Georgian Mathematical Journal
Volume 12 (2005), Number 3, 461–474

ON LINEAR CONJUGATION PROBLEMS WITH A DOUBLY PERIODIC JUMP LINE (THE CASE OF OPEN ARCS)

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Abstract. The linear conjugation problem for the class of exponentially doubly quasi-periodic functions is considered, when the jump line is doubly-periodic and consist of open arcs. Effective solutions are obtained by means of a Cauchy type integral with Weierstrass kernel.

2000 Mathematics Subject Classification: 30E25, 45E, 30E20, 30D30.
Key words and phrases: Linear conjugation problems, elliptic functions, singular integrals.

Introduction

The linear conjugation problem for different classes of functions when the jump line consists of a countable number of contours was considered by numerous authors [1], [3]–[6], [8], [10], [13]. We will consider the linear conjugation problem for the class of exponentially doubly quasi-periodic functions. The case of closed arcs is considered by the author in [9].

1. The Doubly-Periodic and Doubly Quasi-Periodic Functions

Consider a complex z-plane $\mathbb{C}$, $z = x + iy$, and two complex numbers ω_1 and $i\omega_2$ satisfying the condition $\operatorname{Im} \frac{i\omega_2}{\omega_1} > 0$. Let us introduce some definitions.

Definition 1.1. A line L is called *a doubly-periodic line* if it is a union of a countable number of smooth nonintersecting contours L^j_{mn}, $j = 1, 2, \dots, k$; $m, n = 0, \pm 1, \dots$, doubly-periodically distributed with periods $2\omega_1$ and $2i\omega_2$ in the whole z-plane

$$L = \sum_{m,n=-\infty}^{\infty} L_{mn}, \quad L_{mn} = \sum_{j=1}^{k} L^j_{mn}, \quad L^{j_1}_{mn} \cap L^{j_2}_{mn} = \varnothing, \quad j_1 \neq j_2. \tag{1.1}$$

In the sequel we will deal with lines of this type with open contours L^j_{mn}. By S we denote the z-plane cut along L.

Let us recall some definitions from the theory of elliptic functions [2], [7], [10], [14].

Definition 1.2. A set $D \subset \mathbb{C}$ is called *doubly-periodic set* if $z \in D$ implies $z + 2m\omega_1 + 2ni\omega_2 \in D$, $m, n = 0, \pm 1, \dots$. Points z and $z + 2m\omega_1 + 2ni\omega_2$, $m, n = 0, \pm 1, \dots$, are called *congruent points*.

Note that $\mathbb{C} \setminus L$ is doubly-periodic set.

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Definition 1.3. A function $F_0(z)$ defined on the doubly-periodic set D is called *doubly quasi-periodic* with periods $2\omega_1$ and $2i\omega_2$ if the following condition is fulfilled:

$$F_0(z + 2m\omega_1 + 2ni\omega_2) = F_0(z) + m\gamma_1 + n\gamma_2, \quad m, n = 0, \pm 1, \dots . \tag{1.2}$$

γ_1 and γ_2 are definite constants, called *addends.*

If $\gamma_1 = \gamma_2 = 0$, then a function $F_0(z)$ is called a *doubly-periodic function.*

Definition 1.4. A doubly-periodic meromorphic function is called *an elliptic function.*

The parallelogram with vertices $0, 2\omega_1, 2\omega_1 + 2i\omega_2, 2i\omega_2$ is called the *fundamental parallelogram.* The interior of this parallelogram is denoted by S_{00}.

The number of poles of an elliptic function in the fundamental parallelogram is called the order of an elliptic function.

Theorem 1.1 (Liouville)**.** *An elliptic function, holomorphic in every bounded domain of the z-plane, is a constant.*

Theorem 1.2. *A non-zero elliptic function of first order does not exist.*

Theorem 1.3 (Liouville)**.** *Every elliptic function of n-th order takes each of its value n times in the fundamental parallelogram.*

Definition 1.3 implies

Theorem 1.4. *If a doubly quasi-periodic function is differentiable in its domain of definition, then the first derivative of this function is a doubly-periodic function.*

By Theorems 1.1 and 1.4 we immediately get

Theorem 1.5. *A doubly quasi-periodic function, holomorphic in every bounded domain of the z-plane, is representable in the form*

$$F_0(z) = Az + B,$$

where A and B are arbitrary constants, the addends of this function are $\gamma_1 = 2A\omega_1$ and $\gamma_2 = 2Ai\omega_2$.

Let us fix the lattice $T = \{T_{mn} = 2m\omega_1 + 2ni\omega_2,\ m, n = 0, \pm 1, \dots\}$ and recall the definitions of Weierstrass ζ-function and σ-function for this lattice.

Definition 1.5. The function representable by the double series

$$\zeta(z) = \zeta(z; T) = \frac{1}{z} + \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \Big(\frac{1}{z - T_{mn}} + \frac{1}{T_{mn}} + \frac{z}{T_{mn}^2} \Big) \tag{1.3}$$

is called the Weierstrass ζ-function.

Series (1.3) converges uniformly in every closed region of the z-plane not containing the points of T. The Weierstrass ζ-function has the following properties:

1. It is a meromorphic function with simple poles T_{mn}, $m, n = 0, \pm 1, \dots$;

2. $\zeta(z)$ is doubly quasi-periodic,

$$\begin{aligned} &\zeta(z+2\omega_1)=\zeta(z)+\delta_1, \quad \zeta(z+2i\omega_2)=\zeta(z)+\delta_2,\\ &\delta_1=2\zeta(\omega_1), \quad \delta_2=2\zeta(i\omega_2), \quad i\omega_2\delta_1-\omega_1\delta_2=\pi i, \end{aligned} \tag{1.4}$$

where δ_1 and δ_2 are the addends of ζ-function.

