COMPOSITION OPERATORS ON $Q_p$ SPACES

SONGXIAO LI

Abstract. Suppose that $\phi$ is a nonconstant analytic self-mapping of the unit disk $D$. Necessary or sufficient conditions for the composition operator $C_\phi : Q_q \to Q_p$ to be bounded or compact are given in terms of $\phi$. The function-theoretic characterization of the compactness of the composition operator $C_\phi : Q_q \to Q_{p,0}$ is also given.

2000 Mathematics Subject Classification: Primary 47B35. Secondary 30H05.

Key words and phrases: Composition operator, space $Q_p$, space $Q_{p,0}$.

1. Introduction

Let $D$ be the open unit disc in the complex plane and $\partial D$ the unit circle. The one-to-one holomorphic functions that map $D$ onto itself have the form $\lambda \varphi_a$, where $\lambda \in \partial D$ and $\varphi_a$ is the basic conformal automorphism defined by $\varphi_a = \frac{a-z}{1-\overline{a}z}$ for $a \in D$. It is easy to check that the inverse of $\varphi_a$ under composition is $\varphi_a$, i.e., $\varphi_a \circ \varphi_a(z) = z$ for $z \in D$. Also, $|\varphi_a'(z)| = \frac{1-|a|^2}{|1-\overline{a}z|^2}$ and $1 - |\varphi_a(z)|^2 = (1 - |z|^2)|\varphi_a'(z)|$.

For $p \in (0, \infty)$, the weighted Dirichlet-type space $D_p$ is the Hilbert space of all analytic functions on $D$ satisfying

$$
\|f\|_{D_p} = \left\{ \int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p dm(z) \right\}^{\frac{1}{2}} < \infty.
$$

Note, in particular, that $D_1 = H^2$. We say that an analytic function on $D$ belongs to the space $Q_p$ if $\|f\|_{Q_p} = \sup_{a \in D} \|f \circ \varphi_a\|_{D_p} < \infty$. Thus $Q_p$ can be considered as a Möbius invariant version of $D_p$. The subspace $Q_{p,0}$ of $Q_p$ consists of those functions $f$ in $Q_p$ for which $\lim_{|a| \to 1} \|f \circ \varphi_a\|_{D_p} = 0$. From [1] we know that the space $Q_p$ coincides with the space of all analytic functions $f$ such that

$$
\sup_{a \in D} \int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p dm(z) < \infty.
$$

$Q_{p,0}$ can be characterized as the class of all analytic functions $f$ on $D$ such that

$$
\lim_{|a| \to 1} \int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p dm(z) = 0.
$$

It is well known that $Q_1 = BMOA$, $Q_0$ is the classical Dirichlet space $D$ with $\|f\|_D = \left\{ \int_D |f'(z)|^2 dm(z) \right\}^{\frac{1}{2}}$ and that for all $p$, $1 < p < \infty$, $Q_p$ is the Bloch
space \( \mathcal{B} = \left\{ f : f \in H(D), \|f\|_\mathcal{B} = \sup_{a \in D} |f'(z)|(1 - |z|^2) < \infty \right\} \).

Also, \( Q_{1,0} = VMOA \) and, for \( p > 1 \), \( Q_{p,0} = \mathcal{B}_0 \). Finally, \( Q_p \) is a Banach space with norm \( \|f\| = |f(0)| + \|f\|_{Q_p} \), and \( Q_{p,0} \) is its closed subspace.

Denote by \( \Phi \) the set of all nonconstant analytic self-maps \( \phi \) of \( D \). For an analytic self-map \( \phi \) of the unit disk \( D \), the composition operator \( C_\phi \) is defined by \( C_\phi(f) = f(\phi(z)) \) for all analytic functions \( f \) on \( D \). It is interesting to characterize composition operators on various analytic function spaces; the book [2] contains plenty of relevant information. Problems of this kind have recently been studied for composition operators on \( Q_p \) spaces [3, 4]; between Bloch-type spaces and Hardy and Besov spaces [5]; between \( BMOA \) and \( VMOA \) spaces [6]; between \( B \) and \( Q_p \) [7], to mention only some related works.

