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PERIODIC SOLUTIONS OF HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS VIA A CONTINUATION THEOREM

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Abstract. Based on a continuation theorem of Mawhin, periodic solutions are found for difference equations of the form

$$\Delta^k y_n = f(n, y_n, y_{n-1}, \cdots, y_{n-l}), \quad n \in Z.$$

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1. Introduction

There are many reasons for studying nonlinear difference equations of the form

$$\Delta^k x_n = f(n, x_n, x_{n-1}, \dots, x_{n-l}), \quad n \in Z = \{0, \pm 1, \pm 2, \dots\}, \tag{1}$$

where k and ω are positive integers and $f = f(t, u_0, u_1, \dots, u_l)$ is a real continuous function defined on R^{l+2} such that

$$f(t+\omega, u_0, \dots, u_l) = f(t, u_0, \dots, u_l), \quad (t, u_0, \dots, u_l) \in R^{l+2}.$$

For one reason, (1) is a standard numerical scheme for computing solutions of differential equations. As another reason, the well known logistic equation

$$x_{n+1} - x_n = \mu x_n(1 - x_n) \tag{2}$$

is a particular case of (1).

Let us recall that a solution of (1) is a real sequence of the form $\{x_n\}_{n\in Z}$ which renders (1) into an identity after substitution. It is not difficult to see that solutions can be found when an appropriate function f is given. However, one interesting question is whether there are any solutions which are ω-periodic, where a sequence $\{x_n\}_{n\in Z}$ is said to be ω-periodic if $x_{n+\omega} = x_n$ for $n \in Z$. Such questions have been raised in the study of (2) and lead to the chaos concepts.

There are several techniques (see, e.g., [1–5]) which can help to answer such a question. Among these techniques are fixed point theorems such as that of Krasnolselskii, Leggett–Williams, and others; and topological methods such as degree theories. Here we will invoke a continuation theorem of Mawhin for obtaining such solutions. More specifically [6, pp. 39–40], let X and Y be two Banach spaces and $L : \text{Dom } L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim Im } L < +\infty$, and $\text{Im } L$ is closed in Y. If L is a

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Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im}\,(I - Q)$. It follows that $L_{|\text{ Dom } L \cap \text{Ker } P} : (I - P)\,X \to \text{ Im } L$ has an inverse which is denoted by K_P. If Ω is an open and bounded subset of X, then the mapping N is called L-compact on $\bar{\Omega}$ provided that $QN\left(\bar{\Omega}\right)$ is bounded and $K_P\,(I - Q)\,N : \bar{\Omega} \to X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$ there exists an isomorphism $J : \text{Im } Q \to \text{Ker } L$.

Theorem A (Mawhin's continuation theorem). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\bar{\Omega}$. Suppose*
(i) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega, Lx \neq \lambda Nx$; and*
(ii) *for each $x \in \partial\Omega \cap \text{Ker } L, QNx \neq 0$ and $\deg\,(JQN, \Omega \cap \text{Ker }, 0) \neq 0$.*
Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{dom } L$.

Note that if $\omega = 1$, then an ω-periodic solution of (1) is a constant sequence $\{c\}_{n\in Z}$ that satisfies (1). Hence

$$f(n, c, \dots, c) = 0, \quad n \in Z.$$

Conversely, if $c \in R$ such that $f(n, c, \dots, c) = 0$ for $n \in Z$, then the constant sequence $\{c\}_{n\in Z}$ is an ω-periodic solution of (1). For this reason, we will assume in the rest of our discussion that ω is an integer greater than or equal to 2.

For any real sequence $\{u_n\}_{n\in Z}$, we define a nonstandard "summation" operation

$$\bigoplus_{n=\alpha}^{\beta} u_n = \begin{cases} \sum\limits_{n=\alpha}^{\beta} u_n & \alpha \le \beta, \\ 0 & \beta = \alpha - 1, \\ -\sum\limits_{n=\beta+1}^{\alpha-1} u_n & \beta < \alpha - 1. \end{cases}$$

We will need two results and one of them is similar to Rolle's theorem in differential calculus.

