BEYOND MOSER’S CONJECTURE ON GRUNSKY INEQUALITIES

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Abstract. We prove a weakened version of a conjecture concerning the relation between the Grunsky constant and the Teichmüller norm of extensions of univalent functions.

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1. TWO CONJECTURES ON THE GRUNSKY INEQUALITIES

1.1. We consider the class $\Sigma$ of univalent holomorphic functions

$$f(z) = z + b_0 + b_1z^{-1} + \cdots \quad (1.1)$$

mapping conformally the disk $\Delta^* = \{z \in \hat{\mathbb{C}} : |z| > 1\}$ into $\hat{\mathbb{C}} \setminus \{0\}$. Let $\Sigma(k)$ denote the set of $f \in \Sigma$ having a $k$-quasiconformal extension onto the whole Riemann sphere $\hat{\mathbb{C}}$, and let $\Sigma^0 = \bigcup_k \Sigma(k)$. The functions from $\Sigma^0$ are uniquely determined, for example, by adding to (1.1) the condition $f(0) = 0$.

This collection naturally relates to the universal Teichmüller space $T$ modelled as a bounded domain in the complex Banach space $B$ of hyperbolically bounded holomorphic functions in $\Delta^*$ with the norm

$$\|\varphi\|_B = \sup_{\Delta^*} \left((|z|^2 - 1)^2 \right) |\varphi(z)|. \quad (1.2)$$

The elements of this space are the Schwarzian derivatives

$$S_f = (f''/f')' - (f''/f')^2/2$$

of locally univalent holomorphic functions in $\Delta^*$, while the indicated domain $T$ consists of the Schwarzians derivatives of those functions which are univalent in the whole disk $\Delta^*$ and have quasiconformal extensions onto the sphere $\hat{\mathbb{C}}$. Note that $\varphi(z) = O(|z|^{-4})$ as $z \to \infty$.

The fundamental Grunsky theorem states that a holomorphic function in $\Delta^*$ with expansion (1.1) is univalent in this disk if and only if it satisfies the inequality

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}x_mx_n \right| \leq 1, \quad (1.3)$$

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where the Grunsky coefficients \( \alpha_{mn}(f) \) are determined by

\[
\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2,
\]

(1.4)

taking the principal branch of the logarithmic function, and \( x = (x_n) \) ranges over the unit sphere \( S(l^2) \) of the Hilbert space \( l^2 \) of sequences with \( \|x\|^2 = \sum_1^\infty |x_n|^2 \) (cf. [3]).

For \( f \in \Sigma^0 \), the inequality (1.3) is strengthened as follows:

\[
\kappa(f) := \sup \left\{ \left. \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| : x = (x_n) \in S(l^2) \right\} \leq k(f),
\]

(1.5)

where \( k(f) \) denotes the minimal dilatation among quasiconformal extensions of \( f \) from \( \Delta^* \) onto \( \hat{\mathbb{C}} \) (cf. [10]). The quantity \( \kappa(f) \) is called the Grunsky constant of the map \( f \).

A crucial point here is that for a generic function \( f \in \Sigma^0 \), one has in (1.5) the strict inequality

\[
\kappa(f) < k(f).
\]

(1.6)

(see, e.g., [11], [12], [4]), and it is important to know for which functions the equality in (1.6) occurs. This relates to various topics. For example, there is an intrinsic connection with Fredholm eigenvalues of (closed) Jordan curves, which relies on the basic Kühnau–Schiffer theorem on the reciprocity of \( \kappa(f) \) to the least positive Fredholm eigenvalue \( \rho_L \) (cf. [11], [15]).

1.2. In 1985 Jürgen Moser conjectured that the set of functions with \( \kappa(f) = k(f) \) is rather sparse in \( \Sigma^0 \) so that any function \( f \in \Sigma \) is approximated by functions satisfying (1.6), uniformly on compact sets in \( \Delta^* \).

Moser’s conjecture has recently been proved in [9], where two different proofs of this conjecture are given. One of these proofs also shows that approximating functions \( f_n \) can be chosen in such a way that the images of the unit circle \( \partial \Delta = \{ |z| = 1 \} \) are analytic Jordan curves.

There is another, still open, conjecture posed in [9] which states that, in contrast to the above, functions \( f \in \Sigma^0 \) with \( \kappa(f) < k(f) \) cannot be approximated by \( f_n \in \Sigma^0 \) with \( \kappa(f_n) = k(f_n) \) (again in the topology of uniform convergence on compact sets in \( \Delta^* \)).

