\\
&=A_1(x,h)+A_2(x,h)+A_3(x,h)+A_4(x,h).
\end{aligned}$$

1) $A_1(x,h)$:
We have

$$A_1(x,h)=\int\limits_{|z|\geq 1}\Big(\int\limits_0^1\sum_{j=1}^{n}\Big[\frac{\partial u}{\partial x_j}(x+h+tz)-\frac{\partial u}{\partial x_j}(x+tz)\Big]z_j\,dt\Big)K(x+h,z)\,d\mu(z)$$

which implies

$$\Big[\int\limits_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n}|A_1(x,h)|^2\frac{dh}{|h|^{n+2\theta}}\Big]^{\frac{p}{2}}dx\Big]^{\frac{1}{p}}\leq C\sum_{j=1}^{n}\Big\|L_\theta\Big(\frac{\partial u}{\partial x_j}\Big)\Big\|_p\|K\|_\infty\Big[\int\limits_{|z|\geq 1}|z|d\mu(z)\Big].$$

2) $A_2(x,h)$:
In this case we get

$$\begin{aligned}
&\Big[\int\limits_{\mathbb{R}^n}|A_2(x,h)|^2\frac{dh}{|h|^{n+2\theta}}\Big]^{\frac{1}{2}}\\
&\leq\int\limits_{|z|\geq 1}\big|u(x+z)-u(x)\big|\int\limits_{\mathbb{R}^n}\Big[\big|K(x+h,z)-K(x,z)\big|^2\frac{dh}{|h|^{n+2\theta}}\Big]^{\frac{1}{2}}d\mu(z)\\
&\leq H(\theta)\int\limits_{|z|\geq 1}\big|u(x+z)-u(x)\big|\,d\mu(z)
\end{aligned}$$

and

$$\Big[\int\limits_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n}|A_2(x,h)|^2\frac{dh}{|h|^{n+2\theta}}\Big]^{\frac{p}{2}}dx\Big]^{\frac{1}{p}}\leq CH(\theta)\|u\|_p\Big[\int\limits_{|z|\geq 1}|z|\,d\mu(z)\Big].$$

3) $A_3(x,h)$:
It is clear that

$$\Big[\int\limits_{\mathbb{R}^n} |A_3(x,h)|^2 \frac{dh}{|h|^{n+2\theta}}\Big]^{\frac{1}{2}} \leq \sum_{j=1}^{n} \Big[\|K\|_\infty \; L_\theta\Big(\frac{\partial u}{\partial x_j}\Big)(x)\Big]\Big[\int\limits_{|z|\geq 1} |z|\, d\mu(z)\Big]$$

and hence

$$\Big[\int\limits_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n} |A_3(x,h)|^2 \frac{dh}{|h|^{n+2\theta}}\Big]^{\frac{p}{2}} dx\Big]^{\frac{1}{p}} \leq \sum_{j=1}^{n} \Big[\|K\|_\infty \|L_\theta\Big(\frac{\partial u}{\partial x_j}\Big)\|_p\Big]\Big[\int\limits_{|z|\geq 1} |z|\, d\mu(z)\Big].$$

4) $A_4(x,h)$:
In the last case we obtain

$$\Big[\int\limits_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n} |A_4(x,h)|^2 \frac{dh}{|h|^{n+2\theta}}\Big]^{\frac{p}{2}} dx\Big]^{\frac{1}{p}} \leq H(\theta)\Big[\int\limits_{|z|\geq 1} |z|\, d\mu(z)\Big] \sum_{j=1}^{n} \Big\|\frac{\partial u}{\partial x_j}\Big\|_p.$$

Summing up we get

$$\big\|L_\theta(S_1 u)\big\|_p \leq \big[\|K\|_\infty + H(\theta)\big]\Big[\int\limits_{|z|\geq 1} |z|\, d\mu(z)\Big] \times \Big[\|u\|_p + \sum_{j=1}^{n} \Big\|L_\theta\Big(\frac{\partial u}{\partial x_j}\Big)\Big\|_p + \sum_{j=1}^{n} \Big\|\frac{\partial u}{\partial x_j}\Big\|_p\Big].$$

We have

$$\Big\|\frac{\partial u}{\partial x_j}\Big\|_p \leq C\|u\|_{H_p^1}$$

and

$$\Big\|L_\theta\Big(\frac{\partial u}{\partial x_j}\Big)\Big\|_p \leq C\|u\|_{H_p^{\theta+1}}.$$

