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INTERVAL OSCILLATION CRITERIA OF MATRIX DIFFERENTIAL SYSTEMS WITH DAMPING

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Abstract. Using a generalized Riccati transformation, some new oscillation criteria of linear second order matrix differential system with damping are built by the method of integral average. These results are based on the information on a sequence of subintervals of $[t_0, \infty)$.

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1. Introduction

In this paper, we consider oscillatory properties for the linear second order matrix differential system with a damped term

$$Y'' + R(t)Y' + Q(t)Y = 0, \quad t \in [t_0, \infty), \tag{1}$$

where $R(t), Q(t)$, and $Y(t)$ are $n \times n$ continuous matrix functions, $R(t)$ and $Q(t)$ being symmetric. When $R(t) \equiv 0$, system (1) reduces to the linear second order matrix differential system

$$Y'' + Q(t)Y = 0, \quad t \in [t_0, \infty). \tag{2}$$

By M^* we mean the transpose of the matrix M; for any symmetric $n \times n$ matrix M, its eigenvalues are real numbers always denoted by $\lambda_1\{M\} \geq \lambda_2\{M\} \geq \cdots \geq \lambda_n\{M\}$. For any symmetric $n \times n$ matrices A, B, by $A > B$ ($A \geq B$) we mean that $A - B$ is positive (semi-)definite. A solution $Y(t)$ of (1) (or (2)) is said to be nontrivial if $\det Y(t) \neq 0$ for at least one $t \in [t_0, \infty)$, and a nontrivial solution $Y(t)$ of (1) (or (2)) is said to be prepared if

$$\begin{aligned} &Y^*(t)Y'(t) - (Y^*(t))'Y(t) \equiv 0, \\ &Y^*(t)R(t)Y'(t) - (Y^*(t))'R(t)Y(t) \equiv 0, \quad t \in [t_0, \infty). \end{aligned}$$

System (1) (or (2)) is said to be oscillatory on $[t_0, \infty)$ in case the determinant of every nontrivial prepared solution vanishes at least at one point on $[T, \infty)$ for each $T \geq t_0$.

Here, we point out that the definition of a prepared solution coincides with that of the system

$$(P(t)Y'(t))' + Q(t)Y(t) = 0, \quad t \in [t_0, \infty), \tag{3}$$

where $P(t)$ is nonsingular (see [1] for details), because we can transform (1) into system (3) by using the fundamental symmetric solution of $Z' = R(t)Z$.

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There are quite a number of works on oscillation for (2) and (3). It was conjectured by Hinton and Lewis [2] that (2) is oscillatory if

$$\lim_{t\to\infty} \lambda_1 \left\{ \int_{t_0}^{t} Q(s)\, ds \right\} = \infty.$$

This conjecture was partially proved by several authors and finally settled by Byers, Harris and Kwong [3]. Coles [4 , 5] extended this result by applying the weighted average method. Butler, Erbe and Mingarelli [6] showed that (2) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \lambda_1 \left\{ \int_{t_0}^{s} (Q_1(\tau)\, d\tau \right\} ds = \infty$$

provided that

$$\liminf_{t\to\infty} \frac{1}{t} \int_{a}^{t} \int_{a}^{s} \mathrm{tr}(Q(\tau))\, d\tau\, ds > -\infty.$$

Erbe, Kong and Ruan [1] gave the following theorem.

Theorem A. *Suppose that there exists a constant $\alpha > 1$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \lambda_1 \left\{ \int_{t_0}^{t} (t-s)^{\alpha} Q(s)\, ds \right\} = \infty.$$

