A NOTE ON FOURIER COEFFICIENTS OF FUNCTIONS OF GENERALIZED WIENER CLASS

RAJENDRA G. VYAS

Abstract. Let $f$ denote a $2\pi$ periodic function in $L[0, 2\pi]$, and $\hat{f}(n), n \in \mathbb{Z}$, be its Fourier coefficients. For a function $f$ of the generalized Wiener class $\Lambda BV(p(n) \uparrow \infty)$ we have proved that

$$\hat{f}(n) = O\left(\frac{1}{\left(\sum_{i=1}^{\lfloor n \rfloor} \frac{1}{\lambda_i}\right)^{1/p(k(n))}}\right).$$

2000 Mathematics Subject Classification: 42A16.

Key words and phrases: Fourier coefficients, generalized Wiener class, $p$-$\Lambda$-bounded variation.

Let $f$ be a $2\pi$ periodic function in $L[0, 2\pi]$, and $\hat{f}(n), n \in \mathbb{Z}$, be its Fourier coefficients. R. Siddiqi [5] extended the classical result “$f \in BV[0, 2\pi]$ implies its Fourier coefficients $\hat{f}(n) = O(1/|n|)$” for the Wiener class. He proved that “$f \in BV(p, [0, 2\pi])$ ($1 \leq p < \infty$) implies $\hat{f}(n) = O(1/|n|^{1/p})$”. The concept of Wiener class was generalized by H. Kita and K. Yoneda [1] as generalized Wiener class $BV(p(n) \uparrow \infty)$ and also by Shiba [4] as $p$-$\Lambda$-bounded variations ($\Lambda BV(p)$). From these two generalizations, one can define a more generalized class $\Lambda BV(p(n) \uparrow \infty)$ as follows.

Definition. Given a subinterval $I$ of $[0, 2\pi]$, a sequence $\Lambda = \{\lambda_m\}$ ($m = 1, 2, \ldots$) of non-decreasing positive real numbers $\lambda_m$ such that $\sum \frac{1}{\lambda_m}$ diverges and $1 \leq p(n) \uparrow \infty$ as $n \to \infty$, we say that $f \in \Lambda BV(p(n) \uparrow \infty, I)$ (that is, $f$ is a function of $p(n)$-$\Lambda$-bounded variation over $(I)$) if

$$V_{\Lambda}(f, p(n), I) = \sup_{n \geq 1} \sup_{\{I_m\}} V_{\Lambda}(\{I_m\}, f, p(n), I) : \rho\{I_m\} > 2\pi/2^n < \infty,$$

where

$$V_{\Lambda}(\{I_m\}, f, p(n), I) = \left(\sum_{m} \frac{|f(a_m) - f(b_m)|^{p(n)}}{\lambda_m^{1/p(n)}}\right)^{1/p(n)},$$

$$\rho\{I_m\} = \inf_{m} |a_m - b_m|$$

and $\{I_m\}$ is a sequence of nonoverlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$.

Note that if $p(n) = p$ for all $n$, one gets the class $\Lambda BV(p, I)$; if $\lambda_m \equiv 1$ for all $m$, one gets the class $BV(p(n) \uparrow \infty)$; if $p(n) = 1$ for all $n$ and $\lambda_m \equiv 1$ for all
m, one gets the class $\text{BV}(I)$; if $p(n) = 1$ for all $n$ and $\lambda_m \equiv m$ for all $m$, one gets the class Harmonic $\text{BV}(I)$.

Schramm and Waterman [3] estimated the Fourier coefficients of function of $\bigwedge \text{BV}(p)$. They proved the following theorem.

**Theorem A.** If $f \in \bigwedge \text{BV}(p, [0, 2\pi])$ ($1 \leq p < \infty$), then

$$\hat{f}(n) = O\left(1/\left(\sum_{i=1}^{[n]} \frac{1}{\lambda_i}\right)^{1/p}\right).$$

For the generalized Wiener class T. Akhobadze [6] proved the following theorem.

**Theorem B.** If $f \in \text{BV}(p(n) \uparrow \infty, [0, 2\pi])$, then $\hat{f}(n) = O(1/|n|^{1/p(k(n))})$, where $k(n)$ is an integer for which

$$1 + \log_2 |n| < k(n) \leq 2 + \log_2 |n|. \quad (1)$$

Here we have extended these two results and estimated the order of the magnitude of the Fourier coefficients of functions of $\bigwedge \text{BV}(p(n) \uparrow \infty)$.

**Theorem.** Let $f \in \bigwedge \text{BV}(p(n) \uparrow \infty, [0, 2\pi])$, then

$$\hat{f}(n) = O\left(1/\left(\sum_{i=1}^{[n]} \frac{1}{\lambda_i}\right)^{1/p(k(n))}\right),$$

where $k(n)$ is an integer satisfying (1).