1.3. In this paper, we prove the following weakened version of the second conjecture:

**Theorem 1.1.** Any sequence of functions \( f_n \in \Sigma^0 \) with \( \kappa(f_n) = k(f_n) \) and such that the curves \( f_n(|z| = 1) \) are asymptotically conformal, cannot converge locally uniformly in \( \Delta^* \) to a function \( f \in \Sigma \) with \( \kappa(f) < k(f) \).
We recall that a Jordan curve $L \subset \mathbb{C}$ is called \emph{asymptotically conformal} if for any pair of points $a, b \in L$,

$$\max_{z \in L(a,b)} \frac{|a - z| + |z - b|}{|a - b|} \to 1 \quad \text{as} \quad |a - b| \to 0,$$

(1.7)

where $L(a, b)$ denotes the subarc of $L$ with endpoints $a, b$ and of smaller diameter. Such curves are quasicircles (i.e., the images of the circle under quasiconformal homeomorphisms of the sphere $\hat{\mathbb{C}}$) and cannot have corners. All $C^1$-smooth curves are asymptotically conformal. On the other hand, asymptotically conformal curves can be rather pathological (see, e.g., [14], p.249).

For $L = f(\partial \Delta)$, $f \in \Sigma^0$, the condition (1.7) is equivalent, for example, to one of the following:

(i) $S_f(z) = o(|z|^{-2})$ as $|z| \to 1+$;

(ii) $f$ extends quasiconformally onto the disk $\Delta$ with Beltrami coefficient $\mu$ satisfying

$$\operatorname{ess \ sup}_{|z| > r} |\mu(z)| \to 0 \quad \text{as} \quad r \to 1-. $$

2. Proof of Theorem 1.1

We first construct, for any $f \in \Sigma$, the complex isotopy $f(z, t) : \Delta^* \times \Delta \to \hat{\mathbb{C}}$ by

$$f(z, t) = tf(t^{-1}z) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \cdots. \quad (2.1)$$

The Schwarzian derivatives $\varphi(t, z) = S_{f_t}(z)$ of the fiber maps $f_t(z) := f(z, t)$ relate to $S_f$ by $S_{f_t}(z) = t^{-2} S_f(t^{-1}z)$. This equality defines a holomorphic map

$$\Phi_f(t) = \varphi(t, \cdot) : \Delta \to T \quad (2.2)$$

so that if

$$f(z) = z + b_0 + \sum_{n=m}^{\infty} b_n z^{-n}, \quad b_m \neq 0 \ (m \geq 1),$$

then

$$\Phi_f(0) = \Phi'_f(0) = \cdots = \Phi^{(m-1)}_f(0) = 0, \quad \Phi^m_f(0) \neq 0$$

(for details see, e.g., [5]). This shows that the image $\Phi_f(\Delta)$ is a holomorphic disk in $T$ with the single singular point $\Phi_f(0) = 0$.

Further, for all $f_t$ with fixed $|t| = r < 1$, we have the equalities $k(f_t) = k(f_r)$ and $\kappa(f_t) = \kappa(f_r)$. We shall use the notation

$$k_f(t) := k_{f_t}, \quad \kappa_f(t) := \kappa_{f_t}.$$ 

Both functions $k_f(t)$ and $\kappa_f(t)$ are logarithmically subharmonic and continuous on the disk $\Delta$; regarded as radial functions of the variable $r = |t|$, they are strictly monotone increasing and continuous on $[0, 1]$ (see [7], [11], [16]).

Our goal is to establish the relation between $\kappa_f(t)$ and $k_f(t)$ on $\Delta$. This will be derived by examining certain Finsler metrics of the generalized Gaussian curvature $\kappa \leq -4$ on the disk $\Phi_f(\Delta)$. The arguments follow the lines of [8].

**Lemma 2.1.** If $f \in \Sigma^0$ is such that $\kappa(f) < k(f)$, then $\kappa_f(r) < k_f(r)$ for all $0 < r < 1$. 
The proof of this lemma is based on the maximum principle of Minda. To present it, recall that the \textit{generalized Laplacian} $\Delta u$ of an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$
\Delta u(z) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta - u(z) \right\}.
$$

(2.3)

It reduces to the usual Laplacian $4 \partial \bar{\partial}$ for $C^2$ functions and has similar properties: a function $u$ is subharmonic in $\Omega$ if and only if $\Delta u(z) \geq 0$; at a point $z_0$ of a local maximum of an upper semicontinuous function $u$ with $u(z) > -\infty$, we have $\Delta u(z_0) \leq 0$.