Definition 1.6. The function $\sigma(z)$ given by the infinite product

$$\sigma(z)=\sigma(z;T)=z\prod_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\Big(1-\frac{z}{T_{mn}}\Big)\exp\Big\{\frac{z}{T_{mn}}+\frac{z^2}{2T_{mn}^2}\Big\}$$

is called the Weierstrass σ-function.

σ-function is a meromorphic function with simple zeros at the points $z\in T$. The Weierstrass σ-function has the following properties:

$$\begin{aligned} &1. \quad \sigma(z+2\omega_1)=-\sigma(z)\exp(\delta_1 z+\delta_1\omega_1),\\ &2. \quad \sigma(z+2i\omega_2)=-\sigma(z)\exp(\delta_2 z+\delta_2 i\omega_2). \end{aligned} \tag{1.5}$$

Though the σ-function is not doubly quasi-periodic, it can be used in constructing any elliptic function.

Theorem 1.6. *Every elliptic function $F(z)$ of n-th order with zeros $\alpha_2,\dots,\alpha_n$ and poles $\beta_1,\beta_2,\dots,\beta_n$ in the fundamental parallelogram can be represented in the form*

$$F(z)=C\,\frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_n)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\cdots\sigma(z-\beta_n)},$$

where $\alpha_1=(\beta_1+\beta_2+\cdots+\beta_n)-(\alpha_2+\cdots+\alpha_n)$, C is a definite constant.

Let us introduce a new class of functions.

Definition 1.7. A function $\Phi(z)$ defined in the doubly-periodic set is called *exponentially doubly quasi-periodic* if the following conditions are fulfilled

$$\begin{aligned} &1. \quad \Phi(z+2\omega_1)=\Phi(z)\exp(P_{k_1}(z))\,,\\ &2. \quad \Phi(z+2i\omega_2)=\Phi(z)\exp(Q_{k_2}(z)), \end{aligned}$$

where P_{k_1} and Q_{k_2} are the definite polynomials of orders k_1 and k_2, respectively.

We denote this class of functions by $\mathcal{P}_e(k)$, $k=\max(k_1,k_2)$.

In the case $P_{k_1}=Q_{k_2}=0$, the sub-class of the class $\mathcal{P}_e(0)$ is denoted by $\mathcal{P}_0(0)$, this is the class of doubly-periodic functions.

Definition 1.8. A function $\Phi(z)$ is called *sectionally holomorphic, exponentially doubly quasi-periodic* with jump line L if it has the following properties:

1. It is holomorphic in each bounded region not containing points of the line L;

2. $\Phi(z)$ is left and right continuous on L with a possible exception for the ends $c_1,c_2,\dots,c_q$, near which the following condition is fulfilled:

$$|\Phi(z)|\le\frac{C}{|z-c|^{\alpha}}\,,$$

where c is one of the points $c_1, c_2, \dots, c_q$, and C and α are the definite real constants, $\alpha < 1$;

3. The function $\Phi(z)$ is of the class $\mathcal{P}_e(k)$,

$$\Phi(z+2\omega_1)=\Phi(z)\exp\{P_{k_1}(z)\}, \quad \Phi(z+2i\omega_2)=\Phi(z)\exp\{Q_{k_2}(z)\}, \quad z \in S,$$

where P_{k_1}, Q_{k_2} are polynomials of degree k_1, k_2, respectively.

2. A Cauchy Type Integral with the Weierstrass Kernel

Let L_{00} be the pert of$L$ given by (1.1). Consider the function

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\, \zeta(t-z)\, dt, \ z \in S, \tag{2.1}$$

where $\varphi(t)$ is a given doubly-periodic function on L belonging to Muskhelishvili class H^* on L_{00} [11], and ζ is the Weierstrass ζ-function for periods $2\omega_1, 2i\omega_2$ defined by (1.3).

Integrals of this type were first considered by Sedov [13]. Using these integrals he solved several doubly-periodic boundary value problems of hydrodynamics.

The integral given by (2.1) has the properties of an ordinary Cauchy integral and is called a Cauchy type integral with the Weierstrass kernel. The following theorem is true [5].

Theorem 2.1. *The function*

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\, \zeta(t-z)\, dt + Az + B, \tag{2.2}$$

where A and B are arbitrary fixed constants, represents a sectionally holomorphic doubly quasi-periodic function with the jump line L and the addends

$$\gamma_1 = -\frac{\delta_1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\, dt + 2A\omega_1, \quad \gamma_2 = -\frac{\delta_2}{2\pi i} \int\limits_{L_{00}} \varphi(t)\, dt + 2Ai\omega_2. \tag{2.3}$$

Function (2.2) *is doubly-periodic if and only if $A = 0$ and*

$$\int\limits_{L_{00}} \varphi(t) dt = 0.$$

From Theorems 2.1 and 1.5 it follows that function (2.2) is the general solution of the following boundary value problem.

Problem 2.1. Find a sectionally holomorphic, doubly quasi-periodic function $\Phi(z)$ with jump line L satisfying the boundary condition (except the ends of L)

$$\Phi^+(t) - \Phi^-(t) = \varphi(t), \quad t \in L, \tag{2.4}$$

where $\varphi(t)$ is a given doubly periodic function on L, belonging to the Muskhelishvili H^* class on L_{00} [11].

In the sequel we will use the solutions of the following auxiliary problem.