Then system (2) *is oscillatory.*

The above results can be regarded as extensions of the classical oscillation criteria for scalar equations corresponding to (2) and (3). All the criteria are dependent on the values of Q on the whole half-line $[t_0, \infty)$ for some t_0. However, for the scalar equation (2), as implied by the Sturm Separation Theorem, oscillation is essentially an interval property, i.e., if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, $a_i \to \infty$, such that for each i, there exists a solution of (2) which has at least two zeros in $[a_i, b_i]$, then every solution of (2) is oscillatory no matter how "bad" the scalar equation (2)(or (3)) is on the remaining parts of $[t_0, \infty)$. Based on these facts, Kong [7] gives interval type criteria for the oscillation of system (2), which are extensions of those for scalar equations obtained by Kong [11]. Meanwhile, the oscillation of a system with damping has drawn less attention, Zheng [13] gave oscillation criteria for system (1), which generalize Theorem A. In this paper, we are concerned with extending interval oscillation criteria for system (2) to those of the damped linear second order matrix differential system (1). We also notice that in the criteria mentioned above, only the largest eigenvalues of some integrals are involved. It is not clear from them how oscillation is affected by other eigenvalues. Our results will show that not only the largest one, but every single eigenvalue of an integral can be used to determine oscillation. The main results are stated in

Section 2, the proofs are given in Section 3, and two interesting examples are given to illustrate the efficiency of the theorems in Section 4.

2. Main Results

In the sequel we say that a function $h = h(t,s)$ belongs to a function class $\mathcal{H}$ if $h \in C(D, R_+)$, where $D = \{(t,s) : t \geq s \geq t_0\}$, and satisfies $h(t,t) = 0, h(t,s) > 0$ for $(t,s) \in D_0 = \{(t,s) : t > s \geq t_0\}$. Furthermore, h has a continuous derivatives $\frac{\partial h}{\partial t}$ and $\frac{\partial h}{\partial s}$ on D, and there exist two functions $\lambda_1, \lambda_2 \in C(D_0, R_+)$ depending on the function h, such that for all $(t,s) \in D_0$,

$$\frac{\partial h(t,s)}{\partial t} = \lambda_1(t,s)h(t,s) \quad \text{and} \quad \frac{\partial h(t,s)}{\partial s} = -\lambda_2(t,s)h(t,s).$$

Note that $(t-s)^\alpha$ for $\alpha > 1; \ln(\frac{t}{s}); \beta(t-s)$ with $\beta \in C^1(0,\infty), \beta'(t) > 0$ for $t > 0$ and $\beta(0) = 0$ belong to the function class $\mathcal{H}$. In particular, if $h(t,s) = h(t-s) \in \mathcal{H}$, then $\lambda_1(t,s) = \lambda_2(t,s) \triangleq \lambda(t-s)$. We denote by $\mathcal{H}_0$ the class of all functions $h(t-s)$. Let $\rho \in \mathrm{C}^1[t_0,\infty)$ and $\rho > 0$ on $[t_0,\infty)$. We now define the operators $A^\rho(.;\tau,t)$ and $B^\rho(.;t,\tau)$ in terms of h and ρ as

$$A^\rho(g;\tau,t) = \int_\tau^t \rho(s)h(t,s)g(s)ds, \quad t \geq \tau, \tag{4}$$

$$B^\rho(g;t,\tau) = \int_t^\tau \rho(s)h(s,t)g(s)ds, \quad t \leq \tau, \tag{5}$$

where $g \in \mathrm{C}[t_0,\infty)$. It is easy to verify that $A^\rho(.;\tau,t)$ and $B^\rho(.;t,\tau)$ are linear operators and satisfy

$$A^\rho(g';\tau,t) = -\rho(\tau)h(t,\tau)g(\tau) + A^\rho\left(\left[\lambda_2 - \frac{\rho'}{\rho}\right]g;\tau,t\right), \quad t \geq \tau, \tag{6}$$

$$B^\rho(g';t,\tau) = \rho(\tau)h(\tau,t)g(\tau) - B^\rho\left(\left[\lambda_1 + \frac{\rho'}{\rho}\right]g;t,\tau\right), \quad t \leq \tau, \tag{7}$$

with $\lambda_1 = \lambda_1(s,t), \lambda_2 = \lambda_2(t,s)$.