In this paper, we continue to study composition operators on \( Q_p \) spaces and give the function-theoretic characterization of \( C_\phi : Q_q \to Q_p \) as bounded or compact. The compactness of \( C_\phi : Q_q \to Q_{p,0} \) is also investigated. Our compactness conditions for a composition operator \( C_\phi \) depend only on \( \phi \) and can be considered as a little improvement of the known results.

### 2. Composition Operator \( C_\phi : Q_q \to Q_p \)

The aim of this section is to investigate the boundedness and compactness of composition operators \( C_\phi : Q_q \to Q_p \). For this, we give the following lemma.

**Lemma 2.1.** Let \( p, q \in (0, \infty) \) and \( \phi \in \Phi \). Then:

1. the operator \( C_\phi : Q_q \to Q_p \) is bounded if and only if
   \[ \sup_{a \in D, f \in \mathbb{B}_{Q_q}} \int_D \left| f'(\phi(z))\phi'(z)\right|^2(1 - |\varphi_a(z)|^2)^p dm(z) < \infty; \quad (2.1) \]

2. the operator \( C_\phi : Q_q \to Q_p \) is compact if and only if \( \phi \) satisfies (2.1) and
   \[ \lim_{|t| \to 1} \sup_{a \in D, f \in \mathbb{B}_{Q_q}|\phi(z)|=t} \int \left| f'(\phi(z))\phi'(z)\right|^2(1 - |\varphi_a(z)|^2)^p dm(z) = 0, \quad (2.2) \]

where \( \mathbb{B}_{Q_q} = \{ f \in Q_q : \|f\|_{Q_q} \leq 1 \} \).

**Proof.** The lemma can be proved by modifying the arguments of [3, Theorem 5.1], we omit the details. \( \square \)

Although this lemma can be viewed as a characterization of bounded and compact composition operators \( C_\phi : Q_q \to Q_p \), its conditions (2.1) and (2.2) are rather abstract. The following theorem gives a characterization of \( C_\phi \) directly in terms of \( \phi \).

**Theorem 2.2.** Let \( p, q \in (0, \infty) \) and \( \phi \in \Phi \). Then the following conditions are satisfied.
(1) If
\[
\sup_{a \in D} \int_D \frac{\left| \phi'(z) \right|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \infty,
\] (2.3)
then \( C_\phi : Q_q \to Q_p \) is bounded.
Conversely, if \( C_\phi : Q_q \to Q_p \) is bounded, then
\[
\sup_{a \in D} \int_D \frac{\left| \phi'(z) \right|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \infty.
\] (2.4)

(2) If (2.3) holds and
\[
\lim_{|t| \to 1} \sup_{a \in D} \int_{|\phi(z)| > t} \frac{\left| \phi'(z) \right|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) = 0,
\] (2.5)
then \( C_\phi : Q_q \to Q_p \) is compact.
Conversely, if \( C_\phi : Q_q \to Q_p \) is compact, then
\[
\lim_{|t| \to 1} \sup_{a \in D} \int_{|\phi(z)| > t} \frac{\left| \phi'(z) \right|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) = 0.
\] (2.6)

