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ON WEAKLY PRIMARY IDEALS

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Abstract. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied in [1]. Here we study the weakly primary ideals of a commutative ring. We define a proper ideal P of R to be weakly primary if $0 \neq pq \in P$ implies $p \in P$ or $q \in \mathrm{Rad}(P)$, so every weakly prime ideal is weakly primary. Various properties of weakly primary ideals are considered. For example, we show that a weakly primary ideal P that is not primary satisfies $\mathrm{Rad}(P) = \mathrm{Rad}(0)$. Also, we show that an intersection of a family of weakly primary ideals that are not primary is weakly primary.

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1. Introduction

Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [1]. Here we study the weakly primary ideals of a commutative ring. The weakly prime and weakly primary ideals are different concepts. Some of our results are analogous to the results given in [1]. The corresponding results are obtained by modification and here we give a number of results concerning weakly primary ideals and weakly primary submodules (see Section 2).

Before we state some results let us introduce some notation and terminology. Throughout this paper all rings will be commutative with identity. A proper ideal P of R is said to be weakly prime if $0 \neq ab \in P$, then either $a \in P$ or $b \in P$ (see [1]). If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. An R-module M is secondary if $0 \neq M$ and, for each $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent. This implies that $\mathrm{nilrad}(M) = P$ is a prime ideal of R, and M is said to be P-secondary. A secondary ideal of R is just a secondary submodule of the R-module R (see [3]). Let N be an R-submodule of M. Then N is pure in M if any finite system of equations over N which is solvable in M is also solvable in N. So if N is pure in M, then $IN = N \cap IM$ for each ideal I of R. An R-module is absolutely pure if it is pure in every module that contains it as a submodule. An important property of regular rings is that every module is absolutely pure (see [4]).

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2. Weakly Primary Ideals

We define a proper ideal P of a commutative ring R to be weakly primary if $0 \neq pq \in P$ implies $p \in P$ or $q \in \mathrm{Rad}(P)$. So a primary ideal is weakly primary. However, since 0 is always weakly primary (by definition), a weakly primary ideal does not need to be primary. Moreover, since every weakly prime is weakly primary, so over a quasilocal ring (R, P) with $P^2 = 0$, every proper ideal is weakly primary (see [1]).

Proposition 2.1. *Let R be a commutative ring, and let P be a proper ideal of R. Then the following assertion are equivalent.*
(i) *P is a weakly primary ideal of R.*
(ii) *For $a \in R - \mathrm{Rad}(P)$, $(P : Ra) = P \cup (0 : Ra)$.*
(iii) *For $a \in R - \mathrm{Rad}(P)$, $(P : Ra) = P$ or $(P : Ra) = (0 : Ra)$.*

Proof. (i) $\Longrightarrow$ (ii) Let $a \in R - \mathrm{Rad}(P)$. Clearly, $P \cup (0 : Ra) \subseteq (P : Ra)$. For the other inclusion, suppose that $b \in (P : Ra)$, so $ab \in P$. If $ab \neq 0$, then $b \in P$ since P is weakly primary. If $ab = 0$, then $b \in (0 : Ra)$, so we have the equality. (ii) $\Longrightarrow$ (iii) is clear.

(iii) $\Longrightarrow$ (i) Let $0 \neq ab \in P$ with $a \notin \mathrm{Rad}(P)$. Then $b \in (P : Ra) = P \cup (0 : Ra)$ by (iii); hence $b \in P$ since $ab \neq 0$, as required. □

Theorem 2.2. *Let R be a commutative ring, and let P be a weakly primary ideal that is not primary. Then* $\mathrm{Rad}(P) = \mathrm{Rad}(0)$.

Proof. First, we prove that $P^2 = 0$. Suppose that $P^2 \neq 0$; we show that P is primary. Let $pq \in P$, where $p, q \in R$. If $pq \neq 0$, then either $p \in P$ or $q \in \mathrm{Rad}(P)$ since P is weakly primary. So suppose that $pq = 0$. If $pP \neq 0$, then there is an element p' of P such that $pp' \neq 0$, so $0 \neq pp' = p(p' + q) \in P$, and hence P weakly primary gives either $p \in P$ or $(p' + q) \in \mathrm{Rad}(P)$. As $p' \in P \subseteq \mathrm{Rad}(P)$ we have either $p \in P$ or $q \in \mathrm{Rad}(P)$. So we can assume that $pP = 0$. Similarly, we can assume that $qP = 0$. Since $P^2 \neq 0$, there exist $c, d \in P$ such that $cd \neq 0$. Then $(p + c)(q + d) = cd \in P$, so either $p + c \in P$ or $q + d \in \mathrm{Rad}(P)$, and hence either $p \in P$ or $q \in \mathrm{Rad}(P)$. Thus P is primary.