Theorem 1. *Suppose there exist $h \in \mathcal{H}$ and $f, \rho \in \mathrm{C}^1[t_0,\infty)$ with $\rho > 0$ such that for each $T \geq t_0$, there exist a, b and c with $T \leq a < c < b$, and*

$$\frac{1}{h(c,a)}B^\rho(N(s,a);a,c) + \frac{1}{h(b,c)}A^\rho(M(b,s);c,b) \tag{8}$$

has a positive eigenvalue. Then system (1) *is oscillatory, where*

$$M(t,s) = G(s) - \frac{\alpha(s)}{4}\left[R(s) + \left(\lambda_2(t,s) - \frac{\rho'(s)}{\rho(s)}\right)I\right]^2,$$

$$N(s,t) = G(s) - \frac{\alpha(s)}{4}\left[R(s) - \left(\lambda_1(s,t) + \frac{\rho'(s)}{\rho(s)}\right)I\right]^2,$$

$G(t) = \alpha(t)\{Q(t) + f^2(t)I - f(t)R(t) - f'(t)I\}$ *and* $\alpha(t) = \exp(-2\int^t f(s)\,ds)$.

As special cases of Theorem 1, we have

Corollary 1. *Suppose there exist $h \in \mathcal{H}$ and $f, \rho \in \mathrm{C}^1[t_0, \infty)$ with $\rho > 0$ such that for each $T \geq t_0$, there exist a, b and c with $T \leq a < c < b$, and*

$$\begin{aligned} &\lambda_i \left\{ \frac{1}{h(c,a)} B^\rho(G; a, c) + \frac{1}{h(b,c)} A^\rho(G; c, b) \right\} \\ > \ & \frac{1}{4}\lambda_i \Bigg\{ \frac{1}{h(c,a)} B^\rho \left(\alpha(s) \left[R(s) - \left(\lambda_1(s,a) + \frac{\rho'(s)}{\rho(s)} \right) I \right]^2 ; a, c \right) \\ &\qquad + \frac{1}{h(b,c)} A^\rho \left(\alpha(s) \left[R(s) + \left(\lambda_2(b,s) - \frac{\rho'(s)}{\rho(s)} \right) I \right]^2 ; c, b \right) \Bigg\} \end{aligned}$$

for some $i \in \{1, 2, \ldots, n\}$. Then system (1) is oscillatory.

Corollary 2. *Suppose there exist $h \in \mathcal{H}$ and $f, \rho \in \mathrm{C}^1[t_0, \infty)$ with $\rho > 0$ such that for each $T \geq t_0$, there exist a, b and c with $T \leq a < c < b$, and*

$$\frac{1}{h(c,a)} B^\rho(\mathrm{tr}N(s,a); a, c) + \frac{1}{h(b,c)} A^\rho(\mathrm{tr}M(b,s); c, b) > 0. \tag{9}$$

Then system (1) is oscillatory.

Remark 1. Replacing "tr" in Corollary 2 by any positive linear functional, we can obtain new oscillation criteria for system (1).

Corollary 3. *Suppose there exist $h \in \mathcal{H}$ and $f, \rho \in \mathrm{C}^1[t_0, \infty)$ with $\rho > 0$ such that for each $r \geq t_0$,*

$$\limsup_{t \to \infty} B^\rho(\mathrm{tr}N(s,r); r, t) > 0, \tag{10}$$

$$\limsup_{t \to \infty} A^\rho(\mathrm{tr}M(t,s); r, t) > 0. \tag{11}$$

Then system (1) is oscillatory.

Theorem 2. *Suppose there exist $h \in \mathcal{H}_0$ and $f, \rho \in \mathrm{C}^1[t_0, \infty)$ with $\rho > 0$ such that for each $T \geq t_0$, there exist a, c with $T \leq a < c$, and one of the following conditions is fulfilled:*

(I) $B^\rho\left(\frac{\rho(2c-s)}{\rho(s)} M(2c-a, 2c-s) + N(s,a); a, c \right)$ *has a positive eigenvalue;*

(II) *there exist $i \in \{1, 2, \ldots, n\}$ such that*

$$\lambda_i \left\{ B^\rho \left(G(s) + \frac{\rho(2c-s)}{\rho(s)} G(2c-s); a, c \right) \right\} > \frac{1}{4} \lambda_i \left\{ B^\rho \left(U(s); a, c \right) \right\},$$

where

$$\begin{aligned} U(s) = & \frac{\rho(2c-s)}{\rho(s)} \alpha(2c-s) \left[R(2c-s) + \left(\lambda(s-a) - \frac{\rho'(2c-s)}{\rho(2c-s)} \right) I \right]^2 \\ & + \alpha(s) \left[R(s) - \left(\lambda(s-a) + \frac{\rho'(s)}{\rho(s)} \right) I \right]^2 ; \end{aligned}$$

(III) $B^\rho\left(\frac{\rho(2c-s)}{\rho(s)}\mathrm{tr}M(2c-a,2c-s)+\mathrm{tr}N(s,a);a,c\right)>0$.