Proof. (1) Observe that every function \( f \in Q_q, (1 - |z|^2)|f'(z)| \leq \|f\|_{Q_q} \). So if (2.3) holds, then
\[
\sup_{a \in D} \int_D \left| (f \circ \phi)'(z) \right|^2 (1 - |\varphi_a(z)|^2)^p dm(z)
= \sup_{a \in D} \int_D \left| f'((\phi(z)) \right|^2 \left| \phi'(z) \right|^2 (1 - |\varphi_a(z)|^2)^p dm(z)
\leq \|f\|_{Q_q} \sup_{a \in D} \int_D \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) \leq \text{const} \cdot \|f\|_{Q_q}.
\]
Hence \( C_\phi : Q_q \to Q_p \) is bounded.
Conversely, let \( C_\phi : Q_q \to Q_p \) be bounded. Since \( f_\theta(z) = \log \frac{1}{1 - e^{-i \theta} z} \in Q_q \) for all \( \theta \in [0, 2\pi) \), by Lemma 2.1,
\[
\sup_{a \in D} \sup_{\theta \in [0, 2\pi]} \int_D \left| f_\theta'((\phi(z)) \right|^2 \left| \phi'(z) \right|^2 (1 - |\varphi_a(z)|^2)^p dm(z) < \infty.
\]
Then for any \( a \in D \) and any \( \theta \in [0, 2\pi) \),
\[
A_{a, \theta} = \int_D \frac{1}{1 - e^{-i \theta} \phi(z)} \left| \phi'(z) \right|^2 (1 - |\varphi_a(z)|^2)^p dm(z)
= \int_D \frac{1}{1 - e^{-i \theta} \phi(z)} \left| \phi'(z) \right|^2 (1 - |\varphi_a(z)|^2)^p dm(z) < \infty.
\]
Thus we obtain (2.4) by integrating with respect to $\theta$, the Fubini theorem and the Poisson formula.

(2) The proof is similar to (1) and we omit the details. \qed

Next, we consider the composition operator induced by a boundedly valent holomorphic function where image lies inside the polygon inscribed in the unit circle. For this, we need the following lemma.

**Lemma 2.3 [4]**. Let $p \in (0, \infty)$ and $\phi \in \Phi$. Then the following statements are equivalent:

1. $C_\phi : D \rightarrow Q_p$ is compact;
2. $\lim_{|a| \rightarrow 1} \|C_\phi \varphi_a\|_{Q_p} = 0$.

**Theorem 2.4**. Let $p, q \in (0, \infty)$, and $\phi \in \Phi$ be boundedly valent such that $\phi(D)$ lies inside the polygon inscribed in the unit circle. Then the following statements are equivalent:

1. $C_\phi : Q_q \rightarrow Q_p$ is compact;
2. $C_\phi : B \rightarrow B$ is compact;
3. $C_\phi : B_0 \rightarrow B_0$ is compact.

**Proof.** (1) $\Rightarrow$ (2). Suppose that (1) holds. Since $D \subset Q_q$, $C_\phi : D \rightarrow Q_p$ is compact, by Lemma 2.3, we see that $\lim_{|a| \rightarrow 1} \|C_\phi \varphi_a\|_{Q_p} = 0$. Therefore

$$\lim_{|a| \rightarrow 1} \|C_\phi \varphi_a\|_{B} = 0.$$ (2.7)

(see [9]). Since $\phi \in \Phi$ is boundedly valent, $\phi \in D(\subset B_0)$. This fact and (2.7) imply

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)|\phi'(z)|}{1 - |\phi(z)|^2} = 0.$$ (2.8)

Therefore $C_\phi : B_0 \rightarrow B_0$ is compact.

(3) $\Rightarrow$ (1). Suppose that $C_\phi : B_0 \rightarrow B_0$ is compact. By [9], (2.8) holds, hence $\log \frac{1}{w - \phi} \in B_0$ for each $w \in \partial D$. By [10,Theorem 6.1] each boundedly valent function in $B_0$ belong to $Q_{p,0}$, hence $\phi \in Q_{p,0}$ and $\log \frac{1}{w - \phi} \in Q_{p,0}$ for each $w \in \partial D$. Thus

$$\sup_{a \in D} \int_D \frac{|\phi'(z)|^2}{|w - \phi(z)|^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \infty,$$ (2.9)

and

$$\lim_{|a| \rightarrow 1} \int_D \frac{|\phi'(z)|^2}{|w - \phi(z)|^2} (1 - |\varphi_a(z)|^2)^p dm(z) = 0,$$ (2.10)

for each $w \in \partial D$.

