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ON A NORMAL OF THE DISTRIBUTION OF HYPERPLANE ELEMENTS IN AN AFFINE SPACE

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Abstract. Using the Cartan–Laptev invariant analytic method, an invariant affine normal (E) is constructed, which is intrinsically connected with the distribution of hyperplane elements in the $(n+1)$-dimensional affine space.

For the normal (E) we define a second kind normal and an invariant $(n-1)$-dimensional plane lying in the plane of the element and not passing through the center and corresponding to this normal in the Bompiani–Pantazi projectivity. An invariant point of intersection of the two-dimensional plane passing through the normals (L) and (E) with the second kind normal is found.

The construction is carried out without assuming that the nonholonomy tensor is different from zero. Hence both nonholonomic and holonomic distributions are framed by the constructed normal.

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When dealing with differentiable mappings of smooth manifolds which are framed with structures defined by systems of linear, linearly independent differential forms, we observe the generation of fibre manifolds connected with geometric objects. In particular, the latter manifolds include a space of representations of Lie groups [3].

Using the $(n+1)$-dimensional representation of the affine group – of the point affine space A_{n+1} – let us consider the manifold of hyperplane elements in A_{n+1}. This manifold forms a fibre space over the initial affine space whose fibers are sheafs of the hyperplanes corresponding to the points of the base.

As is known [3], the field, which to each point of the space assigns one of the elements corresponding to this point, is called the distribution of hyperplane elements. In other words, the distribution of hyperplane elements is the section in the fibre space of these elements.

This distribution is associated with the Pfaff equation which in the holonomic case becomes completely integrable and defines the family of hypersurfaces enveloping the distribution elements.

In [1] and [2] we discussed the questions as to how the distribution is framed in an invariant manner. In those papers we constructed both affine and projective normals and considered sheafs of these normals. The aim of the present paper is to construct one more invariant affine normal connected intrinsically with the distribution.

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1. Differential equations of the distribution of hyperplane elements in the $(n+1)$-dimensional affine space with respect to the zero order frame have the form

$$\omega_i^{n+1} = L_{i\alpha}\omega^\alpha \quad (i,j,k=\overline{1,n}, \quad \alpha,\beta,\gamma=\overline{1,n+1}), \tag{1}$$

where $L_{ij} \neq L_{ji}$.

Differential continuation of system (1) generates a sequence of fundamental geometric objects

$$(L_{i\alpha}) \subset (L_{i\alpha}, L_{i\alpha\beta}) \subset (L_{i\alpha}, L_{i\alpha\beta}, L_{i\alpha\beta\gamma}) \subset \cdots,$$

by means of which we construct the differential geometry of the distribution.

2. In a second order neighborhood of the distribution, let us consider the objects

$$\begin{aligned} A_{ij} &= L_{ijs}L^{sm}L_{mn+1} - L_{ijn+1} = -L_{ijn+1} - L_{ijs}L^s, \\ A_{in+1} &= -L_{in+1n+1} - L_{in+1l}L^l. \end{aligned}$$

By direct differentiation we get

$$dA_{ij} - A_{il}\omega_j^l - A_{lj}\omega_i^l = A_{ij\alpha}\omega^\alpha, \tag{2}$$

$$dA_{in+1} - A_{ln+1}\omega_i^l - A_{in+1}\omega_{n+1}^{n+1} - A_{il}\omega_{n+1}^l = A_{in+1\alpha}\omega^\alpha. \tag{3}$$

The values A_{ij} form the twice-covariant tensor.

In general, $A_0 = \det(A_{ij}) \neq 0$. Indeed, if we assume that the fundamental objects of the distribution have values

$$(L_{ij}) = \begin{pmatrix} 1 & 3 & 3 & \dots & 3 \\ 1 & 1 & 3 & \dots & 3 \\ 1 & 1 & 1 & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 3 & \dots & 1 \end{pmatrix}; \quad L_{ijk} = 0 \quad \text{for} \quad j \neq k;$$

$$L_{ijj} = \begin{cases} 1 & \text{for} \quad i \neq j, \\ -(4j-1) & \text{for} \quad i = j; \end{cases} \quad L_{ijn+1} = jL_{ij};$$

$$(L_{ijn+1}) = \begin{pmatrix} 1 & 6 & 9 & \dots & 3n \\ 1 & 2 & 9 & \dots & 3n \\ 1 & 2 & 3 & \dots & 3n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{pmatrix};$$

$$L_{sn+1n+1} = -\sum_j jL_{sj} = s(s+1) - \frac{3n(n+1)}{2}.$$

Then the components of the objects we have constructed take the following values:

$$L_0 = \det(L_{ij}) = (-1)^{n+1}2^{n-1}; \quad L^i = -L^{ik}L_{kn+1} = 0;$$

$$A_{ij} = -jL_{ij}; \quad A_0 = \det(A_{ij}) = \begin{vmatrix} -1 & -6 & -9 & \dots & -3n \\ -1 & -2 & -9 & \dots & -3n \\ -1 & -2 & -3 & \dots & -3n \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & -n \end{vmatrix} = (-1)^{2n+1} 2n!;$$

$$A_{in+1} = -i(i+1) + \frac{3n(n+1)}{2}\,.$$

Thus for the given initial values, the determinant A_0 is not equal to zero and therefore it will also differ from zero in some neighborhood of the given initial values, i.e., it will vanish only for some special distributions.

