THE ALGEBRAIC SUM OF TWO ABSOLUTELY NEGLIGIBLE SETS CAN BE AN ABSOLUTELY NONMEASURABLE SET

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Abstract. We prove that there exist two absolutely negligible subsets $A$ and $B$ of the real line $\mathbb{R}$, whose algebraic sum $A + B$ is an absolutely nonmeasurable subset of $\mathbb{R}$. We also obtain some generalization of this result and formulate a relative open problem for uncountable commutative groups.

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Let $\mathbb{R}$ denote the real line and let $\lambda$ be the standard Lebesgue measure on $\mathbb{R}$. In the well-known article by Sierpiński [1] it was demonstrated that there exist two sets $A$ and $B$ in $\mathbb{R}$ satisfying the relations
$$
\lambda(A) = \lambda(B) = 0, \quad A + B \notin \text{dom}(\lambda).
$$

In other words, it was shown that the algebraic sum of two small sets (in the sense of $\lambda$) can be a nonmeasurable set with respect to the same $\lambda$. This result was strengthened in [2] by using purely set-theoretical and combinatorial techniques. Namely, let $\mu$ be a nonzero $\sigma$-finite complete measure on $\mathbb{R}$ quasi-invariant under the group $\Gamma_\mathbb{R}$ of all affine transformations of $\mathbb{R}$ and let $\mathcal{I}(\mu)$ denote the $\sigma$-ideal of all $\mu$-measure zero subsets of $\mathbb{R}$. Then the following two assertions are equivalent:

1) there exist sets $X \in \mathcal{I}(\mu)$ and $Y \in \mathcal{I}(\mu)$ such that $X + Y \notin \mathcal{I}(\mu);$  
2) there exist sets $A \in \mathcal{I}(\mu)$ and $B \in \mathcal{I}(\mu)$ such that $A + B \notin \text{dom}(\mu).$

In particular, suppose that $\mu$ is an extension of $\lambda$ and $\mu$ is quasi-invariant under the group $\Gamma_\mathbb{R}$. Then, taking into account the simple fact that there are sets $X \in \mathcal{I}(\lambda)$ and $Y \in \mathcal{I}(\lambda)$ for which $X + Y = R$, we easily infer that there are sets $A \in \mathcal{I}(\mu)$ and $B \in \mathcal{I}(\mu)$ for which $A + B \notin \text{dom}(\mu)$. Of course, here the sets $A$ and $B$ essentially depend on $\mu$. In the present paper we are going to describe another situation where $A$ and $B$ are fixed small subsets of $\mathbb{R}$ whose algebraic sum is a set with extremely bad properties from the measure-theoretical point of view. First, let us introduce the precise notion of "smallness" which will play a significant role in our further considerations.

Let $(G, +)$ be an arbitrary commutative group and let $Z$ be a subset of $G$. We say that $Z$ is $G$-absolutely negligible in $G$ if, for any $\sigma$-finite $G$-invariant (respectively, $G$-quasi-invariant) measure $\mu$ on $G$, there exists a $G$-invariant (respectively, $G$-quasi-invariant) measure $\mu'$ on $G$ extending $\mu$ and satisfying the relation $\mu'(Z) = 0$. 
Various properties of absolutely negligible sets are discussed in the monograph [3]. Here we need one auxiliary proposition about these sets which gives us their purely algebraic characterization.

**Lemma 1.** Let $Z$ be a subset of a commutative group $(G, +)$. The following two assertions are equivalent:

1) $Z$ is $G$-absolutely negligible in $G$;
2) for any countable family $\{f_i : i \in I\}$ of elements from $G$, there exists a countable family $\{g_j : j \in J\}$ of elements from $G$ such that

$$\bigcap_{j \in J} \left( g_j + \bigcup_{i \in I} (f_i + Z) \right) = \emptyset.$$ 

The proof of Lemma 1 is given in [2] and [3].

Let $(G, +)$ be a commutative group and let $H$ be a subgroup of $G$. Clearly, $H$ can be regarded as a certain group of transformations (in fact, translations) of $G$. As usual, we denote by $G/H$ the family of all $H$-orbits in $G$.

Lemma 1 implies the next auxiliary proposition.

**Lemma 2.** Suppose that a subset $Z$ of an uncountable commutative group $(G, +)$ has the following property: for every countable subgroup $H$ of $G$, the relation

$$\operatorname{card} \left( \{ T \in G/H : \operatorname{card}(T \cap Z) \geq 2 \} \right) < \operatorname{card}(G)$$

is satisfied. Then $Z$ is a $G$-absolutely negligible set in $G$.