Let $\{w_i : 1 \leq i \leq n\}$ be the vertices of the inscribed polygon containing $\phi(D)$. Break up $D$ into a compact set $K$ and finitely many domains $E_i = \{z \in D : |w_i - \phi(z)| < t\}$, $i = 1, 2, \ldots, n$, where $t$ is chosen so that the domains are
disjoint, and so that \(|w_i - \phi(z)| \leq c \cdot (1 - |\phi(z)|^2)\) for some constant \(c > 1\), each \(z \in E_i\) and each \(i, 1 \leq i \leq n\). Then for each \(a \in D\) and each \(i, 1 \leq i \leq n\),

\[
\int_{E_i} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) 
\leq \int_{E_i} \frac{|\phi'(z)|^2}{|w_i - \phi(z)|^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \infty, \quad (2.11)
\]

Hence

\[
\int_{D} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) 
= \sum_{i=1}^{n} \int_{E_i} + \int_{K} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) 
\leq c \cdot \sum_{i=1}^{n} \int_{E_i} \frac{|\phi'(z)|^2}{|w_i - \phi(z)|^2} (1 - |\varphi_a(z)|^2)^p dm(z) 
+ c \cdot \int_{D} |\phi'(z)|^2 (1 - |\varphi_a(z)|^2)^p dm(z). \quad (2.12)
\]

This together with (2.9), (2.10) and \(\phi \in Q_{p,0}\) leads to

\[
\sup_{a \in D} \int_{D} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \infty, \quad (2.13)
\]

and

\[
\lim_{|a| \to 1} \int_{D} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) = 0. \quad (2.14)
\]

By (2.14), for any given \(\epsilon > 0\), there is \(r \in (0, 1)\) such that

\[
\lim_{|a| > r} \int_{D} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \epsilon,
\]

hence for any \(t \in (0, 1)\),

\[
\lim_{|a| > r} \int_{|\phi(z)| > t} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \epsilon. \quad (2.15)
\]

On the other hand, if \(|a| \leq r\), then by (2.13), we have

\[
\int_{D} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\varphi_a(z)|^2)^p dm(z) < \infty,
\]
which yields (since $\lim_{t \to 1} m(\{z \in D : |\phi(z)| > t\}) = 0$)

\[
\lim_{t \to 1} \sup_{|z| \leq r, |\phi(z)| > t} \int_{|\phi(z)| > t} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} (1 - |\phi_a(z)|^2)\,d\mu(z) = 0.
\]  

(2.16)

We get (2.5) by (2.15) and (2.16). Then we finish the proof by Theorem 2.2. \qed

3. Composition Operator $C_\phi : Q_q \to Q_p, 0$

In this section, we give some characterizations of a compact composition operator $C_\phi : Q_q \to Q_p, 0$. First, we give some lemmas.

**Lemma 3.1.** Let $X = Q_p$. Then

(1) every bounded sequence $(f_n)$ in $X$ is uniformly bounded on compact sets;
(2) for any sequence $(f_n)$ in $X$ such that $\|f_n\|_{Q_p} \to 0$, $f_n - f_n(0) \to 0$ uniformly on compact sets.

**Proof.** It is well known that a Bloch function can increase at most as fast as $\log \frac{1}{1 - |z|}$, i.e.,

\[|f_n(z) - f_n(0)| \leq \text{const} \cdot \|f\|_B \log \frac{1}{1 - |z|} \leq \text{const} \cdot \|f\|_{Q_p} \log \frac{1}{1 - |z|}.
\]

Hence follows the result. \qed

**Lemma 3.2 ([8, 11]).** Let $X, Y$ be two Banach spaces of analytic functions on $D$. Suppose that

(1) the point evaluation functionals on $X$ are continuous;
(2) the closed unit ball of $X$ is a compact subset of $X$ in the topology of uniform convergence on compact sets.
(3) $T : X \to Y$ is continuous when $X$ and $Y$ are given the topology of uniform convergence on compact subsets in $D$.