Lemma 1. *If $\{r_n\}_{n\in Z}$ is a real sequence and $r_a = r_b$, where $b - a \geqslant 2$, then there is $l \in \{a, \dots, b - 2\}$ such that $\Delta r_l \cdot \Delta r_{l+1} \le 0$.*

The proof is easy but we include it here for the sake of convenience. Suppose to the contrary that $\Delta r_n \Delta r_{n+1} > 0$ for $n \in \{a, \dots, b-2\}$. There are two cases: (i) $\Delta r_a > 0$, or (ii) $\Delta r_a < 0$. Assume without loss of generality that $\Delta r_a > 0$, we then have $\Delta r_{a+1}, \dots, \Delta r_{b-1} > 0$. Thus $r_a < r_{a+1} < \cdots < r_b$ which is contrary to our assumption that $r_a = r_b$.

Lemma 2. *Let $x = \{x_k\}_{k\in Z}$ be a real ω-periodic sequence. Let m be a positive integer. If*

$$\max_{0\le n\le \omega-1} |\Delta^m x_n| \le D,$$

for some positive number D, then there exist positive numbers $D_1, \dots, D_m$ such that

$$\max_{0\le n\le \omega-1} \left|\Delta^j x_n\right| \le D_j, \quad j = 1, \dots, m.$$

Proof. The case $m = 1$ is trivially true. If $m > 1$, let $\Delta^{m-1}x_\varsigma = \max_{0\le i\le\omega-1} \Delta^{m-1}x_i$ and $\Delta^{m-1}x_\eta = \min_{0\le i\le\omega-1} \Delta^{m-1}x_i$, where $0 \le \varsigma, \eta \le \omega - 1$. Let $\varsigma_1 = \max\{\varsigma, \eta\}$ and $\eta_1 = \min\{\varsigma, \eta\}$. Then

$$\Delta^{m-1}x_\varsigma - \Delta^{m-1}x_\eta = \left|\bigoplus_{i=\eta_1}^{\varsigma_1-1} \Delta^m x_i\right| \le \bigoplus_{i=\eta_1}^{\varsigma_1-1} |\Delta^m x_i| \le \omega D.$$

Since $\{\Delta^{m-2}x_n\}_{n\in Z}$ is ω-periodic, by Lemma 1, we know that $\Delta^{m-1}x_\varsigma \ge 0$ and $\Delta^{m-1}x_\eta \le 0$. Thus,

$$\Delta^{m-1}x_\varsigma \le \Delta^{m-1}x_\eta + \omega D \le \omega D,$$

and

$$\Delta^{m-1}x_\eta \geqslant \Delta^{m-1}x_\varsigma - \omega D \geqslant -D.$$

It follows that

$$-\omega D \le \Delta^{m-1}x_\eta \le \Delta^{m-1}x_n \le \Delta^{m-1}x_\varsigma \le \omega D, \quad 0 \le n \le \omega - 1,$$

or,

$$\left|\Delta^{m-1}x_n\right| \le \omega D, \quad 0 \le n \le \omega - 1.$$

We may thus take $D_{m-1} = \omega D$. By induction, we may show that $D_j = \omega^{m-j}D$ for $j = 1, \dots, m-1$. □

2. Existence Criteria

We will establish existence criteria based on combinations of the following conditions, where D and M are positive constants:

(a_1) $f(t, x_0, x_1, \dots, x_l) > 0$ for $t \in R$ and $x_0, x_1, \dots, x_l \ge D$,
(a_2) $f(t, x_0, x_1, \dots, x_l) < 0$ for $t \in R$ and $x_0, x_1, \dots, x_l \ge D$,
(b_1) $f(t, x_0, x_1, \dots, x_l) < 0$ for $t \in R$ and $x_0, x_1, \dots, x_l \le -D$,
(b_2) $f(t, x_0, x_1, \dots, x_l) > 0$ for $t \in R$ and $x_0, x_1, \dots, x_l \le -D$,
(c_1) $f(t, x_0, x_1, \dots, x_l) \ge -M$ for $(t, x_0, x_1, \dots, x_l) \in R^{l+2}$,
(c_2) $f(t, x_0, x_1, \dots, x_l) \le M$ for $(t, x_0, x_1, \dots, x_l) \in R^{l+2}$.