Then system (1) *is oscillatory.*

3. Proofs

Proof of Theorem 1. Suppose to the contrary that there exists a prepared solution $Y(t)$ of (1) which is not oscillatory. Without loss of generality, we may suppose that $\det Y(t)\neq 0$ for $t\geq t_0$. Define

$$V(t)=a(t)(Y'(t)Y^{-1}(t)+f(t)I),\quad t\geq t_0. \tag{12}$$

Then $V(t)$ is symmetric, this and (1) imply the Riccati equation

$$V'(t)+\frac{1}{\alpha(t)}V^2(t)+R(t)V(t)+G(t)=0,\quad t\geq t_0. \tag{13}$$

From (12) and the definition of a prepared solution, we get that RV is symmetric. Denote $W(t)=V(t)+\frac{\alpha(t)R(t)}{2}$. Let $t_0\leq T\leq a<c<b$. Applying $A^\rho(\cdot;c,t)$ $(c<t\leq b)$ to (13) and using (4), we get

$$A^\rho\left(\left[\lambda_2-\frac{\rho'}{\rho}\right]V;c,t\right)+A^\rho\left(\frac{1}{\alpha}W^2;c,t\right)+A^\rho\left(G-\frac{\alpha}{4}R^2;c,t\right)\leq\rho(c)h(t,c)V(c).$$

This yields

$$A^\rho\left(\frac{1}{\alpha}\left[W+\frac{\alpha}{2}(\lambda_2-\frac{\rho'}{\rho})I\right]^2;c,t\right)+A^\rho(M;c,t)\leq\rho(c)h(t,c)V(c).$$

Note that the first term is nonnegative, so

$$A^\rho(M;c,t)\leq\rho(c)h(t,c)V(c).$$

In particular, assuming that $t=b$, we have

$$\frac{1}{h(b,c)}A^\rho(M(b,s);c,b)\leq\rho(c)V(c). \tag{14}$$

On the other hand, applying $B^\rho(\cdot;t,c)$ $(a\leq t<c)$ to (13) and using (5), we get

$$-B^\rho\left(\left[\lambda_1+\frac{\rho'}{\rho}\right]V;t,c\right)+B^\rho\left(\frac{1}{\alpha}W^2;t,c\right)+B^\rho\left(G-\frac{\alpha}{4}R^2;t,c\right)$$
$$\leq-\rho(c)h(c,t)V(c).$$

This yields

$$B^\rho\left(\frac{1}{\alpha}\left[W-\frac{\alpha}{2}\left(\lambda_1+\frac{\rho'}{\rho}\right)I\right]^2;t,c\right)+B^\rho(N;t,c)\leq-\rho(c)h(c,t)V(c).$$

Also, the first term is nonnegative, so

$$B^\rho(N;t,c)\leq-\rho(c)h(c,t)V(c).$$

In particular, assuming that $t = a$, we obtain

$$\frac{1}{h(c,a)} B^{\rho}(N(s,a);a,c) \le -\rho(c)V(c). \tag{15}$$

Now, (14) and (15) imply that the matrix defined by (8) is negative semi-definite, then all the eigenvalues are non-positive, which contradicts the hypothesis. This completes the proof of Theorem 1. □

The *proofs* of Corollary 1 and Corollary 2 are based on the knowledge of eigenvalues and traces of the given matrices (see [15] for details), here we omit the details. □

Proof of Corollary 3. For any $T \ge t_0$, let $a = T$. We choose $r = a$ in (10). Then there exists $c \ge a$ such that

$$B^{\rho}(\mathrm{tr}N(s,a);a,c) > 0. \tag{16}$$

In (11), we choose $r = c$. Then there exists $b \ge c$ such that

$$A^{\rho}(\mathrm{tr}M(b,s);c,b) > 0. \tag{17}$$

Combining (16) and (17) we obtain (9). The conclusion thus comes from Corollary 2. □