Then $T$ is a compact operator if and only if, given a bounded sequence $(f_n)$ in $X$ such that $f_n \to 0$ uniformly on compact sets, the sequence $(Tf_n)$ converges to zero in the norm of $Y$.

**Lemma 3.3.** Let $\phi \in \Phi$ and $p, q \in (0, \infty)$. Then $C_\phi : Q_q \to Q_p$ is a compact composition operator if and only if, for any bounded sequence $(f_n)$ in $Q_q$ with $(f_n) \to 0$ uniformly on compact sets as $n \to \infty$, $\|C\phi f_n\|_{Q_p} \to 0$ as $n \to \infty$.

**Proof.** By Lemma 3.1 it is easy to see that (1) and (3) of Lemma 3.2 hold. To complete the proof, we have only to show that (2) of lemma 3.2 holds for $T = C_\phi$. Let $f_n \in B_{Q_q}$, then by Lemma 3.1, $(f_n)$ is uniformly bounded on compact sets. Therefore, by Montel’s theorem, there is a subsequence $(f_{n_k})$ that converges to $f$ analytic in $D$ and both $f_{n_k} \to f$ and $f'_{n_k} \to f'$ uniformly on
compact subsets of $D$. It follows that $f \in Q_q$ from the estimates
\[
\int_{D} |f'(z)|^2(1 - |\varphi_a(z)|^2)^q dm(z) = \int_{D} \lim_{k \to \infty} |f'_{n_k}(z)|^2(1 - |\varphi_a(z)|^2)^q dm(z)
\leq \liminf_{k \to \infty} \int_{D} |f'_{n_k}(z)|^2(1 - |\varphi_a(z)|^2)^q dm(z) \leq \liminf_{k \to \infty} \|f_{n_k}\|_{Q_q}^2 < \infty.
\]

Thus Lemma 3.2 implies that $C_{\phi} : Q_q \to Q_p$ is a compact operator if and only if, for any bounded sequence $(f_n)$ in $Q_q$ space with $f_n \to 0$ uniformly on compact sets as $n \to \infty$, $\|f_n(\phi(0))\|_p + \|C_{\phi}f_n\|_{Q_p} \to 0$ as $n \to \infty$, which is clearly equivalent to the statement of this lemma.

**Theorem 3.4.** Let $\phi \in \Phi$ such that $C_{\phi}(Q_q) \subseteq Q_{p,0}$. Then $C_{\phi} : Q_q \to Q_{p,0}$ is a compact composition operator if and only if
\[
\lim \sup_{|a|-1 \|f\|_{Q_q} < 1} \int_{D} |f'(\phi(z))|^2|\phi'(z)|^2(1 - |\varphi_a(z)|^2)^p dm(z) = 0. \tag{3.1}
\]

**Proof.** We adopt the methods of [12]. First suppose $C_{\phi} : Q_q \to Q_{p,0}$ is compact. Then $A = \text{cl}(\{f \circ \phi \in Q_{p,0} : \|f\|_{Q_q} < 1\})$ is a compact subset of $Q_{p,0}$. Let $\epsilon > 0$ be given. Since a compact set in a metric space is completely bounded, there is $f_1, f_2, \ldots, f_N \in Q_{p,0}$ such that each function in $A$ lies at most $\epsilon$ distant from $B$, where $B = \{f_1, f_2, \ldots, f_N\}$. That is there exists $j \in J = \{1, 2, \ldots, N\}$ such that
\[
\|g - f_j \circ \phi\|_{Q_p} < \frac{\epsilon}{4}, \tag{3.2}
\]
for $g \in A$. On the other hand, since $\{f_j \circ \phi : j \in J\} \subseteq Q_{p,0}$, there exists $\delta > 0$ such that for all $j \in J$ and $|a| > 1 - \delta$, 
\[
\int_{D} |(f_j \circ \phi)'(z)|^2(1 - |\varphi_a(z)|^2)^p dm(z) < \frac{\epsilon}{4}. \tag{3.3}
\]