Theorem 1. *Suppose either one of the following set of conditions hold:*
(i) (a_1), (b_1) *and* (c_1), *or,*
(ii) (a_2), (b_2) *and* (c_1), *or,*
(iii) (a_1), (b_1) *and* (c_2), *or,*
(iv) (a_2), (b_2) *and* (c_2).
Then (1) *has an ω-periodic solution.*

We only give the proof in case (a_1), (b_1) and (c_1) hold, since the other cases can be treated in similar manners.

First of all, it is easy to see if $\{x_n\}_{n\in Z}$ is an ω-periodic solution of the equation

$$\Delta^{k-1}x_n = \Delta^{k-1}x_0 + \bigoplus_{i=0}^{n-1} f(i, x_0, x_1, \dots, x_l), \quad n \in Z, \tag{3}$$

then $\{x_n\}_{n\in Z}$ is an ω-periodic solution of (1). We will therefore seek an ω-periodic solution of (3).

Let X_ω be the Banach space of all real ω-periodic sequences of the form $x = \{x_n\}_{n\in Z}$, and endowed with the usual linear structure as well as the norm $\|x\|_1 = \max_{0\le i\le \omega-1} |x_i|$. Let Y_ω be the Banach space of all real sequences of the form $y = \{y_n\}_{n\in Z} = \{n\alpha + h_n\}_{n\in Z}$ such that $y_0 = 0$, where $\alpha \in R$ and $\{h_n\}_{n\in Z} \in X_\omega$, and endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$. Let the zero element of X_ω and Y_ω be denoted by θ_1 and θ_2 respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$ respectively by

$$(Lx)_n = \Delta^{k-1}x_n - \Delta^{k-1}x_0, \quad n \in Z. \tag{4}$$

and

$$(Nx)_n = \bigoplus_{i=0}^{n-1} f\left(i, , x_0, x_1, \ldots, x_l\right), \quad n \in Z. \tag{5}$$

Let

$$\bar{h}_n = \bigoplus_{i=0}^{n-1} f\left(i, x_0, x_1, \ldots, x_l\right) - \frac{n}{\omega} \bigoplus_{i=0}^{\omega-1} f\left(i, x_0, x_1, \ldots, x_l\right), \quad n \in Z. \tag{6}$$

Since $\bar{h} = \{\bar{h}_n\}_{n\in Z} \in X_\omega$ and $\bar{h}_0 = 0$, N is a well-defined operator from X_ω to Y_ω. Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$ respectively by

$$(Px)_n = x_0, \quad n \in Z, \tag{7}$$

for $x = \{x_n\}_{n\in Z} \in X_\omega$ and

$$(Qy)_n = n\alpha, \quad n \in Z, \tag{8}$$

for $y = \{n\alpha + h_n\}_{n\in Z} \in Y_\omega$.

Lemma 3. *Let the mapping L be defined by* (4)*. Then*

$$\text{Ker } L = \left\{x \in X_\omega \mid x_n = x_0,\ n \in Z,\ x_0 \in R\right\}. \tag{9}$$

Proof. It suffices to show that if $\{x_k\}_{k\in Z}$ is an ω-periodic real sequence which satisfies $\Delta^{k-1}x_n = \Delta^{k-1}x_0$ for all $n \in Z$, then $\{x_n\}_{n\in Z}$ is a constant sequence. Indeed, the case $k = 1$ is true since $\Delta^0 x_n = x_n$. If

$$\Delta x_n = \Delta x_0, \quad n \in Z,$$

then after summations on both sides, we see that

$$x_n = x_0 + \bigoplus_{k=0}^{n-1} \Delta x_0, \quad n \in Z.$$

But in view of $x_\omega = x_0$, we must have $\Delta x_0 = 0$, so that $x_n = x_0$ for $n \in Z$. The other cases are proved similarly. □