Problem 2.2. Find a doubly-periodic function $\Phi(z)$ with jump line L, sectionally holomorphic everywhere with a possible exception for the points $\beta_1+2m\omega_1+2ni\omega_2, \beta_2+2m\omega_1+2ni\omega_2, \dots, \beta_q+2m\omega_1+2ni\omega_2$; $m,n=0,\pm1,\dots$; $\beta_1,\beta_2,\dots,\beta_q \in S_{00}-L_{00}; \beta_i \neq \beta_j$, where (except the ends of L) it may have simple poles, and also satisfies the following boundary condition

$$\Phi^+(t)-\Phi^-(t)=\varphi(t), \quad t\in L,$$

where $\varphi(t)$ is the given doubly periodic function on L of the Muskhelishvili class H^* on L_{00}.

The solution of this problem is given by the author in [9].

Theorem 2.2. *A solution of Problem* 2.2 *exists and, in the case of* $q>1$, *the general solution is given by*

$$\begin{aligned}\Phi(z)=&\frac{1}{2\pi i}\int\limits_{L_{00}}\varphi(t)\left[\zeta(t-z)-\zeta(\beta_1-z)\right]dt\\&+C_1\,\frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_q)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\cdots\sigma(z-\beta_q)}+C_2,\end{aligned} \tag{2.5}$$

where C_1, C_2 *are arbitrary constants and the constants* $\alpha_1,\alpha_2,\dots,\alpha_q$ *satisfy the condition*

$$\alpha_1+\alpha_2+\cdots+\alpha_q=\beta_1+\beta_2+\cdots+\beta_q.$$

In case of $q=1$, *a solution of Problem* 2.2 *is given by* (2.5), *where* $C_1=0$.

3. Boundary Value Problems in the Case of Open Arcs

Let L be the doubly-periodic line defined in Section 1 and assume that L_{00} is the single open contour with ends c_1, c_2. Direction from c_1 to c_2 is chosen as the positive one.

In the sequel, we will consider functions of the class $\mathcal{P}_e(0)$, i.e., the class of functions $\Phi(z)$ defined on doubly-periodic set with periods $2\omega_1$ and $2i\omega_2$ satisfying the conditions

$$\Phi(z+2\omega_1)=\Phi(z)\exp\gamma_1, \quad \Phi(z+2i\omega_2)=\Phi(z)\exp\gamma_2, \tag{3.1}$$

where γ_1 and γ_2 are the definite constants.

Problem 3.1. Find a sectionally holomorphic function $\Phi(z)$ of the class $\mathcal{P}_e(0)$ with jump line L, satisfying the following condition

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t), \quad t\in L, \tag{3.2}$$

where $G(t)$ is a doubly-periodic function given on L, belonging to the Hölder class on L_{00}, $G(t)\neq 0$, while $g(t)$ is a function of the class $\mathcal{P}_e(0)$ given on L, which belongs to the Hölder class on L_{00},

1. $g(t+2\omega_1)=g(t)\exp\gamma_1$,
2. $g(t+2i\omega_2)=g(t)\exp\gamma_2, \quad t\in L.$

Condition (3.2) holds everywhere on L except the ends.

We classify solutions of Problem 3.1 with respect to the ends of the line L and find solutions in the Muskhelishvili–Kveselava classes [11], but first we need to consider an auxiliary problem.

Let us introduce a new class of functions $\widetilde{\mathcal{P}_e(0)}$.

Definition 3.1. A function $\Phi_0(z)$ defined in S is said to be of the class $\widetilde{\mathcal{P}_e(0)}$ if

1. it belongs to the class $\mathcal{P}_e(0)$;
2. it may have simple isolated zeros not belonging to the line L;
3. it may have simple poles at the isolated points $a_1 + 2m\omega_1 + 2ni\omega_2$, $a_2 + 2m\omega_1 + 2ni\omega_2, \dots, a_q + 2m\omega_1 + 2ni\omega_2$; $a_1, a_2, \dots, a_q \in S_{00} \setminus L_{00}$ (q is the definite non-negative integer), $m, n = 0, \pm 1, \pm 2, \dots$;
4. $\Phi_0(z)$ is sectionally holomorphic with jump line L except the points $a_1 + 2m\omega_1 + 2ni\omega_2$, $a_2 + 2m\omega_1 + 2ni\omega_2, \dots, a_q + 2m\omega_1 + 2ni\omega_2$.

Problem 3.2. Find a function $\Phi_0(z)$ of the class $\widetilde{\mathcal{P}_e(0)}$ satisfying the boundary condition

$$\Phi_0^+(t) = G(t)\Phi_0^-(t), \quad t \in L, \tag{3.3}$$

where $G(t)$ ($G(t) \neq 0$) is a doubly-periodic function given on L belonging to the Hölder class on L_{00}.

Condition (3.3) holds everywhere on L except the ends.

We solve this problem by the Muskhelishvili method [11].

Let $\ln G(t)$ be any branch of this function continuous on L_{00}. It is clear that $\ln G(t)$ belongs to the class H on L_{00} and there exist the following limiting values;

$$\lim_{t \to c_k} \ln G(t) = \ln G(c_k), \quad t \in L_{00}, \quad k = 1, 2.$$

Consider the integral

$$\gamma(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} [\ln G(t)]\, \zeta(t - z)\, dt. \tag{3.4}$$

By the results given in Section 2, the integral given by (3.4) is a Cauchy type integral having limits from the left and from the right of L except the ends of the line L. Hence the function $\exp(\gamma(z))$ satisfies the boundary condition (3.3) except the ends.

The function $\gamma(z)$ has the following behavior near the ends of the line of integration [11, Ch. 1]:

$$\gamma(z) = (\alpha_k + i\beta_k) \ln(z - c_k) + \gamma_{0k}(z), \quad k = 1, 2,$$

where $\gamma_{0k}(z)$ remains bounded near c_k and takes a certain value there, α_k and β_k ($k = 1, 2$) are real constants given by

$$\alpha_1 + i\beta_1 = -\frac{\ln G(c_1)}{2\pi i}, \quad \alpha_2 + i\beta_2 = \frac{\ln G(c_2)}{2\pi i}. \tag{3.5}$$

Therefore near the ends c_1 and c_2 we have

$$\exp(\gamma(z)) = (z - c_k)^{\alpha_k + i\beta_k}\Omega_k(z), \quad k = 1, 2,$$

where $\Omega_k(z)$ is a non-vanishing bounded function which assumes the definite value at the point c_k.