Proof of Theorem 2. Let $b = 2c - a$. Then $h(b-c) = h(c-a)$, and for any function $g \in \mathrm{L}[a,b]$, we have

$$\int_c^b g(s)ds = \int_a^c g(2c-s)ds.$$

Now, Theorem 1 and Corollaries 1 and 2 imply that system (1) of oscillatory. □

4. Examples

Example 1. Consider the following 2-dimensional systems

$$Y'' + R(t)Y' + Q(t)Y = 0, \quad t \ge 1, \tag{18}$$

where $R(t) = \frac{1}{t^2} I_2$,

$$Q(t) = \begin{cases} \mathrm{diag}\left(\frac{\gamma}{t^3}, \frac{\mu}{t^3}\right)(t-3n), & 3n \le t \le 3n+1, \\ \mathrm{diag}\left(\frac{\gamma}{t^3}, \frac{\mu}{t^3}\right)(-t+3n+2), & 3n+1 < t \le 3n+2, \\ F(t), & 3n+2 < t \le 3n+3, \end{cases}$$

$\gamma \ge \mu > 0$, $F(t)$ is an arbitrary 2×2 matrix function such that $Q(t)$ is a continuous function, $n \in N_0 = \{1, 2, \cdots\}$. Suppose that $\gamma > \frac{1}{4}$. Take $h(s-a) =$

$(s-a)^2$, $f(t)=-1/(2t)$ and $\rho\equiv 1$ in Theorem 1. It is easy to verify that

$$G(t)=\begin{cases}\begin{bmatrix}\frac{1}{t}(\gamma-\frac{1}{4})+\frac{l_1}{2t^2} & 0\\ 0 & \frac{1}{t}(\mu-\frac{1}{4})+\frac{l_2}{2t^2}\end{bmatrix}, & 3n\le t\le 3n+1,\\ \begin{bmatrix}-\frac{1}{t}(\gamma+\frac{1}{4})+\frac{l_3}{2t^2} & 0\\ 0 & -\frac{1}{t}(\mu+\frac{1}{4})+\frac{l_4}{2t^2}\end{bmatrix}, & 3n+1\le t\le 3n+2,\\ F^*(t), & 3n+2<t\le 3n+3,\end{cases}$$

where $l_1=1-6n\gamma, l_2=1-6n\mu, l_3=1+(6n+4)\gamma, l_4=1+(6n+4)\mu$ are constants, $F^*(t)$ are some matrix functions such that $G(t)$ is continuous for $t\in[1,\infty)$. For each $T\ge 0$, let n be large enough so that $3n\ge T$. Let $a=3n, c=3n+1, b=3n+2$. Since $R(t)$ and $Q(t)$ are diagonal matrices, we obtain that one of the eigenvalues of $\frac{1}{h(c,a)}B^\rho(N(s,a);a,c)+\frac{1}{h(b,c)}A^\rho(M(b,s);c,b)$ is

$$\begin{aligned}&\lambda_i\left\{\frac{1}{h(c,a)}B^\rho(N(s,a);a,c)+\frac{1}{h(b,c)}A^\rho(M(b,s);c,b)\right\}\\
&=\int_{3n}^{3n+1}(s-3n)^2\left[\frac{1}{s}\left(\gamma-\frac{1}{4}\right)+\frac{l_1}{2s^2}-\frac{s}{4}\left(\frac{1}{s^2}-\frac{2}{s-3n}\right)^2\right]ds\\
&\quad+\int_{3n+1}^{3n+2}(3n+2-s)^2\left[-\frac{1}{s}\left(\gamma+\frac{1}{4}\right)+\frac{l_3}{2s^2}-\frac{s}{4}\left(\frac{1}{s^2}-\frac{2}{3n+2-s}\right)^2\right]ds\\
&=\int_{3n}^{3n+1}(s-3n)^2\left[\frac{1}{s}\left(\gamma-\frac{1}{4}\right)+\frac{l_1}{2s^2}-\frac{s}{4}\left(\frac{1}{s^2}-\frac{2}{s-3n}\right)^2\right]ds\\
&\quad+\int_{3n}^{3n+1}(s-3n)^2\Bigg[\frac{\gamma+\frac{1}{4}}{s-6n-2}+\frac{l_3}{2(6n+2-s)^2}\\
&\qquad\qquad-\frac{6n+2-s}{4}\left(\frac{1}{(6n+2-s)^2}-\frac{2}{s-3n}\right)^2\Bigg]ds\\
&>0\qquad\left(\text{since }\gamma>\frac{1}{4}\right).\end{aligned}$$