**Lemma 2.2** ([13]). If a function $u : \Omega \to [-\infty, +\infty)$ is upper semicontinuous in a domain $\Omega \subset \mathbb{C}$ and its generalized Laplacian satisfies the inequality $\Delta u(z) \geq Ku(z)$ with some positive constant $K$ at any point $z \in \Omega$, where $u(z) > -\infty$, then if

$$
\limsup_{z \to \zeta} u(z) \leq 0 \quad \text{for all } \zeta \in \partial \Omega,
$$

then either $u(z) < 0$ for all $z \in \Omega$ or $u(z) = 0$ for all $z \in \Omega$.

We also recall that the (generalized) Gaussian curvature $\kappa_\lambda$ of an upper semicontinuous Finsler metric $ds = \lambda(t)|dt|$ on the domain $\Omega$ is defined by

$$
\kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2},
$$

(2.4)

where $\Delta$ denotes the generalized Laplacian (2.3).

One defines in a similar way also the sectional holomorphic curvature of a Finsler metric on complex Banach manifolds $X$ as the supremum of the curvatures (2.4) over appropriate collections of holomorphic maps $\Delta \to X$ for a given tangent direction (for details, we refer, e.g., to [2]). We shall use only the fact that the holomorphic curvature of the Kobayashi metric of the universal Teichmüller space $T$ equals $-4$ everywhere (cf. [1], [7]).

**Proof of Lemma 2.1.** The Grunsky coefficients $\alpha_{mn}$ of functions $f \in \Sigma^0$, together with the Taylor coefficients $b_k$ of these functions, depend holomorphically on their Schwarzian derivatives $\varphi = S_f \in B$. Therefore we have a family of holomorphic maps

$$
h_\mathbf{x}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(\varphi)x_mx_n : T \to \Delta
$$

(2.5)

parametrized by the points $\mathbf{x} = (x_n) \in S(t^2)$.

Using these maps, we define, on the unit disk $\Delta$ the conformal metrics $ds = \lambda_{h_\mathbf{x} \circ \Phi_f}(t)|dt|$ as the pull-backs of the hyperbolic metric $\lambda_0(\zeta)|d\zeta|$ in $\Delta$ of curvature $-4$ under the maps (2.5), where $\varphi \in \Phi_f(\Delta)$; explicitly,

$$
\lambda_{h_\mathbf{x} \circ \Phi_f}(t) := (h_\mathbf{x} \circ \Phi_f)^*(\lambda_0) = \frac{|(h_\mathbf{x} \circ \Phi_f)'(t)|}{1 - |h_\mathbf{x} \circ \Phi_f(t)|^2}.
$$

(2.6)
Let $\lambda_d$ denote the restriction to $\Phi_f(\Delta)$ of the Kobayashi–Teichmüller metric $d_T$ on the space $T$. It induces, on $\Delta$, a Finsler metric $\Phi_f^*(\lambda_d)$ of curvature $-4$.

The assertion of Lemma 2.1 follows from comparing the latter metric with the upper envelope of metrics (2.6). Namely, assume that for some $r = r_0 < 1$, we have

$$\kappa_{f_0}(r_0) = \kappa_f(r_0).$$

Fix $t_0 = r_0 e^{i\theta_0}$ and consider a maximizing sequence $\{x_n\}_{n=1}^\infty \subset S(l^2)$, for which

$$\sup \{ \lambda_{h_n \circ \Phi_f}(t_0) : x \in S(l^2) \} = \lim_{n \to \infty} \lambda_{h_n \circ \Phi_f}(t_0).$$

The functions $h_{x_n} \circ \Phi_f$ converge to a holomorphic function $g_0$ on $\Delta$ which determines a $C^2$ smooth conformal metric $\lambda_{g_0}(t) = g_0^*(\lambda_0)$ of curvature $-4$ in a sufficiently small neighborhood $U(t_0)$ of the distinguished point $t_0$. The relations (1.6) and (2.7) yield

$$\lambda_{g_0}(t_0) = \lambda_d(t_0), \quad \lambda_{g_0}(t) < \lambda_d(t) \quad \text{for } t \in U(t_0) \setminus \{t_0\},$$

which means that $\lambda_{g_0}$ is a supporting metric for $\lambda_d$ at the point $t_0$.