According to Muskhelishvili [11], we select integers λ_1 and λ_2 satisfying the conditions

$$-1 < \alpha_1 + \lambda_1 < 1, \quad -1 < \alpha_2 + \lambda_2 < 1. \tag{3.6}$$

The ends for which α_k is an integer are called special ends, for this ends λ_k is uniquely determined: $\lambda_k = -\alpha_k$.

For non-special ends the numbers λ_k are determined apart from the terms ± 1 and, in fact, λ_k can be chosen such that $\alpha_k + \lambda_k > 0$ or $\alpha_k + \lambda_k < 0$.

The choice of λ_k will be completely determined if one more condition is introduced. This condition will now be stated.

Solutions of Problem 3.2 will be admitted which become infinite at the ends with degree less than 1. Sometimes it is required that the unknown solution be bounded at non-special ends.

Let c_1 and c_2 be non-special ends and let us consider the solutions of the generalized Muskhelishvili–Kveselava classes [11], i.e.,

1. $\widetilde{h}_0$ is the class of solutions having singularities less than 1 at the points c_1, c_2; in this case $-1 < \alpha_k + \lambda_k < 0$ $(k = 1, 2)$;
2. $\widetilde{h}(c_1)$ is the class of solutions bounded at the point c_1; in this case $0 < \alpha_1 + \lambda_1 < 1$;
3. $\widetilde{h}(c_2)$ is the class of solutions bounded at the point c_2; in this case $-1 < \alpha_1 + \lambda_1 < 0$;
4. $\widetilde{h}_2$ is the class of solutions bounded at the points c_1 and c_2; in this case $0 < \alpha_k + \lambda_k < 1$ $(k = 1, 2)$.

Now let us agreed to choose numbers λ_k in such a way that at the non-special ends, where the solutions of a given class are bounded, $\alpha_k + \lambda_k > 0$, while at the remaining non-special ends $\alpha_k + \lambda_k < 0$ (at the special ends $\alpha_k + \lambda_k = 0$, $k = 1$ or $k = 2$).

In defining the classes of solutions attention is not paid to the special ends because, as it will be seen later, each solution of Problem 3.2 is necessarily bounded near all special ends.

We will consider five cases.

1. For $\alpha_1 > 0$, $\alpha_2 < 0$ $(\lambda_1 \le 0, \lambda_2 \ge 0, |\lambda_1| + |\lambda_2| \ne 0)$, a solution of Problem 3.2 of the corresponding class will be given as

$$\Phi_0(z) = C\, \frac{\sigma^{\lambda_2}(z - c_2)}{\sigma^{-\lambda_1}(z - c_1)} \cdot \frac{\sigma(z - b_1)}{\sigma(z - a_1)} \cdots \frac{\sigma(z - b_{-\lambda_1})}{\sigma(z - a_{\lambda_2})}\, \exp\{\gamma(z) + Az + B\}, \tag{3.7}$$

where C and B are arbitrarily chosen constants and the constants $A, b_1, b_2, \dots,$ $b_{-\lambda_1}, a_1, \dots, a_{\lambda_2}$ satisfy the conditions

$$\begin{aligned} &-\delta_1(b_1+\dots+b_{-\lambda_1}+\lambda_2 c_2+\lambda_1 c_1-a_1-\dots-a_{\lambda_2}) \\ &\qquad -\frac{\delta_1}{2\pi i}\int\limits_{L_{00}}\ln G(t)dt+2A\omega_1=\gamma_1, \\ &-\delta_2(b_1+\dots+b_{-\lambda_1}+\lambda_1 c_1+\lambda_2 c_2-a_1-\dots-a_{\lambda_2}) \\ &\qquad -\frac{\delta_2}{2\pi i}\int\limits_{L_{00}}\ln G(t)dt+2Ai\omega_2=\gamma_2, \end{aligned} \tag{3.8}$$

$a_1, \dots, a_{\lambda_2}, b_1, \dots, b_{-\lambda_1} \notin L$, $a_i \neq a_j$, $b_i \neq b_j$, $i \neq j$.

Indeed, by Theorem 3.2 the function $\Phi_0(z)$ satisfies the boundary condition (3.3). Condition (3.6) ensures that $\Phi_0(z)$ may become infinite at the ends with degree less than 1 for the corresponding class, and conditions (3.8) and (1.5) ensure that $\Phi_0(z)$ belongs to the class $\mathcal{P}_e(0)$, i.e., satisfies conditions (3.1).

Let us introduce the notation

$$a_0=\lambda_1 c_1+\lambda_2 c_2+\frac{\gamma_1 i\omega_2-\gamma_2\omega_1}{\pi i}+\frac{1}{2\pi i}\int\limits_{L_{00}}\ln G(t)\,dt.$$

From (3.8) and (1.4) we obtain

$$a_1+\dots+a_{\lambda_2}-b_1-\dots b_{-\lambda_1}=a_0, \tag{3.8$_1$}$$

$$A=\frac{\gamma_2\delta_1-\gamma_1\delta_2}{2\pi i}\,. \tag{3.8$_2$}$$

Remark 1. If c_1 or c_2 is a special end, then near special ends, $\Phi_0(z)$ belongs to the class H_ε^*, remaining bounded there.