Clearly, $\mathrm{Rad}(0) \subseteq \mathrm{Rad}(P)$. As $P^2 = 0$, we get $P \subseteq \mathrm{Rad}(0)$; hence $\mathrm{Rad}(P) \subseteq \mathrm{Rad}(0)$, as required. □

Recall that, in general, the intersection of a family of primary ideals is not primary, but we have the following result.

Theorem 2.3. *Let R be a commutative ring, and let $\{P_i\}_{i \in I}$ be a family of weakly primary ideals of R that are not primary. Then $P = \bigcap_{i \in I} P_i$ is a weakly primary ideal of R.*

Proof. First, we show that $\mathrm{Rad}(P) = \bigcap_{i \in I} \mathrm{Rad}(P_i)$. Clearly, $\mathrm{Rad}(P) \subseteq \bigcap_{i \in I} \mathrm{Rad}(P_i)$. For the other inclusion suppose that $r \in \bigcap_{i \in I} \mathrm{Rad}(P_i)$, so $r^m = 0$ for some m since $\bigcap_{i \in I} \mathrm{Rad}(P_i) = \mathrm{Rad}(0)$ by Theorem 2.2. It follows that $r^m \in P_i$ for each $i \in I$, and hence $r \in \mathrm{Rad}(P)$.

As $\mathrm{Rad}(P) = \mathrm{Rad}(0) \neq R$, we have that P is a proper ideal of R. Suppose that $a, b \in R$ are such that $0 \neq ab \in P$ but $b \notin P$. Then there is an element $j \in I$ such that $b \notin P_j$ and $ab \in P_j$. It follows that $a \in \mathrm{Rad}(P_j) = \mathrm{Rad}(P)$ since P_i is weakly primary, as needed. □

Corollary 2.4. *Let R be a commutative ring, and let $\{P_i\}_{i\in I}$ be a family of weakly prime ideals of R that are not prime. Then $P = \bigcap_{i\in I} P_i$ is a weakly prime ideal of R.*

Proof. By [1, Theorem 1], $P_i^2 = 0$, $(i \in I)$, so $\mathrm{Rad}(P_i) = \mathrm{Rad}(0)$ for every $i \in I$. Since $\mathrm{Rad}(P) = \mathrm{Rad}(0) \neq R$, we have that P is a proper ideal of R. Now it is easy to see that P is a weakly prime ideal of R. □

The following lemma is well-known, but we give it for the sake of references.

Lemma 2.5. *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Then the following hold:*
(i) *If I_1 is an ideal of R_1, then $\mathrm{Rad}(I_1 \times R_2) = \mathrm{Rad}(I_1) \times R_2$.*
(ii) *If I_2 is an ideal of R_2, then $\mathrm{Rad}(R_1 \times I_2) = R_1 \times \mathrm{Rad}(I_2)$.*

Theorem 2.6. *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Then the following hold:*
(i) *If P_1 is a primary ideal of R_1, then $P_1 \times R_2$ is a primary ideal of R.*
(ii) *If P_2 is a primary ideal of R_2, then $R_1 \times P_2$ is a primary ideal of R.*
(iii) *If P is a weakly primary ideal of R, then either $P = 0$ or P is primary.*

Proof. (i) Let $(a, b)(c, d) = (ac, bd) \in P_1 \times R_2$ where $(a, b), (c, d) \in R$, so either $a \in P_1$ or $c \in \mathrm{Rad}(P_1)$ since P_1 is primary. It follows that either $(a, b) \in P_1 \times R_2$ or $(c, d) \in \mathrm{Rad}(P_1) \times R_2 = \mathrm{Rad}(P_1 \times R_2)$ by Lemma 2.5. Thus $P_1 \times R_2$ is primary. Likewise, $R_1 \times P_2$ is primary.

(ii) This proof is similar to that in case (i) and we omit it.