Let

$$M = \{ \sup \lambda_d(t) : t \in U(t_0) \}.$$ 

Then $\lambda_d(t) + \lambda_{g_0}(t) \leq 2M$. Consider the function

$$u = \log \frac{\lambda_{g_0}}{\lambda_d}.$$ 

Then (cf. [13], [2]) for $t \in U(t_0)$,

$$\Delta u(t) = \log \lambda_{g_0}(t) - \lambda_d(t) = 4(\lambda_{g_0}^2 - \lambda_d^2) \geq 8M(\lambda_{g_0} - \lambda_d).$$

The elementary estimate

$$M \log(t/s) \geq t - s \quad \text{for } 0 < s \leq t < M$$

with equality for $s = t$ implies that

$$M \log \frac{\lambda_{g_0}(t)}{\lambda_d(t)} \geq \lambda_{g_0}(t) - \lambda_d(t),$$

and hence

$$\Delta u(t) \geq 4M^2 u(t).$$

Applying to $u$ the maximum principle of Minda (Lemma 2.2), we obtain that, in view of the first equality in (2.8), the metric $\lambda_{g_0}$ must coincide with $\lambda_d$ in the whole disk $\Delta$, which is equivalent to the equality of the functions $\kappa_f(r)$ and $k_f(r)$ for $0 \leq r < 1$. By continuity, these functions must coincide also at $r = 1$, but this contradicts the assumption of Lemma 2.1.

**Lemma 2.3.** Let $f \in \Sigma^0$ with $\kappa(f) = k(f)$ map the unit circle $\partial \Delta$ onto an analytic curve. Then $\kappa_f(r) = k_f(r)$ on the whole interval $[0, 1]$. 

Proof. According to the well-known properties of conformal maps of domains with analytic boundaries, the function $f$ extends to a conformal map $\hat{f}$ of some disk $\Delta^* := \{|z| > a\}$, $a < 1$. It follows that for $z \in \Delta^*$,
\[
f(z) = \hat{f}_a(z) := a\hat{f}(a^{-1}z).
\]
The assumption $\kappa_f = k_f$ yields
\[
\kappa_{\hat{f}}(a) = k_{\hat{f}}(a). \tag{2.9}
\]
We now use the upper semi-continuous envelope of the metrics (2.6)
\[
\lambda_\kappa(t) = \limsup_{t' \to t} \sup_{x \in S(l^2)} \lambda_{b_{x(\Phi f(t'))}}(t')
\]
Similarly to [8], one obtains that this metric is logarithmically subharmonic on $\Delta$ and its curvature $\kappa_{\lambda_\kappa}(t) \leq -4$.

On the other hand, together with all metrics (2.6), $\lambda_\kappa$ is majorized by the Kobayashi–Teichmüller metric $\lambda_d$. Therefore, the equality (2.9) means that the function $\log \lambda_{\lambda_\kappa}$ attains, at the point $t = a$, its maximal value on the disk $\Delta$. By Minda’s principle (applied similarly to the above), the latter is possible only if $\kappa_{\hat{f}}(t) = k_{\hat{f}}(t)$ for all $t \in \Delta$ or, equivalently, if $\kappa_f(t) = k_f(t)$.

Lemma 2.4. Every function $f \in \Sigma^0$ with $\kappa_f = k_f$ mapping the unit circle $\partial \Delta$ onto an asymptotically conformal curve is approximated locally uniformly on $\Delta$ by functions $f_n \in \Sigma^0$ with $\kappa_{f_n} = k_{f_n}$ mapping the circle $\partial \Delta$ onto analytic curves.

Proof. Due to [6], the function $f$ has (a unique) extremal quasiconformal extension $\hat{f}^{\mu_0}$ to $\Delta$ which is of the Teichmüller–Kühnau type; its Beltrami coefficient $\mu_0 = k_0|\psi_0|/\psi_0$ is determined by a quadratic differential $\psi_0$ of the form
\[
\psi_0(z) = \frac{1}{\pi} \sum_{m+n=2}^\infty \sqrt{mn} x_m x_n z^{m+n-2} \quad \text{with} \quad x = (x_n) \in l^2,
\]
which has, in $\Delta$, only zeros of even order. It is uniquely determined, letting $\sum_1^\infty |x_n|^2 = 1$.

Now choose in $[0, 1]$ a sequence $\{r_n\}$ approaching 1 and put
\[
\psi_n(z) = c_n\psi_0(r_n z),
\]
choosing $c_n > 0$ so that $||\psi_n||_{L^1(\Delta)} = 1$. The Beltrami coefficients $\mu_n = k_0|\psi_n|/\psi_n$, extended by zero to $\Delta^*$, determine quasiconformal automorphisms $f_n := f^{\mu_n}$ of $\hat{\mathbb{C}}$ whose restrictions to $\Delta^*$ belong to $\Sigma^0$ and satisfy
\[
\kappa(f_n) = k(f_n), \quad n = 1, 2, \ldots,
\]
because every $\psi_n$ also has, in $\Delta$, only zeros of even order (cf. [12], [4]).