2. In the case $\alpha_1 < 0$, $\alpha_2 > 0$ ($\lambda_1 \geq 0, \lambda_2 \leq 0$, $|\lambda_1|+|\lambda_2| \neq 0$), a solution of the corresponding class is given by

$$\Phi_0(z)=\frac{\sigma^{\lambda_1}(z-c_1)}{\sigma^{-\lambda_2}(z-c_2)}\cdot\frac{\sigma(z-b_1)\cdots\sigma(z-b_{-\lambda_2})}{\sigma(z-a_1)\cdots\sigma(z-a_{\lambda_1})}\cdot\exp\{\gamma(z)+Az+B\}, \tag{3.9}$$

where C and B are arbitrary constants, A is given by (3.8$_2$) and the constants $a_1, \dots, a_{\lambda_1}, b_1, \dots, b_{-\lambda_2}$ satisfy the conditions

$$\begin{gathered} a_1+\dots+a_{\lambda_1}-b_1-\dots-b_{-\lambda_2}=a_0, \\ a_1,\dots,a_{\lambda_1},b_1,\dots,b_{-\lambda_2}\notin L, \quad a_i\neq a_j, \quad b_i\neq b_j \;\; (i\neq j). \end{gathered}$$

3. In the case $\alpha_1 > 0$, $\alpha_2 > 0$ ($\lambda_1 \leq 0$, $\lambda_2 \leq 0$, $|\lambda_1|+|\lambda_2| \neq 0$), a solution of the corresponding class is

$$\Phi_0(z)=C\,\frac{\sigma(z-b_1)\cdots\sigma(z-b_{-\lambda_1-\lambda_2})}{\sigma^{-\lambda_1}(z-c_1)\sigma^{-\lambda_2}(z-c_2)}\,\exp\{\gamma(z)+Az+B\}, \tag{3.10}$$

where C and B are arbitrary constants, A is given by (3.8$_2$) and the constants $b_1, \dots, b_{-\lambda_1-\lambda_2}$ satisfy the conditions

$$b_1+\dots+b_{-\lambda_1-\lambda_2}=-a_0,$$

$$b_1,\dots,b_{-\lambda_1-\lambda_2}\notin L,\quad b_i\neq b_j\quad (i\neq j).$$

4. In the case $\alpha_1<0$, $\alpha_2<0$ ($\lambda_1\geq 0$, $\lambda_2\geq 0$)

$$\Phi_0(z)=C\,\frac{\sigma^{\lambda_1}(z-c_1)\sigma^{\lambda_2}(z-c_2)}{\sigma(z-a_1)\cdots\sigma(z-a_{\lambda_1+\lambda_2})}\,\exp\{\gamma(z)+Az+B\},\tag{3.11}$$

where C and B are arbitrary constants, A is given by (3.8_2) and the constants $a_1,\dots,a_{\lambda_1+\lambda_2}$ satisfy the conditions

$$a_1+\cdots+a_{\lambda_1+\lambda_2}=a_0,$$
$$a_1,\dots,a_{\lambda_1+\lambda_2}\notin L,\quad a_i\neq a_j\quad (i\neq j).$$

5. If $-1<\alpha_k<1$ ($\lambda_1=0$, $\lambda_2=0$), a solution of the corresponding class is given by

$$\Phi_0(z)=C\,\frac{\sigma(z-b_1)}{\sigma(z-a_1)}\,\exp\{\gamma(z)+Az+B\},$$

where C and B are arbitrary constants, A is given by (3.8_2) and a_1,b_1 satisfy the condition

$$a_1-b_1=\frac{\gamma_1 i\omega_2-\gamma_2\omega_1}{\pi i}+\frac{1}{2\pi i}\int\limits_{L_{00}}\ln G(t)\,dt,\quad a_1,b_1\notin L.$$

From representations (3.7), (3.9), (3.10) and (3.11) we can obtain a solutions of Problem 3.2 with a minimal number of poles:

1. If $-\lambda_1-\lambda_2>1$ or $-\lambda_1-\lambda_2=1$, $-a_0\notin L$, then

$$\begin{aligned}\Phi_0(z)=C\sigma^{\lambda_1}(z-c_1)\sigma^{\lambda_2}(z-c_2)\sigma(z-b_1)\cdots\sigma(z-b_{-\lambda_1-\lambda_2})\\ \times\exp\{\gamma(z)+Az+B\},\end{aligned}\tag{3.12}$$

where $b_1+\cdots+b_{-\lambda_1-\lambda_2}=-a_0$, $b_i\notin L$, $b_i\neq b_j$ ($i\neq j$).

2. If $-\lambda_1-\lambda_2=1$, $-a_0\in L$, then

$$\Phi_0(z)=C\sigma^{\lambda_1}(z-c_1)\sigma^{\lambda_2}(z-c_2)\,\frac{\sigma(z-b_1)\sigma(z-b_2)}{\sigma(z-a_1)}\cdot\exp\{\gamma(z)+Az+B\},\tag{3.13}$$

where $a_1-b_1-b_2=a_0$, $a_1,b_1,b_2\notin L$, $b_1\neq b_2$.

3. If $-\lambda_1-\lambda_2<1$ or $\lambda_1+\lambda_2=1$, $a_0\notin L$, then

$$\Phi_0(z)=C\,\frac{\sigma^{\lambda_1}(z-c_1)\sigma^{\lambda_2}(z-c_2)}{\sigma(z-a_1)\cdots\sigma(z-a_{\lambda_1+\lambda_2})}\,\exp\{\gamma(z)+Az+B\},\tag{3.14}$$

where $a_1+\cdots+a_{\lambda_1+\lambda_2}=a_0$, $a_i\notin L$, $a_i\neq a_j$ ($i\neq j$).