(ii) Let $P = P_1 \times P_2$ be a weakly primary ideal of R. We can assume that $P \neq 0$. So there is an element (a, b) of P with $(a, b) \neq (0, 0)$. Then $(0, 0) \neq (a, 1)(1, b) \in P$ gives either $(a, 1) \in P$ or $(1, b) \in \mathrm{Rad}(P)$. If $(a, 1) \in P$, then $P = P_1 \times R_2$. We show that P_1 is primary; hence P is primary by (i). Let $cd \in P_1$, where $c, d \in R_1$. Then $(0, 0) \neq (c, 1)(d, 1) = (cd, 1) \in P$, so either $(c, 1) \in P$ or $(d, 1) \in \mathrm{Rad}(P) = \mathrm{Rad}(P_1) \times R_2$ by Lemma 2.5 and hence either $c \in P_1$ or $d \in \mathrm{Rad}(P_1)$. If $(1, b) \in \mathrm{Rad}(P)$, then $(1, b^n) \in P$ for some n, so $P = R_1 \times P_2$. By a similar argument, $R_1 \times P_2$ is primary. □

Theorem 2.7. *Let I be a secondary ideal of a commutative ring R. Then if Q is a weakly primary ideal* (*resp. weakly prime ideal*) *of R, then $I \cap Q$ is secondary.*

Proof. Let I be a P-secondary ideal of R, and let $a \in R$. If $a \in P$, then $a^n(I \cap Q) \subseteq a^n I = 0$ for some integer n. If $a \notin P$, then $aI = I$. We show that $a(I \cap Q) = I \cap Q$. It is enough to show that $I \cap Q \subseteq a(I \cap Q)$. Suppose that $c \in I \cap Q$. We may assume that $c \neq 0$. Then there exists $b \in I$ such that $0 \neq c = ab \in Q$ since $aI = I$. Therefore either $a \in \mathrm{Rad}(Q)$ or $b \in Q$.

If $a \in \mathrm{Rad}(Q)$, then $a^k \in Q$ for some integer k, so $a^k I = I \subseteq Q$, and hence $a(I \cap Q) = aI = I = I \cap Q$. Otherwise, $c = ab \in a(I \cap Q)$, as required. □

Proposition 2.8. *Let P and Q be weakly primary ideals of a commutative ring R and S a multiplicatively closed set of R with* $\mathrm{Rad}(P) \cap S = \varnothing$. *Then $S^{-1}P$ is weakly primary in $S^{-1}R$. In particular, if P is not primary, then* $(S^{-1}P)^2 = 0$.

Proof. As $\mathrm{Rad}(P) \cap S = \varnothing$, $S^{-1}P \neq S^{-1}R$. Suppose $0 \neq (r/s)(a/t) = (ra)/st \in S^{-1}P$ with $a/t \notin S^{-1}P$ where $r, a \in R$ and $s, t \in S$. Then there exist $b \in P$ and $u \in S$ such that $(ra)/st = b/u$, so $0 \neq rauv = stbv \in P$ for some $v \in S$. For $0 \neq uva \in P$, if $a \notin P$ and P are weakly primary, then $uv \in \mathrm{Rad}(P) \cap S$, a contradiction. So we may assume that $uva \notin P$; hence $r \in \mathrm{Rad}(P)$. Thus $r/s \in S^{-1}\mathrm{Rad}(P) = \mathrm{Rad}(S^{-1}P)$, as required.

Finally, $(S^{-1}P)^2 = S^{-1}P^2 = 0$. □

Theorem 2.9. *Let R be a commutative ring and let I be a secondary ideal of R. Then if Q is a proper weakly primary* (*resp. weakly prime*) *subideal of I, then Q is secondary.*

Proof. (i) Let I be P-secondary, and let $a \in R$. If $a \in P$, then there is an integer m such that $a^m Q \subseteq a^m I = 0$. If $a \notin P$, then $aI = I$, so $a^k I = I$ for every integer k. We want to show that $aQ = Q$. Clearly, $aQ \subseteq Q$. For the other inclusion, suppose that $q \in Q$. We may assume that $q \neq 0$. Then there is an element $b \in I$ such that $0 \neq q = ab \in Q$, so if Q is weakly primary, then either $a \in \mathrm{Rad}(Q)$ or $b \in Q$. If $a \in \mathrm{Rad}(Q)$, then $a^n \in Q$ for some n, so $a^n I = I \subseteq Q$, a contradiction. Thus $b \in Q$, and hence $q = ab \in aQ$, as required. □

Proposition 2.10. *Let $I \subseteq P$ be proper ideals of a commutative ring R. Then the following hold:*

(i) *If P is weakly primary, then P/I is weakly primary.*

(ii) *If I and P/I are weakly primary* (*resp. weakly prime*), *then P is weakly primary* (*resp. weakly prime*).

Proof. (i) Let $0 \neq (a + I)(b + I) = ab + I \in P/I$ where $a, b \in R$, so $ab \in P$. If $ab = 0 \in I$, then $(a + I)(b + I) = 0$, a contradiction. So if P is weakly primary, then either $a \in P$ or $b \in \mathrm{Rad}(P)$; hence either $a + I \in P/I$ or $b^n + I = (b + I)^n \in P/I$ for some integer n, as required.