It follows finally that

$$\big\|L_\theta(S_1 u)\big\|_p \leq C\big[\|K\|_\infty + H(\theta)\big]\|u\|_{H_p^{\theta+1}} \int\limits_{|z|\geq 1} |z|\, d\mu(z).$$

B) Estimation of $L_\theta(S_2 u)$.
Here we write

$$S_2 u(x+h) - S_2 u(x) = B_1(x,h) + B_2(x,h),$$

where

$$B_1(x,h) = \int\limits_{|z|\leq 1} \Big[u(x+z+h) - u(x+z) - u(x+h) + u(x) - \sum_{j=1}^{n} z_j \Big(\frac{\partial u}{\partial x_j}(x+h) - \frac{\partial u}{\partial x_j}(x)\Big)\Big] K(x+h,z)\, d\mu(z)$$

and

$$B_2(x,h) = \int\limits_{|z|\le 1} \Big[u(x+z)-u(x)-\sum_{j=1}^{n} z_j\Big(\frac{\partial u}{\partial x_j}(x)\Big)\big(K(x+h,z)-K(x,z)\big)\Big]\, d\mu(z).$$

First we put $f^j_{tz}(x) = \frac{\partial u}{\partial x_j}(x+tz) - \frac{\partial u}{\partial x_j}(x)$ and observe that

$$|B_1(x,h)| \le \sum_{j=1}^{n} \|K\|_\infty \int\limits_{|z|\le 1} \int\limits_0^1 |z|\,\big|f^j_{tz}(x+h) - f^j_{tz}(x)\big|\, dt\, d\mu(z).$$

Hence we get

$$\Big[\int\limits_{\mathbb{R}^n} \frac{|B_1(x,h)|^2}{|h|^{n+2\theta}}\, dh\Big]^{\frac{1}{2}} \le \sum_{j=1}^{n} \|K\|_\infty \int\limits_{|z|\le 1} |z| \int\limits_0^1 L_\theta(f^j_{tz})(x)\, dt\, d\mu(z).$$

Applying now Lemma 2 with $s = \alpha - 1$ and $\gamma = \theta$, we obtain

$$\big\|L_\theta(f^j_{tz})\big\|_p \le C|tz|^{\alpha-1}\Big\|\frac{\partial u}{\partial x_j}\Big\|_{H_p^{\theta+\alpha-1}}$$

and, consequently,

$$\Big[\int\limits_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n} \frac{|B_1(x,h)|^2}{|h|^{n+2\theta}}\, dh\Big]^{\frac{p}{2}} dx\Big]^{\frac{1}{p}} \le C\|K\|_\infty \int\limits_{|z|\le 1} |z|^\alpha\, d\mu(z) \sum_{j=1}^{n} \Big\|\frac{\partial u}{\partial x_j}\Big\|_{H_p^{\theta+\alpha-1}}.$$

On the other hand, we have

$$\big|B_2(x,h)\big| \le \sum_{j=1}^{n} \int\limits_0^1 \int\limits_{|z|\le 1} |z|\,\Big|\frac{\partial u}{\partial x_j}(x+tz) - \frac{\partial u}{\partial x_j}(x)\Big|\,\big|K(x+h,z)-K(x,z)\big|\, d\mu(z)\, dt.$$

Using (H2), it follows that

$$\Big[\int\limits_{\mathbb{R}^n} \frac{|B_2(x,h)|^2}{|h|^{n+2\theta}}\, dh\Big]^{\frac{1}{2}} \le \sum_{j=1}^{n} H(\theta) \int\limits_{|z|\le 1} |z| \int\limits_0^1 \Big|\frac{\partial u}{\partial x_j}(x+tz) - \frac{\partial u}{\partial x_j}(x)\Big|\, d\mu(z)\, dt.$$

Using again Lemma 2 with $s = \alpha - 1$ and $\gamma = 0$, we obtain

$$\Big[\int\limits_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n} \frac{|B_2(x,h)|^2}{|h|^{n+2\theta}}\, dh\Big]^{\frac{p}{2}} dx\Big]^{\frac{1}{p}} \le CH(\theta) \sum_{j=1}^{n} \Big\|\frac{\partial u}{\partial x_j}\Big\|_{H_p^{\alpha-1}}.$$

Therefore we have shown that

$$\big\|L_\theta(S_2 u)\big\|_p \le C\big(H(\theta) + \|K\|_\infty\big)\, \|u\|_{H_p^{\theta+\alpha}}$$

holds.