4. If $\lambda_1+\lambda_2=1$, $a_0\in L$, then

$$\Phi_0(z)=C\frac{\sigma^{\lambda_1}(z-c_1)\sigma^{\lambda_2}(z-c_2)\sigma(z-b_1)}{\sigma(z-a_1)\sigma(z-a_2)}\cdot\exp\{\gamma(z)+Az+B\},\tag{3.15}$$

where $a_1+a_2-b_1=a_0$; $a_1,a_2,b_1\notin L$, $a_1\neq a_2$.

5. If $-\lambda_1-\lambda_2=0$, then

$$\Phi_0(z)=C\sigma^{\lambda_1}(z-c_1)\sigma^{\lambda_2}(z-c_2)\,\frac{\sigma(z-b_1)}{\sigma(z-a_1)}\,\exp\{\gamma(z)+Az+B\},\tag{3.16}$$

where $a_1 - b_1 = a_0$; $a_1, b_1 \notin L$.

Remark 2. The sum $-\lambda_1 - \lambda_2$ does not depend on a choice of the branch of the function $\ln G(t)$ [11, Ch. 4].

Let us check the uniqueness. Let $\Phi_1(z)$ be a possible solution of Problem 3.2 with simple poles at the points

$$a_1^0 + 2m\omega_1 + 2ni\omega_2, \dots, a_{n_1}^0 + 2m\omega_1 + 2ni\omega_2; \quad a_1^0, \dots, a_{n_1}^0 \notin L, \quad a_i^0 \neq a_j^0,$$
$$m, n = 0, \pm 1, \pm 2, \dots \ ,$$

n_1 is a natural number. We assume that if $\Phi_0(z)$ has poles, some poles of the function $\Phi_1(z)$ are equal to some poles of the function $\Phi_0(z)$, i.e., $a_i = a_i^0$, $i = 1, \dots, p$ $(p \leq n_1)$.

By condition (3.3), we have

$$\Big(\frac{\Phi_1(t)}{\Phi_0(t)}\Big)^+ = \Big(\frac{\Phi_1(t)}{\Phi_0(t)}\Big)^-, \quad t \in L,$$

where $\Phi_0(z)$ is given by (3.12), (3.13), (3.14), (3.15) or (3.16) and this condition is fulfilled except the ends.

Consequently, the function $\frac{\Phi_1(z)}{\Phi_0(z)}$ is doubly-periodic holomorphic in every bounded domain of the z-plane with simple poles $a_1^0 + 2m\omega_1 + 2ni\omega_2, \dots, a_{n_1}^0 + 2m\omega_1 + 2ni\omega_2$, $b_1 + 2m\omega_1 + 2ni\omega_2, \dots, b_q + 2m\omega_1 + 2ni\omega_2$ $(0 \leq q \leq -\lambda_1 - \lambda_2)$, where $q = -\lambda_1 - \lambda_2$ in the case of (3.12), $q = 2$ in the case of (3.13), $q = 1$ in the case (3.15), (3.16), and with the poles $a_{p+1}^0 + 2m\omega_1 + 2ni\omega_2, \dots, a_{n_1}^0 + 2m\omega_1 + 2ni\omega_2$ in the case of (3.14). Hence Theorem 1.6 implies:

1. If $-\lambda_1 - \lambda_2 > 1$ or $-\lambda_1 - \lambda_2 = 1$, $-a_0 \notin L$, then

$$\begin{aligned}\Phi^*(z) &= \frac{\Phi_1(z)}{\Phi_0(z)} \\ &= C_0 \frac{\sigma(z-\alpha_1)\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_{n_1-\lambda_1-\lambda_2})}{\sigma(z-b_1)\cdots\sigma(z-b_{-\lambda_1-\lambda_2})\sigma(z-a_1^0)\cdots\sigma(z-a_{n_1}^0)}\,,\end{aligned} \tag{3.12*}$$

where C_0 is an arbitrary constant, the constants $\alpha_1, \dots, \alpha_{n_1-\lambda_1-\lambda_2}$ satisfy the conditions

$$\alpha_1 + \cdots + \alpha_{n_1-\lambda_1-\lambda_2} = b_1 + \cdots + b_{-\lambda_1-\lambda_2} + a_1^0 + \cdots + a_{n_1}^0,$$
$$\alpha_1, \dots, \alpha_{n_1-\lambda_1-\lambda_2} \notin L, \quad \alpha_i \neq \alpha_j \quad (i \neq j).$$

2. If $-\lambda_1 - \lambda_2 = 1$, $-a_0 \in L$, then

$$\Phi^*(z) = \frac{\Phi_1(z)}{\Phi_0(z)} = C\,\frac{\sigma(z-a_1)}{\sigma(z-b_1)\sigma(z-b_2)} \cdot \frac{\sigma(z-\alpha_1)\cdots\sigma(z-\alpha_{n_1+1})}{\sigma(z-a_1^0)\cdots\sigma(z-a_{n_1}^0)}\,, \tag{3.13*}$$

where

$$a_1 + \alpha_1 + \cdots + \alpha_{n_1+1} = b_1 + b_2 + a_1^0 + \cdots + a_{n_1}^0,$$
$$\alpha_1, \dots, \alpha_{n_1+1} \notin L, \quad \alpha_i \neq \alpha_j \quad (i \neq j)$$

(the case $a_1 = a_1^0$ is not excluded).