Therefore for each $|a| > 1 - \delta$ and $f \in Q_q$ such that $\|f\|_{Q_q} < 1$ there exists $j \in J$ such that
\[
\int_{D} |(f \circ \phi)'(z)|^2(1 - |\varphi_a(z)|^2)^p dm(z)
\leq 2 \int_{D} |(f \circ \phi - f_j \circ \phi)'(z)|^2(1 - |\varphi_a(z)|^2)^p dm(z)
+ 2 \int_{D} |(f_j \circ \phi)'(z)|^2(1 - |\varphi_a(z)|^2)^p dm(z) < 2 \frac{\epsilon}{4} + 2 \frac{\epsilon}{4} = \epsilon.
\]

By (3.2) and (3.3), this implies (3.1).

Now, let (3.1) hold and $f_n \in B_{Q_q}$. By Lemma 3.1 and Montel’s theorem, there is a subsequence $\{f_{n_k}\}$ which converges to a function $g$ which is analytic in $D$ and both $f_{n_k} \to g$ and $f'_{n_k} \to g'$ uniformly on compact subsets of $D$. By
our hypothesis and Fatou’s lemma, we see that $C_\phi g \in Q_{p,0}$. We remark that $C_\phi$ is a compact composition operator by showing that $\|C_\phi(f_n - g)\|_{Q_p} \to 0$ as $k \to \infty$. In order to simplify the notation we additionally assume, without loss of generality, $g \equiv 0$ and show
\[
\lim_{n \to \infty} \|C_\phi f_n\|_{Q_p} = 0. \tag{3.4}
\]
For this purpose, let $S(h, \theta) = \{z \in D : |z - e^{i\theta}| < h\}$, where $\theta \in [0, 2\pi), h \in (0, 1)$, we use the equivalent $Q_p$-norm as follows (cf. [13]):
\[
\|f\|_{Q_p}^2 \sim \sup_{h \in (0,1), \theta \in [0,2\pi)} \int_{S(h, \theta)} |f'(z)|^2 (1 - |z|^2)^p dm(z).
\]
Thus our hypothesis is equivalent to
\[
\lim_{|h| \to 0} \sup_{\|f\|_{Q_p} < 1} \frac{1}{h^p} \int_{S(h, \theta)} |f'(\phi(z))|^2 |\phi'(z)|^2 (1 - |z|^2)^p dm(z) = 0. \tag{3.5}
\]
By (3.5), given $\epsilon > 0$, there exists $\delta > 0$ such that, if $n \in N, \theta \in [0, 2\pi)$ and $h < \delta$, then
\[
\frac{1}{h^p} \int_{S(h, \theta)} |(f_n \circ \phi)'(z)|^2 (1 - |z|^2)^p dm(z) < \epsilon. \tag{3.6}
\]
Fix $h_0 < \delta, \theta \in [0, 2\pi), n \in N$, and $h \geq \delta$. It is easy to verify that there exists $\{\theta_1, \theta_2, \ldots, \theta_N\} \subset [0, 2\pi)$ such that $S(h, \theta)$ is the union of sets $\{S(h_0, \theta_j) : j = 1, 2, \ldots, N\}$ and $K$, a compact subset of $D$. Hence
\[
\frac{1}{h^p} \int_{S(h, \theta)} |(f_n \circ \phi)'(z)|^2 (1 - |z|^2)^p dm(z)
\leq \sum_{j=1}^N \frac{1}{h_0^p} \int_{S(h_0, \theta_j)} |(f_n \circ \phi)'(z)|^2 (1 - |z|^2)^p dm(z) + \frac{1}{h_0^p} \int_K |(f_n \circ \phi)'(z)|^2 (1 - |z|^2)^p dm(z)
\leq I + II. \tag{3.7}
\]
Since $f_n' \to 0$ uniformly on $K$, as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for $n \geq N$
\[
II \leq \frac{\epsilon}{h_0^p} \int_K (1 - |z|^2)^p dm(z) \leq \text{const} \cdot \epsilon. \tag{3.8}
\]
Moreover, (3.6) implies
\[
I \leq \sum_{j=1}^N \epsilon = \text{const} \cdot \epsilon. \tag{3.9}
\]
Hence (3.6), (3.7), (3.8) and (3.9) give (3.4). Thus Lemma 3.3 yields that $C_\phi : Q_q \to Q_{p,0}$ is a compact operator. \qed
Theorem 3.5. Let $p, q \in (0, \infty)$ and $\phi \in \Phi$ such that $C_\phi(Q_q) \subseteq Q_{p,0}$, if