4. Proof of Theorem 2

Step 1: The proof of part 1) follows from Theorem 1 by restriction arguments.

Step 2: Let $\frac{n}{p}+1<\alpha\le 2$. We show that

$$S:\ H_p^\alpha(\Omega)\to L_p(\Omega)$$

is a compact mapping. We recall that $K(x,z)=0$ if $x+z\notin\overline{\Omega}$. Hence there exists a compact set $M\subset\mathbb{R}^n$ such that

$$Su(x)=\int\limits_M\left[u(x+z)-u(x)-\sum_{j=1}^n z_j\frac{\partial u}{\partial x_j}(x)\right]K(x,z)\,d\mu(z)$$

can be interpreted as a mapping acting on functions u defined on $\overline{\Omega}$.

Now we introduce for $0<\varepsilon<1$ a family of truncation operators given by

$$S_{\Phi_\varepsilon}u(x)=\int\limits_M\left[u(x+z)-u(x)-\sum_{j=1}^n z_j\frac{\partial u}{\partial x_j}(x)\right]K(x,z)\Phi_\varepsilon(x,z)\,d\mu(z),$$

where $\Phi_\varepsilon\in C^\infty(\overline{\Omega}\times\mathbb{R}^n)$ such that

$$\Phi_\varepsilon(x,z)=0\quad\text{if}\quad |x-z|\le\varepsilon,$$

and

$$\Phi_\varepsilon(x,z)=1\quad\text{if}\quad |x-z|\ge 2\varepsilon.$$

Furthermore, by Lemma 3 there exists $C_0>0$ such that

$$\big|K(x,z)-K(y,z)\big|<C_0|x-y|^{\theta_0}$$

holds for all $x,y\in M$ and all $z\in\mathbb{R}^n$. Hence we can show that

$$S_{\Phi_\varepsilon}:\ C^1(\overline{\Omega})\to C(\overline{\Omega})$$

is bounded. Since $\frac{n}{p}+1<\alpha\le 2$, the embedding $H_p^\alpha(\Omega)\hookrightarrow C^1(\overline{\Omega})$ is compact and the embedding $C(\overline{\Omega})\hookrightarrow L^p(\Omega)$ is continuous. We establish that

$$S_{\Phi_\varepsilon}:\ H_p^\alpha(\Omega)\to L_p(\Omega)$$

is compact. As in the proof of Theorem 1, we get

$$\|S_{\Phi_\varepsilon}u\|_p\le C\|K\|_\infty\,\|u\|_{H_p^\alpha(\Omega)}.$$

Furthermore, we have by Lebesgue's Theorem,

$$S_{\Phi_\varepsilon}\to S\quad\text{as}\quad\varepsilon\downarrow 0$$

with respect to the operator norm in $\mathcal{L}(H_p^\alpha(\Omega),L_p(\Omega))$. Because of the fact that the compact operators are a closed subspace in $\mathcal{L}(H_p^\alpha(\Omega),L_p(\Omega))$ it follows that

$$S:\ H_p^\alpha(\Omega)\to L_p(\Omega)$$

is compact for $\frac{n}{p}+1<\alpha$.

Step 3: Now we can finish the proof of Part 2). From Step 1 we can derive that

$$S:\ H_p^{\alpha+\theta}(\Omega)\to H_p^\theta(\Omega)$$

is bounded for all $1 < p < \infty$ and all $0 < \theta < \theta_0$. Using the result of Step 2 we have

$$S: \; H_p^\alpha(\Omega) \to L_p(\Omega)$$

is compact if $\frac{n}{p} + 1 < \alpha$.

Now we can apply a result concerning the complex interpolation of compact linear operators, see [2], in order to obtain our result. Indeed, $H_p^\alpha(\Omega)$ is reflexive and it is known that $H_p^\alpha(\Omega) = [L^p(\Omega), H_p^{\alpha+\theta}(\Omega)]_\sigma$ where $\sigma = 1 - \frac{\alpha}{\alpha+\theta}$.

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