Let us assume that

$$A_0 = \det(A_{ij}) \neq 0.$$

This allows us to introduce the values of A^{ij} satisfying the conditions

$$A^{ik}A_{kj} = \delta^i_j, \quad A^{ik}A_{ji} = \delta^k_j.$$

By virtue of equation (2) we can write

$$dA^{ij} + A^{il}\omega^j_l + A^{lj}\omega^i_l = A^{ii}_\alpha \omega^\alpha. \tag{4}$$

Using the components of the fundamental second order object of the distribution (1) we construct another object

$$E_k = L_{kn+1n+1} - L^l(L^s L_{kls} + L_{kn+1}L_{ln+1}),$$

whose components satisfy the equations

$$dE_k - E_l\omega^l_k - E_k\omega^{n+1}_{k+1} + 2A_{kl}\omega^l_{n+1} = E_{k\alpha}\omega^\alpha. \tag{5}$$

3. The distribution of hyperplane elements in the $(n+1)$-dimensional affine space is called framed if at each point the vector

$$\vec{\nu} = \nu^i \vec{e}_i + \vec{e}_{n+1}$$

is given, which does not belong to the hyperplane at this point.

The invariant field of normals of this distribution is defined by the geometric object (ν^i) whose components satisfy the differential equations [1]

$$d\nu^i + \nu^l\omega^i_l - \nu^i\omega^{n+1}_{n+1} + \omega^i_{n+1} = \nu^i_\alpha \omega^\alpha. \tag{6}$$

Using the components A^{ik} and E_k, let us compose the system of values

$$E^i = \frac{1}{2}\,A^{ik}E_k. \tag{7}$$

After differentiating (7) with (4) and (5) taken into account, we find that

$$dE^i + E^l\omega^i_l - E^i\omega^{n+1}_{n+1} + \omega^i_{n+1} = E^i_\alpha\omega^\alpha.$$

Thus the constructed object E^i satisfies differential equations of form (6). Therefore at each point M it defines an invariant vector passing through the point M and not lying in the plane of the element.

Thus we have obtained the invariant vector field

$$\vec{E} = E^i \vec{e}_i + \vec{e}_{n+1}$$

which is intrinsically connected with the distribution.

The values

$$Q^i = -A^{ik} A_{kn+1}$$

also define the invariant affine normal of the distribution.

Note that the normal (E^i) lies in the sheaf of normals

$$\xi^i(\alpha) = L^i - \alpha q^i, \tag{8}$$

where

$$q^i = L^i - Q^i,$$

and emerges from the sheaf (8) for $\alpha = \frac{1}{2}$, i.e., has the form

$$E^i = \frac{1}{2}(L^i + Q^i).$$

4. To each straight line passing through the point M the Bompiani–Pantazi projectivity assigns the hyperplane element characteristic corresponding to a displacement of the point M along this straight line. The hyperplane characteristic is defined by the equation [1]

$$x^l L_{l\alpha} \omega^\alpha + \omega^{n+1} = 0.$$

If we require that M displaces along a curve tangent to the distribution of normals (ν^i): $\omega^i = \nu^i \omega^{n+1}$, then we will have

$$\pi_i x^i - 1 = 0, \quad x^{n+1} = 0,$$

where

$$\pi_i = -L_{ik} \nu^k - L_{in+1}. \tag{9}$$

The values π_i form the geometric object whose components satisfy the following system of differential equations:

$$d\pi_i - \pi_l \omega_i^l = \pi_{i\alpha} \omega^\alpha.$$

There arises a one-to-one correspondence between the first kind normals and the $(n-1)$-dimensional planes lying in the plane of the element and not passing through the center (second kind normals).

In the Bompiani–Pantazi projectivity, to the one-parametric sheaf of normals $\xi^i(\alpha)$ there corresponds the sheaf of parallel $(n-1)$-dimensional planes lying in the plane of the element.

Indeed, if we define the object $\pi_i(x)$ for sheaf (8), then by (9) we obtain

$$\pi_i(\alpha) = -L_{ik} \xi^k(\alpha) - L_{in+1} = -L_{ik}(L^k - \alpha q^k) - L_{in+1} = \alpha q_i.$$

Thus, in the Bompiani–Pantazi projectivity, to the sheaf of normals $\xi^i(\alpha)$ there corresponds the sheaf of second kind normals

$$\alpha q_i x^i - 1 = 0, \quad x^{n+1} = 0,$$

i.e., the sheaf of parallel $(n-1)$-dimensional planes lying in the plane of the element.

In particular, an equation of the second kind normal that corresponds to the normal (E^i) has the form

$$\frac{1}{2}\,q^i x^i - 1 = 0, \quad x^{n+1} = 0. \tag{10}$$

Let us consider the convolution

$$q = L^{ik} q_i q_k,$$

and write the differential equation of the object q

$$dq - q\omega_{n+1}^{n+1} = q_\alpha \omega^\alpha.$$

Hence q is a relative invariant [1].

The two-dimensional plane passing through the normals (L^i) and (Q^i)

$$x^i = L^i x^{n+1} + \lambda(Q^i - L^i) = L^i x^{n+1} - \lambda q^i,$$

intersects the $(n-1)$-dimensional plane (10) at the invariant point

$$x^i = \frac{2}{q}\,q^i, \quad x^{n+1} = 0.$$

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