(ii) Let $0 \neq ab \in P$ where $a, b \in R$, so $(a + I)(b + I) \in P/I$. For $ab \in I$, if I is weakly primary, then either $a \in I \subseteq P$ or $b \in \mathrm{Rad}(I) \subseteq \mathrm{Rad}(P)$. So we may assume that $ab \notin I$. Then either $a + I \in P/I$ or $b^m + I \in P/I$ for some integer m. It follows that either $a \in P$ or $b \in \mathrm{Rad}(P)$, as needed. □

Theorem 2.11. *Let P and Q be weakly primary ideals of a commutative ring R that are not primary. Then $P + Q$ is a weakly primary ideal of R. In particular,* $\mathrm{Rad}(P + Q) = \mathrm{Rad}(P)$.

Proof. By Theorem 2.2, we have $\mathrm{Rad}(P) + \mathrm{Rad}(Q) = \mathrm{Rad}(0) \neq R$, so $P + Q$ is a proper ideal of R by [2, p. 53]. Since $(P + Q)/Q \cong Q/(P \cap Q)$, we get that $(P+Q)/Q$ is weakly primary by Propositin 2.10 (i). Now the assertion follows from Proposition 2.10 (ii). Finally, by [2, p. 53] we have $\mathrm{Rad}(P+Q) = \mathrm{Rad}(\mathrm{Rad}(P)+\mathrm{Rad}(Q)) = \mathrm{Rad}(\mathrm{Rad}(0)) = \mathrm{Rad}(0)$. $\square$

Now we state and prove a version of Nakayama's lemma.

Theorem 2.12. *Let P be a weakly primary ideal of a commutative ring R that is not primary. Then the following hold:*

(i) *$P \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R.*

(ii) *If M is an R-module and $PM = M$, then $M = 0$.*

(iii) *If M is an R-module and N is a submodule of M such that $PM+N = M$, then $M = N$.*

Proof. (i) Let $a \in P$. We may assume that $a \neq 0$. It is enough to show that for every $b \in R$, the element $1 - ba$ is the unit of R. As $P^2 = 0$, we have $1 = 1 - b^2a^2 = (1-ba)(1+ba)$, so $a \in J(R)$, as required.

(ii) since $PM = M$, we have $M = PM = P^2M = 0$.

(iii) This follows from (ii). $\square$

Corollary 2.13. *Let R be a commutative ring. Then the following hold:*

(i) *Let P a non-zero proper ideal of an integral domain R. Then P is weakly primary if and only if P is primary.*

(ii) *If P is a pure weakly primary ideal of R that is not primary, then $P = 0$.*

(iii) *If R is regular, then the only weakly primary ideals of R that are not primary can only be* 0.

Proof. (i) Let P be a weakly primary ideal of R, and let $pq \in P$ where $p, q \in R$. For $pq \neq 0$, if P is weakly primary, then either $p \in P$ or $q \in \mathrm{Rad}(P)$. If $pq = 0$, then $p = 0 \in P$ or $q = 0 \in \mathrm{Rad}(P)$ since R is a domain.

(ii) Let P be a weakly primary ideal of R that is not primary. Since P is a pure ideal, we get $P = P^2 = 0$ by Theorem 2.2.

(iii) This follows from (ii) since over R, every ideal is a pure ideal. $\square$

A proper submodule N of a module M over a commutative ring R is said to be a weakly primary submodule if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^nM \subseteq N$ for some n. Clearly, every primary submodule of a module is a weakly primary submodule. However, since 0 is always weakly primary (by definition), a weakly primary submodule need not be primary.

Compare the following theorem with Theorem 2.6.

Theorem 2.14. *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Let M_i be an R_i-module and let $M = M_1 \times M_2$ be the R-module with action $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ where $r_i \in R_i$ and $m_i \in M_i$.*

(i) *If P_1 is a primary submodule of M_1, then $P_1 \times M_2$ is a primary submodule of M.*

(ii) *If P_2 is a primary submodule of M_2, then $R_1 \times P_2$ is a primary submodule of M.*

(iii) *If P is a weakly primary submodule of M, then either $P = 0$ or P is primary.*

Proof. (i) Let $(a,b)(m_1,m_2) = (am_1,bm_2) \in P_1 \times M_2$ where $(a,b) \in R$ and $(m_1,m_2) \in M$, so either $a^n \in (P_1 : M_1)$ for some n or $m_1 \in P_1$ since P_1 is primary. It follows that either $(a,b)^n = (a^n,b^n) \in (P_1 : M_1) \times (M_2 : M_2) = (P_1 \times M_2 : M)$ for some n or $(m_1,m_2) \in P_1 \times M_2$, as required.