**Proof.** Take any countable family $\{f_i : i \in I\} \subset G$ and denote by $F$ the subgroup of $G$ generated by this family. Since $\operatorname{card}(F) \leq \omega$ and $\operatorname{card}(G) > \omega$, we can choose an element $h \in G \setminus F$. Further, denote by $H$ the subgroup of $G$ generated by $h$ and $\{f_i : i \in I\}$. Obviously, $\operatorname{card}(H) \leq \omega$. According to our assumption, we have

$$\operatorname{card} \left( \{ T \in G/H : \operatorname{card}(T \cap Z) > 1 \} \right) < \operatorname{card}(G).$$

Let us put

$$P = \bigcup\{ T \in G/H : \operatorname{card}(T \cap Z) \leq 1 \}, \ Z' = Z \cap P.$$ 

Then $\operatorname{card}(Z \setminus Z') < \operatorname{card}(G)$ and, in view of Lemma 1, it is sufficient to demonstrate that

$$\bigcap_{g \in H} \left( g + \bigcup_{f \in F} (f + Z') \right) = \emptyset.$$ 

Suppose to the contrary that there exists an element

$$z \in \bigcap_{g \in H} \left( g + \bigcup_{f \in F} (f + Z') \right).$$ 

Taking into account the definition of $Z'$, we infer that there exists a unique element $z' \in Z'$ for which the inclusion

$$H + z \subset F + z'$$

is violated.
is valid. Consequently, we can write
\[ z \in F + z', \quad F + z = F + z', \quad H + z \subset F + z. \]
The latter inclusion implies at once that \( h + z = f + z \) for some \( f \in F \). Therefore, we get \( h = f \) and \( h \in F \) which contradicts the choice of \( h \). The obtained contradiction ends the proof of Lemma 2. \( \square \)

Let \((G, +)\) be a commutative group and let \( Z \) be a subset of \( G \). We say that \( Z \) is \( G \)-absolutely nonmeasurable in \( G \) if, for any nonzero \( \sigma \)-finite \( G \)-quasi-invariant measure \( \mu \) on \( G \), we have \( Z \not\in \text{dom}(\mu) \) (i.e., \( Z \) is nonmeasurable with respect to \( \mu \)). It is known that in every uncountable commutative group \((G, +)\) there are \( G \)-absolutely nonmeasurable sets. In this connection, see [3] where a more general fact is proved stating that every uncountable solvable group \((G, \cdot)\) contains \( G \)-absolutely nonmeasurable subsets.

Let us mention that the structure of absolutely nonmeasurable sets can be rather simple in some infinite-dimensional vector spaces (considered as commutative groups). Namely, the following proposition is valid.

**Lemma 3.** Let \( E \) be an infinite-dimensional separable Hilbert space (over \( R \)) and let \( K \) be an arbitrary open ball in \( E \). Then \( K \) is an \( E \)-absolutely nonmeasurable subset of \( E \).

The proof of Lemma 3 is presented in [3]. This lemma easily implies the well-known fact that \( E \) does not admit a nonzero \( \sigma \)-finite Borel measure quasi-invariant under the group of all translations of \( E \) (see, e.g., [4]).

**Lemma 4.** Suppose that \((G_1, +)\) and \((G_2, +)\) are two isomorphic commutative groups. Then the following assertions are equivalent:
1) there exist \( G_1 \)-absolutely negligible subsets \( X \) and \( Y \) of \( G_1 \) whose algebraic sum \( X + Y \) is \( G_1 \)-absolutely nonmeasurable in \( G_1 \);
2) there exist \( G_2 \)-absolutely negligible subsets \( A \) and \( B \) of \( G_2 \) whose algebraic sum \( A + B \) is \( G_2 \)-absolutely nonmeasurable in \( G_2 \).

We omit a trivial proof of Lemma 4.

Now, we are able to establish the following statement.

**Theorem 1.** There exist two \( R \)-absolutely negligible subsets of \( R \) such that their algebraic sum is an \( R \)-absolutely nonmeasurable set in \( R \).

**Proof.** Fix an infinite-dimensional separable Hilbert space \((E, \| \cdot \|)\) and denote
\[ K = \{ e \in E : \| e \| < 2 \}. \]
By virtue of Lemma 3, the open ball \( K \) is an \( E \)-absolutely nonmeasurable subset of \( E \). Taking into account Lemma 4 and the fact that \( E \) and \( R \) are isomorphic as commutative groups, it is sufficient to show that there exist two \( E \)-absolutely negligible sets \( X \) and \( Y \) in \( E \) for which the equality \( X + Y = K \) holds true. We are going to define the required sets \( X \) and \( Y \) by using the method of transfinite induction.
Lemma 2). This completes the proof of Theorem 1.