Since $\lim_{n \to \infty} \mu_n(z) = \mu_0(z)$ for all $z \in \hat{\mathbb{C}}$, the maps $f_n$ are convergent to $f^{\mu_0}$ in the spherical metric on $\hat{\mathbb{C}}$. Lemma 2.4 follows. \hfill \Box
Proof of Theorem 1.1. Suppose, on the contrary, that for some \( f_0 \in \Sigma^0 \), which satisfies (1.6), there exists a sequence of functions \( f_n \in \Sigma^0, \ n = 1, 2, \ldots \), with \( \kappa(f_n) = k(f_n) \), locally uniformly convergent to \( f_0 \) and such that the curves \( f_n(\partial \Delta) \) are asymptotically conformal. Applying to \( f_n \), if needed, the approximation Lemma 2.4, we can assume that the sequence \( \{f_n\} \) consists of the functions having analytic images \( f_n(\partial \Delta) \). Then we can use Lemma 2.3, which implies for all \( r \in [0,1] \) the equality

\[
\kappa_{f_n}(r) = k_{f_n}(r).
\]

As is well known, for each \( f \in \Sigma \), the homotopy maps \( f_t(z) \) belong to \( \Sigma(|t|^2) \) (see [5]). Thus, for a fixed \( t \in \Delta \), the family \( \{f_t : f \in \Sigma\} \) is compact in the spherical metric on \( \hat{\mathbb{C}} \). Applying this to the maps \( f_{n,t} \) generated by \( f_n \), \( n = 0, 1, 2, \ldots \), one concludes that for every \( |t| < 1 \) the homotopy maps \( f_{n,t} \) are convergent to \( f_{0,t} \) uniformly on the closed disk \( D^* \).

Now fix two values \( r_0 \) and \( r_1 \) close to 1 so that \( r_0 < r_1 \). Then, by Lemmas 2.1 and 2.3, we have for all \( r \in (0,1) \) the relations

\[
\kappa_{f_{0,r_0}}(r) < k_{f_{0,r_0}}(r), \quad \kappa_{f_{n,r_0}}(r) = k_{f_{n,r_0}}(r). \tag{2.10}
\]

On the other hand, it follows from the above that for any fixed integer \( m > 1 \), the derivatives of \( f_{n,r_1} \) of orders up to \( m \) are uniformly convergent to the corresponding derivatives of \( f_{0,r_1} \) on the closed disk \( D^* \), and therefore,

\[
\lim_{n \to \infty} \|S_{f_{n,r_0}} - S_{f_{0,r_0}}\|_B = 0. \tag{2.11}
\]

Since the Teichmüller and Grunsky functions \( k(f) \) and \( \kappa(f) \) are both continuous with respect to the convergence of \( S_f \in T \) in the norm (1.2), the equality (2.11) implies

\[
\lim_{n \to \infty} \kappa_{f_{n,r_0}}(r) = \lim_{n \to \infty} k_{f_{n,r_0}}(r) = \kappa_{f_{0,r_0}}(r) = k_{f_{0,r_0}}(r),
\]

and we reach a contradiction to the first inequality in (2.10). This completes the proof of the theorem. \( \square \)

3. Remark

The above arguments imply in fact something more: functions \( f \in \Sigma \) with \( \kappa(f) < k(f) \) cannot be approximated by functions \( f_n \in \Sigma^0 \) with \( \kappa(f_n) = k(f_n) \), provided \( f_n \) admit quasiconformal extensions \( \tilde{f}_n^\mu \) to \( \hat{\mathbb{C}} \) of the Teichmüller-Kühnau type, i.e., with \( \mu = k|\psi|/\psi \), where \( \psi \) belong to the set \( A_1^2 \subset L_1(\Delta) \) of integrable holomorphic functions in \( \Delta \) with zeros of even order.

As is well-known (see [4]), a function \( f \in \Sigma^0 \) satisfies \( \kappa(f) = k(f) \) if and only if its extremal extensions \( f^\mu \) to \( \hat{\mathbb{C}} \) satisfy

\[
\sup \left\{ \left| \int \int_{\Delta} \mu \psi \, dx \, dy \right| : \|\mu\|_\infty, \|\psi\|_1 = 1 \right\}. \tag{3.1}
\]

So the question remains open whether the conjecture is true for \( f_n \in \Sigma^0 \), for which the supremum in (3.1) is not attained on \( A_1^2 \).
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