$$
\lim_{|a| \to 1} \int_D \frac{\left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p}{(1 - |\phi(z)|^2)^2} \, dm(z) = 0,
$$

then $C_\phi : Q_q \to Q_{p,0}$ is a compact composition operator.

Conversely, if $C_\phi : Q_q \to Q_{p,0}$ is a compact composition operator, then

$$
\lim_{|a| \to 1} \int_D \frac{\left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p}{1 - |\phi(z)|^2} \, dm(z) = 0.
$$

Proof. First let $C_\phi : Q_q \to Q_{p,0}$ be a compact composition operator. Since $f_\theta(z) = \log \frac{1}{1 - e^{-i\theta} z} \in Q_q$ for all $\theta \in [0, 2\pi)$, by Theorem 3.4,

$$
\lim_{|a| \to 1} \sup_{\theta \in [0, 2\pi)} \int_D \left|f_\theta(\phi(z))\right|^2 \left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p \, dm(z) = 0.
$$

For any $\epsilon > 0$, then there exists $\delta > 0$, such that for any $a \in D$ with $|a| > 1 - \delta$ and any $\theta \in [0, 2\pi)$,

$$
A_\theta = \int_D \left|f_\theta(\phi(z))\right|^2 \left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p \, dm(z) 
$$

$$
= \int_D \frac{1}{|1 - e^{-i\theta} \phi(z)|^2} \left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p \, dm(z) < \epsilon.
$$

Thus, we obtain (3.11) by integrating with respect to $\theta$, the Fubini theorem and the Poisson formula.

Suppose that (3.10) holds, for any function $f \in Q_q$,

$$
\int_D \left|f'(\phi(z))\right|^2 \left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p \, dm(z)
$$

$$
\leq \int_D \|f\|_{Q_q} \frac{\left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p}{(1 - |\phi(z)|^2)^2} \, dm(z)
$$

$$
\leq \|f\|_{Q_q} \int_D \frac{\left|\phi'(z)\right|^2 (1 - |\varphi_a(z)|^2)^p}{(1 - |\phi(z)|^2)^2} \, dm(z) \to 0
$$

as $|a| \to 1$ by our hypothesis. Hence $C_\phi : Q_q \to Q_{p,0}$ is a compact composition operator by Theorem 3.4, we finish the proof. \qed

Acknowledgements

The author thanks the referee for several helpful comments and suggestions on improvements. The author thanks M. Tjani for sending him some papers. The research is supported in part by the National Natural Science Foundation of China (No. 10371051) and the NSF of the ZheJiang Province of China (No. 102025).
REFERENCES


(Received 25.04.2004; revised 28.02.2005)

Author’s address:
Department of Mathematics
JiaYing University
MeiZhou, 514015, GuangDong
China
E-mail: lsx@mail.zjxu.edu.cn