(ii) This proof is similar to that in case (i) and we omit it.

(iii) Let $P = P_1 \times P_2$ be a weakly primary submodule of M. We may assume that $P \neq 0$, so either $P_1 \neq 0$ or $P_2 \neq 0$, say $P_2 \neq 0$. Therefore, there is a non-zero element p_2 of P_2. Let $r \in (P_1 : M_1)$ and $c \in M_1$. Then $(0,0) \neq (r,1)(c,p_2) \in P$, so if P is weakly primary, then either $(r,1)^m = (r^m,1) \in (P : M) = (P_1 : M_1) \times (P_2 : M_2)$ for some m or $(c,p_2) \in P = P_1 \times P_2$, and hence either $1 \in (P_2 : M_2)$ or $c \in P_1$. Then either $0 \times M_2 \subseteq P$, so $P = P_1 \times M_2$ or $M_1 \times 0 \subseteq P$, so $P = M_1 \times P_2$.

First suppose that $P = P_1 \times M_2$. We show that P_1 is a primary submodule of M_1; hence P is primary by (i). Let $tp \in P_1$ where $t \in R_1$, $p \in M_1$. Then $(0,0) \neq (t,1)(p,p_2) \in P$, so $(t^k,1) \in (P_1 : M_1) \times (P_2 : M_2)$ for some k or $(p,p_2) \in P$, so $t^k \in (P_1 : M_1)$ for some k or $m \in P_1$. Thus P_1 is primary. The case where $P = M_1 \times P_2$ is similar. □

We next give two other characterizations of weakly primary submodules.

Theorem 2.15. *Let R be a commutative ring, M an R-module, and P a proper submodule of M. Then the following statements are equivalent:*

(i) *P is a weakly primary submodule of M.*

(ii) *For $m \in M - P$,* $\mathrm{Rad}(P : Rm) = \mathrm{Rad}(P : M) \cup (0 : Rm)$.

(iii) *For $m \in M - P$,* $\mathrm{Rad}(P : Rm) = \mathrm{Rad}(P : M)$ *or* $(0 : Rm) = \mathrm{Rad}(N : Rm)$.

Proof. (i) $\Rightarrow$ (ii) Let $a \in \mathrm{Rad}(P : Rm)$ where $m \in M - P$. Then $a^k m \in P$ for some k. If $a^k m \neq 0$, then $a^k \in (P : M)$ since P is weakly primary; hence $a \in \mathrm{Rad}(N : M)$. If $a^k m = 0$, then assume that s is the smallest integer with $a^s m = 0$. If $s = 1$, then $a \in (0 : Rm)$. Otherwise, $a \in \mathrm{Rad}(P : M)$, so $\mathrm{Rad}(P : Rm) \subseteq \mathrm{Rad}(P : M) \cup (0 : Rm) = H$. For the other inclusion assume that $b \in H$. Clearly, if $b \in (0 : Rm)$, then $b \in \mathrm{Rad}(P : Rm)$. If $b \in \mathrm{Rad}(P : M)$, then $b^t \in (P : M) \subseteq (P : Rm)$ for some t, so $b \in \mathrm{Rad}(P : Rm)$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Suppose that $0 \neq rm \in P$ with $r \in R$ and $m \in M - P$. Then $r \in (P : Rm) \subseteq \mathrm{Rad}(P : Rm)$ and $r \notin (0 : Rm)$. It follows from (iii) that $r \in \mathrm{Rad}(N : Rm) = \mathrm{Rad}(N : M)$, as required. □

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References

1. D. D. Anderson and E. Smith, Weakly prime ideals. *Houston J. Math.* **29**(2003), No. 4, 831–840 (electronic).

2. M. D. Larsen and P. J. McCarthy, Multiplicative theory of ideals. *Pure and Applied Mathematics,* Vol. 43. *Academic Press, New York–London,* 1971.
3. I. G. Macdonald, Secondary representation of modules over a commutative ring. *Symposia Mathematica,* Vol. XI (*Convegno di Algebra Commutativa, INDAM, Rome,* 1971), 23–43. *Academic Press, London,* 1973.
4. M. Prest, Model theory and modules. *London Mathematical Society Lecture Note Series,* 130. *Cambridge University Press, Cambridge,* 1988.

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