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THE ALGEBRAIC SUM OF TWO ABSOLUTELY NEGLIGIBLE SETS CAN BE AN ABSOLUTELY NONMEASURABLE SET

ALEXANDER KHARAZISHVILI

Abstract. We prove that there exist two absolutely negligible subsets A and B of the real line R, whose algebraic sum $A + B$ is an absolutely nonmeasurable subset of R. We also obtain some generalization of this result and formulate a relative open problem for uncountable commutative groups.

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Let R denote the real line and let λ be the standard Lebesgue measure on R. In the well-known article by Sierpiński [1] it was demonstrated that there exist two sets A and B in R satisfying the relations

$$\lambda(A) = \lambda(B) = 0, \ A + B \notin \mathrm{dom}(\lambda).$$

In other words, it was shown that the algebraic sum of two small sets (in the sense of λ) can be a nonmeasurable set with respect to the same λ. This result was strengthened in [2] by using purely set-theoretical and combinatorial techniques. Namely, let μ be a nonzero σ-finite complete measure on R quasi-invariant under the group Γ_R of all affine transformations of R and let $\mathcal{I}(\mu)$ denote the σ-ideal of all μ-measure zero subsets of R. Then the following two assertions are equivalent:

1) there exist sets $X \in \mathcal{I}(\mu)$ and $Y \in \mathcal{I}(\mu)$ such that $X + Y \notin \mathcal{I}(\mu)$;

2) there exist sets $A \in \mathcal{I}(\mu)$ and $B \in \mathcal{I}(\mu)$ such that $A + B \notin \mathrm{dom}(\mu)$.

In particular, suppose that μ is an extension of λ and μ is quasi-invariant under the group Γ_R. Then, taking into account the simple fact that there are sets $X \in \mathcal{I}(\lambda)$ and $Y \in \mathcal{I}(\lambda)$ for which $X + Y = R$, we easily infer that there are sets $A \in \mathcal{I}(\mu)$ and $B \in \mathcal{I}(\mu)$ for which $A + B \notin \mathrm{dom}(\mu)$. Of course, here the sets A and B essentially depend on μ. In the present paper we are going to describe another situation where A and B are fixed small subsets of R whose algebraic sum is a set with extremely bad properties from the measure-theoretical point of view. First, let us introduce the precise notion of "smallness" which will play a significant role in our further considerations.

Let $(G, +)$ be an arbitrary commutative group and let Z be a subset of G. We say that Z is G-absolutely negligible in G if, for any σ-finite G-invariant (respectively, G-quasi-invariant) measure μ on G, there exists a G-invariant (respectively, G-quasi-invariant) measure μ' on G extending μ and satisfying the relation $\mu'(Z) = 0$.

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Various properties of absolutely negligible sets are discussed in the monograph [3]. Here we need one auxiliary proposition about these sets which gives us their purely algebraic characterization.

Lemma 1. *Let Z be a subset of a commutative group $(G,+)$. The following two assertions are equivalent:*

1) *Z is G-absolutely negligible in G;*

2) *for any countable family $\{f_i : i \in I\}$ of elements from G, there exists a countable family $\{g_j : j \in J\}$ of elements from G such that*

$$\bigcap_{j\in J}\left(g_j+\bigcup_{i\in I}(f_i+Z)\right)=\varnothing.$$

The proof of Lemma 1 is given in [2] and [3].

Let $(G,+)$ be a commutative group and let H be a subgroup of G. Clearly, H can be regarded as a certain group of transformations (in fact, translations) of G. As usual, we denote by G/H the family of all H-orbits in G.

Lemma 1 implies the next auxiliary proposition.

Lemma 2. *Suppose that a subset Z of an uncountable commutative group $(G,+)$ has the following property: for every countable subgroup H of G, the relation*

$$\mathrm{card}\left(\{T\in G/H : \mathrm{card}(T\cap Z)\geq 2\}\right)<\mathrm{card}(G)$$

is satisfied. Then Z is a G-absolutely negligible set in G.