Consequently, by Theorem 1, we obtain that system (18) is oscillatory for $\gamma>\frac{1}{4}$. Nevertheless, $F(t)$ can be chosen as a "bad" term of $Q(t)$ such that either the upper limit in Theorem 1 of paper [13] exists or tends to $-\infty$. So our theorems are improvements of those criteria.

Example 2. Consider the following 2-dimensional systems

$$Y''+R(t)Y'+Q(t)Y=0,\quad t\ge 0,\tag{19}$$

where $R(t) = \cos t I_2, Q(t) = \begin{pmatrix} 2\sin t - \cos t & -\sin t + \cos t \\ 2\sin t - 2\cos t & -\sin t + 2\cos t \end{pmatrix}$. Taking $h(t,s) = (t-s)^2$, $f(t) \equiv 0$ and $\rho(t) = \exp(\sin t)$ in Theorem 2 (II), we obtain $\lambda(t,s) = \frac{2}{t-s}$, $G(t) = Q(t)$. For each $T \geq 0$, let n be large enough such that $a = 2n\pi \geq T$ and $c = 2(n+1)\pi + \frac{\pi}{2}$, then $\sin(2c-s) = \sin s$, and $\rho(t) = \rho(2c-t)$.

$$\begin{aligned} & B^\rho\left(G(s) + \frac{\rho(2c-s)}{\rho(s)} G(2c-s); a, c\right) \\ &= \int_a^c (s-a)^2 \exp(\sin s)(Q(s) + Q(2c-s))ds \\ &= \int_{2n\pi}^{2(n+1)\pi+\frac{\pi}{2}} (s-2n\pi)^2 \exp(\sin s) \begin{bmatrix} 4\sin s & -2\sin s \\ 4\sin s & -2\sin s \end{bmatrix} ds \\ &= \begin{bmatrix} 4A & -2A \\ 4A & -2A \end{bmatrix}, \end{aligned}$$

where

$$A \triangleq \int_{2n\pi}^{2(n+1)\pi+\frac{\pi}{2}} (s-2n\pi)^2 \sin s \, \exp(\sin s) ds = \int_0^{\frac{5\pi}{2}} s^2 \sin s \, \exp(\sin s) ds.$$

Hence one of its eigenvalues is

$$\lambda_i \left\{ B^\rho\left(G(s) + \frac{\rho(2c-s)}{\rho(s)} G(2c-s); a, c\right)\right\} = 2\int_0^{\frac{5\pi}{2}} s^2 \sin s \, \exp(\sin s) ds.$$

On the other hand, we compute that

$$\begin{aligned} & \frac{1}{4}\lambda_i \left\{B^\rho\left(U(s); a, c\right)\right\} \\ =\; & \frac{1}{4}\int_a^c (s-a)^2 \exp(\sin s) \left[\left(\frac{2}{s-a}\right)^2 + \left(\frac{2}{s-a}\right)^2\right] ds \\ =\; & 2 \int_{2n\pi}^{2(n+1)\pi+\frac{\pi}{2}} \exp(\sin s) ds = 2\int_0^{\frac{5\pi}{2}} \exp(\sin s) ds. \end{aligned}$$

Since

$$\int_0^{\frac{5\pi}{2}} s^2 \sin s \exp(\sin s) ds > \int_0^{\frac{5\pi}{2}} \exp(\sin s) ds, \tag{20}$$

system (19) is oscillatory by (20) and Theorem 2 (II).

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