Now, putting

\[ y = y_\xi \]

we easily deduce that \( x + y = k_\xi \). Indeed, it suffices to put \( C = k_\xi \).

Let us define \( x_\xi = x \) and \( y_\xi = y \). Proceeding in this fashion, we are able to construct the \( \alpha \)-sequences \( \{x_\xi : \xi < \alpha\} \) and \( \{y_\xi : \xi < \alpha\} \) with properties (1)–(4).

Now, putting

\[ X = \{x_\xi : \xi < \alpha\}, \quad Y = \{y_\xi : \xi < \alpha\}, \]

we easily deduce that \( X + Y = K \) (in view of (1) and (2)). We also deduce that both \( X \) and \( Y \) are \( E \)-absolutely negligible subsets of \( E \) (in view of (3), (4) and Lemma 2). This completes the proof of Theorem 1.

Actually, the preceding argument yields a much stronger result. Namely, we can assert that there exists an \( E \)-absolutely negligible set \( C \subset E \) such that \( C + C = K \). Indeed, it suffices to put \( C = X \cup Y \) where \( X \) and \( Y \) are the above-mentioned \( E \)-absolutely negligible subsets of \( E \).

**Theorem 2.** There are two subsets \( A \) and \( B \) of \( R \) having the following property: for every nonzero \( \sigma \)-finite \( R \)-invariant \( (R\text{-quasi-invariant}) \) measure \( \mu \) on \( R \), there exists an \( R \)-invariant \( (R\text{-quasi-invariant}) \) measure \( \mu' \) on \( R \) extending \( \mu \) and such that

\[ \mu'(A) = \mu'(B) = 0, \quad A + B \not\in \text{dom}(\mu'). \]
Proof. It suffices to take as $A$ and $B$ any two $R$-absolutely negligible subsets of $R$ whose algebraic sum $A + B$ is $R$-absolutely nonmeasurable in $R$ (the existence of such subsets is stated by Theorem 1).

Let $(G_1, +)$ and $(G_2, +)$ be commutative groups and let $\phi : G_1 \to G_2$ be a surjective homomorphism. It is not difficult to verify that:

(a) if a set $Y \subset G_2$ is $G_2$-absolutely negligible, then the set $X = \phi^{-1}(Y)$ is $G_1$-absolutely negligible;
(b) if a set $Y \subset G_2$ is $G_2$-absolutely nonmeasurable, then the set $X = \phi^{-1}(Y)$ is $G_1$-absolutely nonmeasurable.

From (a), (b) and Theorem 1 we easily derive (under CH) that in every uncountable vector space $E$ over the field $\mathbb{Q}$ of all rational numbers there exist two $E$-absolutely negligible sets whose algebraic sum is $E$-absolutely nonmeasurable. In connection with this fact, the following open problem is of certain interest (cf. [5]).

**Problem.** Let $(G, +)$ be an arbitrary uncountable commutative group. Do there exist two $G$-absolutely negligible sets $A$ and $B$ in $G$ whose algebraic sum $A + B$ is $G$-absolutely nonmeasurable in $G$?

As indicated above, the answer to this question is positive (under CH) for all uncountable vector spaces over $\mathbb{Q}$.

**Remark.** Let $E$ be a topological space and let $X$ be a subset of $E$. We recall that $X$ is a universal measure zero set if $\mu^*(X) = 0$ for every $\sigma$-finite diffused Borel measure $\mu$ on $E$.

A set $Y \subset E$ is absolutely nonmeasurable (in the topological sense) if, for any nonzero $\sigma$-finite diffused Borel measure $\mu$ on $E$, we have $Y \notin \text{dom}(\mu')$, where $\mu'$ denotes the completion of $\mu$.

It is well known that there exist uncountable universal measure zero subsets of $R$ (the classical construction of such subsets due to Luzin is presented, e.g., in [6]).

Also, if $E$ is an uncountable Polish space and $Y \subset E$, then the following two assertions are equivalent:

i) $Y$ is absolutely nonmeasurable in the topological sense;
ii) $Y$ is a Bernstein subset of $E$.

A detailed information about the properties of Bernstein sets can be found in [6].

Here it is reasonable to point out a topological version of Theorem 2. Namely, assuming Martin’s Axiom, there exist two subsets $A$ and $B$ of $R$ which are universal measure zero (actually, they are generalized Luzin subsets of $R$) and whose algebraic sum $A + B$ is absolutely nonmeasurable in the topological sense.

Note that this result essentially needs additional set-theoretical axioms since there are models of set theory in which $\omega_1$ is strictly less than the cardinality continuum and in which the cardinality of any universal measure zero subset of $R$ does not exceed $\omega_1$. 
References


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