Proof. Take any countable family $\{f_i : i \in I\} \subset G$ and denote by F the subgroup of G generated by this family. Since $\mathrm{card}(F) \leq \omega$ and $\mathrm{card}(G) > \omega$, we can choose an element $h \in G \setminus F$. Further, denote by H the subgroup of G generated by h and $\{f_i : i \in I\}$. Obviously, $\mathrm{card}(H) \leq \omega$. According to our assumption, we have

$$\mathrm{card}\left(\{T\in G/H : \mathrm{card}(T\cap Z)>1\}\right)<\mathrm{card}(G).$$

Let us put

$$P=\cup\{T\in G/H : \mathrm{card}(T\cap Z)\leq 1\},\ \ Z'=Z\cap P.$$

Then $\mathrm{card}(Z \setminus Z') < \mathrm{card}(G)$ and, in view of Lemma 1, it is sufficient to demonstrate that

$$\bigcap_{g\in H}\left(g+\bigcup_{f\in F}(f+Z')\right)=\varnothing.$$

Suppose to the contrary that there exists an element

$$z\in\bigcap_{g\in H}\left(g+\bigcup_{f\in F}(f+Z')\right).$$

Taking into account the definition of Z', we infer that there exists a unique element $z' \in Z'$ for which the inclusion

$$H+z\subset F+z'$$

is valid. Consequently, we can write

$$z \in F + z', \quad F + z = F + z', \quad H + z \subset F + z.$$

The latter inclusion implies at once that $h + z = f + z$ for some $f \in F$. Therefore, we get $h = f$ and $h \in F$ which contradicts the choice of h. The obtained contradiction ends the proof of Lemma 2. □

Let $(G, +)$ be a commutative group and let Z be a subset of G. We say that Z is G-absolutely nonmeasurable in G if, for any nonzero σ-finite G-quasi-invariant measure μ on G, we have $Z \notin \mathrm{dom}(\mu)$ (i.e., Z is nonmeasurable with respect to μ). It is known that in every uncountable commutative group $(G, +)$ there are G-absolutely nonmeasurable sets. In this connection, see [3] where a more general fact is proved stating that every uncountable solvable group $(G, \cdot)$ contains G-absolutely nonmeasurable subsets.

Let us mention that the structure of absolutely nonmeasurable sets can be rather simple in some infinite-dimensional vector spaces (considered as commutative groups). Namely, the following proposition is valid.

Lemma 3. *Let E be an infinite-dimensional separable Hilbert space* (*over R*) *and let K be an arbitrary open ball in E. Then K is an E-absolutely nonmeasurable subset of E.*

The proof of Lemma 3 is presented in [3]. This lemma easily implies the well-known fact that E does not admit a nonzero σ-finite Borel measure quasi-invariant under the group of all translations of E (see, e.g., [4]).

Lemma 4. *Suppose that $(G_1, +)$ and $(G_2, +)$ are two isomorphic commutative groups. Then the following assertions are equivalent:*

1) *there exist G_1-absolutely negligible subsets X and Y of G_1 whose algebraic sum $X + Y$ is G_1-absolutely nonmeasurable in G_1;*

2) *there exist G_2-absolutely negligible subsets A and B of G_2 whose algebraic sum $A + B$ is G_2-absolutely nonmeasurable in G_2.*

We omit a trivial proof of Lemma 4.

Now, we are able to establish the following statement.

Theorem 1. *There exist two R-absolutely negligible subsets of R such that their algebraic sum is an R-absolutely nonmeasurable set in R.*

Proof. Fix an infinite-dimensional separable Hilbert space $(E, \|\cdot\|)$ and denote

$$K = \{e \in E : \|e\| < 2\}.$$

By virtue of Lemma 3, the open ball K is an E-absolutely nonmeasurable subset of E. Taking into account Lemma 4 and the fact that E and R are isomorphic as commutative groups, it is sufficient to show that there exist two E-absolutely negligible sets X and Y in E for which the equality $X + Y = K$ holds true. We are going to define the required sets X and Y by using the method of transfinite induction.

Let α be the least ordinal number of cardinality continuum, let $\{k_\xi : \xi < \alpha\}$ be an enumeration of all elements from K and let $\{H_\xi : \xi < \alpha\}$ be an enumeration of all countable subgroups of the additive group E. For any $\xi < \alpha$, denote by G_ξ the subgroup of E generated by the set $\cup\{H_\zeta : \zeta < \xi\}$. Now, construct by transfinite recursion two α-sequences $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$ of elements from E satisfying the following conditions:

(1) $\|x_\xi\| < 1$ and $\|y_\xi\| < 1$ for each $\xi < \alpha$;

(2) $x_\xi + y_\xi = k_\xi$ for each $\xi < \alpha$;

(3) $(G_\xi + x_\xi) \cap (G_\xi + \{x_\zeta : \zeta < \xi\}) = \varnothing$ for any $\xi < \alpha$;

(4) $(G_\xi + y_\xi) \cap (G_\xi + \{y_\zeta : \zeta < \xi\}) = \varnothing$ for any $\xi < \alpha$.