3. If $-\lambda_1-\lambda_2<-1$ or $-\lambda_1-\lambda_2=-1$, $a_0\notin L$, then

$$\begin{aligned}\Phi^*(z)&=\frac{\Phi_1(z)}{\Phi_0(z)}\\&=C\,\frac{\sigma(z-a_{p+1})\cdots\sigma(z-a_{\lambda_1+\lambda_2})\sigma(z-\alpha_1)\cdots\sigma(z-\alpha_{n_1-\lambda_1-\lambda_2})}{\sigma(z-a^0_{p+1})\cdots\sigma(z-a^0_{n_1})}\,,\end{aligned}\tag{3.14*}$$

where

$$\alpha_1+\cdots+\alpha_{n_1-\lambda_1-\lambda_2}+\alpha_{p+1}+\cdots+a_{\lambda_1+\lambda_2}=a^0_{p+1}+\cdots+a^0_{n_1},$$
$$\alpha_1,\dots,\alpha_{n_1-\lambda_1-\lambda_2}\notin L,\quad \alpha_i\neq\alpha_j\quad(i\neq j).$$

4. If $-\lambda_1-\lambda_2=-1$ and $a_0\in L$, then

$$\Phi^*(z)=\frac{\Phi_1(z)}{\Phi_0(z)}=C\,\frac{\sigma(z-a_1)\sigma(z-a_2)\sigma(z-\alpha_1)\cdots\sigma(z-\alpha_{n_1-1})}{\sigma(z-b_1)\sigma(z-a^0_1)\cdots\sigma(z-a^0_{n_1})}\,,\tag{3.15*}$$

where $n_1\geq 1$,

$$a_1+a_2+\alpha_1+\cdots+\alpha_{n_1-1}=b_1+a^0_1+\cdots+a^0_{n_1},$$
$$\alpha_1,\dots,\alpha_{n_1-1}\notin L,\quad \alpha_i\neq\alpha_j\quad(i\neq j)$$

(the case $a_1=a^0_1$, $a_2=a^0_2$ is not excluded).

5. If $-\lambda_1-\lambda_2=0$, then

$$\Phi^*(z)=\frac{\Phi_1(z)}{\Phi_0(z)}=c\,\frac{\sigma(z-a_1)\sigma(z-\alpha_1)\cdots\sigma(z-\alpha_{n_1})}{\sigma(z-b_1)\sigma(z-a^0_1)\cdots\sigma(z-a^0_{n_1})}\,,\tag{3.16*}$$

where

$$a_1+\alpha_1+\cdots+\alpha_{n_1}=b_1+a^0_1+\cdots+a^0_{n_1},$$
$$\alpha_1,\dots,\alpha_{n_1}\notin L,\quad \alpha_i\neq\alpha_j\quad(i\neq j)$$

(the case $a_1=a^0_1$ is not excluded).

Remark 3. If $\Phi_1(z)$ has no poles, then $\Phi^*(z)=C$ in the case of (3.14*), (3.15*), (3.16*).

The above reasoning implies

Theorem 3.1. *There exist solutions of Problem* 3.2 *and solutions of a given class with a minimal number of poles given by*

1. (3.12) *if* $-\lambda_1-\lambda_2>1$ *or* $-\lambda_1-\lambda_2=1$, $-a_0\notin L$;
2. (3.13) *if* $-\lambda_1-\lambda_2=1$, $-a_0\in L$;
3. (3.14) *if* $-\lambda_1-\lambda_2<-1$ *or* $-\lambda_1-\lambda_2=-1$, $a_0\notin L$;
4. (3.15) *if* $-\lambda_1-\lambda_2=-1$, $a_0\in L$;
5. (3.16) *if* $-\lambda_1-\lambda_2=0$.

A general solution of Problem 3.2 *is given by* $\Phi_1(z)=\Phi_0(z)\Phi^*(z)$, *where* $\Phi^*(z)$ *is given by* (3.12*), (3.13*), (3.14*), (3.15*) *or* (3.16*).

Let us return to Problem 3.1.

In this case, the ends will be called special if they are the special ends of Problem 3.2. According to Muskhelishvili [11], all possible solutions of Problem

3.1 are divided into classes $\widetilde{h}_0$, $\widetilde{h}(c_1)$, $\widetilde{h}(c_2)$, $\widetilde{h}_2$ by means of the non-special ends c_1 or c_2, near which the solutions of a given class must be bounded.

A solution of Problem 3.1 belonging to a given class can now be sought.

Consider the function $\frac{\Phi(z)}{\Phi_0(z)}$, where $\Phi(z)$ is a solution of Problem 3.2 and $\Phi_0(z)$ is given by (3.12), (3.13), (3.14), (3.15) or (3.16). Taking into account (3.2) and (3.3), we obtain

$$\frac{\Phi^+(t)}{\Phi_0^+(t)} - \frac{\Phi^-(t)}{\Phi_0^-(t)} = \frac{g(t)}{\Phi_0^+(t)}\,, \quad t \in L. \tag{3.17}$$

Condition (3.17) holds except the ends of L.

By condition (3.1) the function $\frac{\Phi(z)}{\Phi_0(z)}$ is doubly-periodic sectionally holomorphic with jump line L except:

1. the points $b_1 + 2m\omega_1 + 2ni\omega_2, \dots, b_{-\lambda_1-\lambda_2} + 2m\omega_1 + 2ni\omega_2$ in the case of (3.12);
2. the points $b_1 + 2m\omega_1 + 2ni\omega_2, b_2 + 2m\omega_1 + 2ni\omega_2$ in the case of (3.13);
3. the points $b_1 + 2m\omega_1 + 2ni\omega_2$ in the case of (3.15), (3.16);

In the case of (3.14) the function $\frac{\Phi(z)}{\Phi_0(z)}$ has no poles.

By Theorems 2.1 and 2.2 we formulate the following statement.