**Remark.** Here, $p(n) = p$ for all $n$, reduces the class $\bigwedge \text{BV}(p(n) \uparrow \infty)$ to the class $\bigwedge \text{BV}(p)$, and $\left(\sum_{i=1}^{[n]} \frac{1}{\lambda_i}\right)^{1/p(k(n))}$ reduces to $\left(\sum_{i=1}^{[n]} \frac{1}{\lambda_i}\right)^{1/p}$, that is we get Theorem A as a particular case. Similarly, $\lambda_m \equiv 1$ for all $m$, reduces the class $\bigwedge \text{BV}(p(n) \uparrow \infty)$ to the class $\text{BV}(p(n) \uparrow \infty)$ and we get Theorem B as a particular case. Thus the theorem generalizes a non-lacunary analogue of our earlier result [2, Theorem 5].

**Proof of the Theorem.** We know that

$$\hat{f}(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(x)e^{-inx} dx,$$

$$\hat{f}(n) = -\frac{1}{2\pi} \int_{0}^{2\pi} f\left(x + \frac{\pi}{n}\right)e^{-inx} dx,$$

$$\hat{f}(n) = -\frac{1}{2\pi} \int_{0}^{2\pi} (T_{\frac{\pi}{n}} f)(x)e^{-inx} dx, \quad \text{where} \quad (T_{\frac{\pi}{n}} f)(x) = f\left(x + \frac{\pi}{n}\right).$$
Then
\[ |\hat{f}(n)| = \frac{1}{4\pi} \left| \int_0^{2\pi} \left( f(x) - \left( T_{n\pi} f \right)(x) \right) e^{-inx} \, dx \right|. \]  
(1.1)

Because of the periodicity of \( f(x) \), we have for any positive integer \( j \)
\[ \left| \int_0^{2\pi} \left( T_{jn\pi} f - T_{(j-1)n\pi} f \right)(x) \, dx \right| = \left| \int_0^{2\pi} \left( f(x) - \left( T_{n\pi} f \right)(x) \right) \, dx \right|, \]
this together with (1.1) implies
\[ |\hat{f}(n)| \leq \frac{1}{4\pi} \int_0^{2\pi} \left| \left( T_{jn\pi} f - T_{(j-1)n\pi} f \right)(x) \right| \, dx. \]  
(1.2)

For a given natural number \( |n| \), let \( k(n) \) be an integer such that
\[ \frac{\pi}{2|n|} \leq \frac{2\pi}{2k(n)} \leq \frac{\pi}{|n|}, \]
i.e., (1) is true. Let \( q(k(n)) \) be such that \( \frac{1}{p(k(n))} + \frac{1}{q(k(n))} = 1 \), then by the Hölder inequality, from (1.2) we get
\[ |\hat{f}(n)| \leq \frac{1}{2\pi} \left( \int_0^{2\pi} \left| \left( T_{jn\pi} f - T_{(j-1)n\pi} f \right)(x) \right|^{p(k(n))} \, dx \right)^{1/p(k(n))} \left( \int_0^{2\pi} 1 \, dx \right)^{1/q(k(n))} \]
\[ = \left( \frac{1}{2\pi} \right)^{1/p(k(n))} \left( \int_0^{2\pi} \left| \left( T_{jn\pi} f - T_{(j-1)n\pi} f \right)(x) \right|^{p(k(n))} \, dx \right)^{1/p(k(n))}. \]

Then
\[ |\hat{f}(n)|^{p(k(n))} \leq \frac{1}{2\pi} \left( \int_0^{2\pi} \left| \left( T_{jn\pi} f - T_{(j-1)n\pi} f \right)(x) \right|^{p(k(n))} \, dx \right). \]  
(1.3)

Dividing both sides of equation (1.3) by \( \lambda_j \) and then performing summation from \( j = 1 \) to \( |n| \), we get
\[ |\hat{f}(n)|^{p(k(n))} \left( \sum_{j=1}^{|n|} \frac{1}{\lambda_j} \right) \leq \frac{1}{2\pi} \left( \int_0^{2\pi} \sum_{j=1}^{|n|} \left| \left( T_{jn\pi} f - T_{(j-1)n\pi} f \right)(x) \right|^{p(k(n))} \frac{1}{\lambda_j} \, dx \right). \]

Hence \( |\hat{f}(n)|^{p(k(n))} \leq \frac{V_{x,f,p(n),[0,2\pi]}(\sum_{j=1}^{|n|} \frac{1}{\lambda_j})}{(\sum_{j=1}^{|n|} \frac{1}{\lambda_j})}. \) This proves the theorem. \( \square \)
References


(Received 9.08.2004)

Author’s address:
Department of Mathematics
Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara-390002, Gujarat
India
E-mail: drrgvyas@yahoo.com