Suppose that, for an ordinal $\xi < \alpha$, the partial ξ-sequences $\{x_\zeta : \zeta < \xi\}$ and $\{y_\zeta : \zeta < \xi\}$ have already been constructed. Let us put

$$Z_\xi = \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\},$$
$$K_\xi = \{e \in E : \|e - k_\xi\| < 1\},$$
$$D = \{e \in E : \|e\| < 1\}.$$

Note that

$$K_\xi = D + k_\xi,$$
$$\mathrm{card}(G_\xi + Z_\xi) \le \mathrm{card}(\xi) + \omega < \mathrm{card}(E),$$
$$\mathrm{card}(K_\xi \cap D) = \mathrm{card}(E).$$

Consequently, there are two points $x \in D$ and $y \in D$ such that

$$(G_\xi + x) \cap (G_\xi + Z_\xi) = \varnothing,$$
$$(G_\xi + y) \cap (G_\xi + Z_\xi) = \varnothing,$$
$$x + y = k_\xi.$$

Let us define $x_\xi = x$ and $y_\xi = y$. Proceeding in this fashion, we are able to construct the α-sequences $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$ with properties (1)–(4). Now, putting

$$X = \{x_\xi : \xi < \alpha\}, \quad Y = \{y_\xi : \xi < \alpha\},$$

we easily deduce that $X + Y = K$ (in view of (1) and (2)). We also deduce that both X and Y are E-absolutely negligible subsets of E (in view of (3), (4) and Lemma 2). This completes the proof of Theorem 1. □

Actually, the preceding argument yields a much stronger result. Namely, we can assert that there exists an E-absolutely negligible set $C \subset E$ such that $C + C = K$. Indeed, it suffices to put $C = X \cup Y$ where X and Y are the above-mentioned E-absolutely negligible subsets of E.

Theorem 2. *There are two subsets A and B of R having the following property: for every nonzero σ-finite R-invariant (R-quasi-invariant) measure μ on R, there exists an R-invariant (R-quasi-invariant) measure μ' on R extending μ and such that*

$$\mu'(A) = \mu'(B) = 0, \quad A + B \notin \mathrm{dom}(\mu').$$

Proof. It suffices to take as A and B any two R-absolutely negligible subsets of R whose algebraic sum $A+B$ is R-absolutely nonmeasurable in R (the existence of such subsets is stated by Theorem 1). □

Let $(G_1,+)$ and $(G_2,+)$ be commutative groups and let $\phi : G_1 \to G_2$ be a surjective homomorphism. It is not difficult to verify that:

(a) if a set $Y \subset G_2$ is G_2-absolutely negligible, then the set $X = \phi^{-1}(Y)$ is G_1-absolutely negligible;

(b) if a set $Y \subset G_2$ is G_2-absolutely nonmeasurable, then the set $X = \phi^{-1}(Y)$ is G_1-absolutely nonmeasurable.

From (a), (b) and Theorem 1 we easily derive (under **CH**) that in every uncountable vector space E over the field Q of all rational numbers there exist two E-absolutely negligible sets whose algebraic sum is E-absolutely nonmeasurable. In connection with this fact, the following open problem is of certain interest (cf. [5]).

Problem. Let $(G,+)$ be an arbitrary uncountable commutative group. Do there exist two G-absolutely negligible sets A and B in G whose algebraic sum $A+B$ is G-absolutely nonmeasurable in G?

As indicated above, the answer to this question is positive (under **CH**) for all uncountable vector spaces over Q.

Remark. Let E be a topological space and let X be a subset of E. We recall that X is a universal measure zero set if $\mu^*(X) = 0$ for every σ-finite diffused Borel measure μ on E.

A set $Y \subset E$ is absolutely nonmeasurable (in the topological sense) if, for any nonzero σ-finite diffused Borel measure μ on E, we have $Y \notin \mathrm{dom}(\mu')$, where μ' denotes the completion of μ.

It is well known that there exist uncountable universal measure zero subsets of R (the classical construction of such subsets due to Luzin is presented, e.g., in [6]).

Also, if E is an uncountable Polish space and $Y \subset E$, then the following two assertions are equivalent:

i) Y is absolutely nonmeasurable in the topological sense;

ii) Y is a Bernstein subset of E.

A detailed information about the properties of Bernstein sets can be found in [6].

Here it is reasonable to point out a topological version of Theorem 2. Namely, assuming Martin's Axiom, there exist two subsets A and B of R which are universal measure zero (actually, they are generalized Luzin subsets of R) and whose algebraic sum $A+B$ is absolutely nonmeasurable in the topological sense.

Note that this result essentially needs additional set-theoretical axioms since there are models of set theory in which ω_1 is strictly less than the cardinality continuum and in which the cardinality of any universal measure zero subset of R does not exceed ω_1.

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Author's Address:

I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0143
Georgia
E-mail: kharaz2@yahoo.com