Theorem 3.2. 1. *For $-\lambda_1-\lambda_2 > 1$ (or $-\lambda_1-\lambda_2 = 1$, $-a_0 \notin L$), the solutions of a given class of Problem* 3.1 *exist and are given by*

$$\Phi(z) = \frac{\Phi_0(z)}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Phi_0^+(t)} \big[\zeta(t-z) - \zeta(b_1 - z)\big]\,dt + C_1\Phi_0(z)\,\frac{\sigma(z-\alpha_1^*)\cdots\sigma(z-\alpha^*_{-\lambda_1-\lambda_2})}{\sigma(z-b_1)\cdots\sigma(z-b_{-\lambda_1-\lambda_2})} + C_2\Phi_0(z)\,, \tag{3.18}$$

where $\Phi_0(z)$ is given by (3.12), *the constants $\alpha_1^*, \dots, \alpha^*_{-\lambda_1-\lambda_2}$ satisfy the condition*

$$\alpha_1^* + \cdots + \alpha^*_{-\lambda_1-\lambda_2} = b_1 + \cdots + b_{-\lambda_1-\lambda_2}.$$

If $-\lambda_1-\lambda_2 > 1$, then C_1, C_2 are arbitrary constants.

If $-\lambda_1-\lambda_2 = 1$, $-a_0 \notin L$, then $C_1 = 0$ and C_2 is an arbitrary constant.

2. *For of $-\lambda_1-\lambda_2 < -1$ (or $\lambda_1+\lambda_2 = 1$, $a_0 \notin L$), a solution of a given class of Problem* 3.1 *exists if and only if*

$$\int\limits_{L_{00}} \frac{g(t)\,dt}{\Phi_0^+(t)} = 0,$$

$$\int\limits_{L_{00}} \frac{g(t)}{\Phi_0^+(t)}\,[\zeta(t-a_1) - \zeta(t-a_k)]dt = 0, \quad k = 2, \dots, \lambda_1+\lambda_2,$$

and is given by

$$\Phi(z) = \frac{\Phi_0(z)}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Phi_0^+(t)} \big[\zeta(t-z) - \zeta(t-a_1)\big]\,dt, \tag{3.19}$$

where $\Phi_0(z)$ is given by (3.14).

3. *For $-\lambda_1-\lambda_2=1$, $-a_0\in L$, the solutions of a given class of Problem* 3.1 *exist and are given by*

$$\Phi(z)=\frac{\Phi_0(z)}{2\pi i}\int\limits_{L_{00}}\frac{g(t)}{\Phi_0^+(t)}\big[\zeta(t-z)-\zeta(b_1-z)-\zeta(t-a_1)+\zeta(b_1-a_1)\big]\,dt$$
$$+\,C_1\Phi_0(z)\,\frac{\sigma(z-a_1)\sigma(z+a_0)}{\sigma(z-b_1)\sigma(z-b_2)}\,, \tag{3.20}$$

where $\Phi_0(z)$ is given by (3.13) *and C_1 is an arbitrary constant.*

4. *For $\lambda_1+\lambda_2=1$ and $a_0\in L$, a solution of a given class of Problem* 3.1 *exists if and only if*

$$\int\limits_{L_{00}}\frac{g(t)}{\Phi_0^+(t)}\,[\zeta(t-a_1)-\zeta(b_1-a_1)]dt=\int\limits_{L_{00}}\frac{g(t)}{\Phi_0^+(t)}\,[\zeta(t-a_2)-\zeta(b_1-a_2)]dt,$$

and is given by (3.20)*, where $C_1=0$, and $\Phi_0(z)$ is given by* (3.15)*.*

5. *For $-\lambda_1-\lambda_2=0$ and $a_0\neq 0$, a solution of a given class of Problem* 3.1 *exists and is given by* (3.20)*, where $\Phi_0(z)$ is given by* (3.16) *and $C_1=0$.*

6. *For $-\lambda_1-\lambda_2=0$ and $a_0=0$, a solution of a given class of Problem* 3.1 *exists if and only if*

$$\int\limits_{L_{00}}\frac{g(t)}{\Phi_0^+(t)}\,dt=0$$

and is given by

$$\Phi(z)=\frac{\Phi_0(z)}{2\pi i}\int\limits_{L_{00}}\frac{g(t)}{\Phi_0^+(t)}\,\zeta(t-z)\,dt+C_1\Phi_0(z),$$

where C_1 is an arbitrary constant and $\Phi_0(z)$ is given by (3.16) *for $a_1=b_1$.*

Remark 4. A doubly-periodic solution of Problem 3.1 exists for $\gamma_1=2\pi ik_1$, $\gamma_2=2\pi ik_2$, where k_1 and k_2 are the definite integers.

Remark 5. Near special ends, a solution of Problem 3.1 remains bounded with a possible exception for those ends where the numbers β_1 (β_2) are zero; near the latter numbers the solutions may be bounded, but they will certainly be almost bounded (of the logarithmic type) [11].

Remark 6. In the case where the contour L_{00} consists of several non-intersected open arcs L_{00}^j, $j=1,2,\dots,k$, a solution of the corresponding Problem 3.2 is representable by $\Phi_0(z)=\Phi_1(z)\Phi_2(z)\cdots\Phi_k(z)$, where $\Phi_1(z),\Phi_2(z),\dots,\Phi_k(z)$ are the solutions of the corresponding Problems of 3.2 type for L_{00}^j, $j=1,2,\dots,k$, respectively, we will chose different zeros and poles for every $\Phi_j(z)$, $j=1,2,\dots,k$. So Theorems 3.1 and 3.2 are true for this case, too.

Remark 7. In the case where line of integration is a segment, an integral with Weierstrass kernel of type (2.1) was first considered by Sedov for boundary value problems of hydrodynamics [13]. This type of integrals was used by Chibrikova [5] for solving doubly-periodic boundary value problems.

Acknowledgement

The author would like to thank Professors V. Paatashvili and D. Shulaia for reading the first version of this paper and providing numerous suggestions.

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(Received 19.07.2004; revised 15.03.2005)

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