Lemma 4. *Let the mapping L be defined by* (4)*. Then*

$$\text{Im } L = \left\{y \in X_\omega \mid y_0 = 0\right\} \subset Y_\omega. \tag{10}$$

Proof. It suffices to show that for each $y = \{y_n\}_{n\in Z} \in X_\omega$ that satisfies $y_0 = 0$, there is a $x = \{x_n\}_{n\in Z} \in X_\omega$ such that

$$y_n = \Delta^{k-1}x_n - \Delta^{k-1}x_0, \quad n \in Z. \tag{11}$$

Indeed, if $k = 1$, that is, if

$$y_n = x_n - x_0, \quad n \in Z,$$

we may let $x_n = y_n$ for $n \in Z$. Similarly, if $k \geqslant 2$, we may let

$$x_n = \bigoplus_{n_{k-2}=0}^{n-1} \bigoplus_{n_{k-3}=0}^{n_{k-2}-1} \cdots \bigoplus_{n_1=0}^{n_2-1} \bigoplus_{i=0}^{n_1-1} y_i, \quad n \in Z,$$

as can be easily checked. □

It follows that Im L is closed in Y_ω. Thus the following Lemma is true.

Lemma 5. *The mapping L defined by* (4) *is a Fredholm mapping of index zero.*

Indeed, from Lemma 3, Lemma 4 and the definition of Y_ω, dim Ker L = codim Im $L = 1 < +\infty$. From (10), we see that Im L is closed in Y_ω. Hence L is a Fredholm mapping of index zero.

Lemma 6. *Let the mapping L, P and Q be defined by* (4), (7) *and* (8) *respectively. Then* Im P = Ker L *and* Im L = Ker Q.

Indeed, from Lemma 3, Lemma 4 and the defining conditions (7) and (8), it is easy to see that Im P = Ker L and Im L = Ker Q.

Next we recall that a subset S of a Banach space X is relatively compact if, and only if, for each $\varepsilon > 0$, it has a finite ε-net.

Lemma 7. *A subset S of X_ω is relatively compact if, and only if, S is bounded.*

Proof. If S is relatively compact in X_ω, then it is easy to see that S is bounded. Conversely, if the subset S of X_ω is bounded, then there is a subset

$$\Gamma := \big\{x \in X_\omega | \quad \|x\|_1 \le H\big\},$$

where H is a positive constant, such that $S \subset \Gamma$. It suffices to show that Γ is relatively compact in X_ω. To see that note that for each $\varepsilon > 0$, we may choose numbers $y_0 < y_1 < \cdots < y_l$ such that $y_0 = -H$, $y_l = H$ and $y_{i+1} - y_i < \varepsilon$ for $i = 0, \ldots, l-1$. Then

$$\big\{v = \{v_n\}_{n\in Z} \in X_\omega \mid v_j \in \{y_0, y_1, \ldots, y_{l-1}\},\ j = 0, \ldots, \omega - 1\big\}$$

is a finite ε-net of Γ. □

Lemma 8. *A subset W of Y_ω is relatively compact if, and only if, W is bounded.*

Proof. If W is relatively compact in Y_ω, then it is easy to see that W is bounded. Conversely, if the subset W of Y_ω is bounded, then there is a subset

$\Lambda := \{y = \{n\alpha + \bar{h}_n\}_{n\in Z} \in Y_\omega | \quad \|y\|_2 \leq A,\ \bar{h} = \{\bar{h}_n\}_{n\in Z} \in X_\omega \quad \text{and} \quad \bar{h}_0 = 0\}$,

where A is a positive constant, such that $W \subset \Lambda$. It suffices to show that Λ is relatively compact in Y_ω. To see this, note that for each $\varepsilon > 0$, we may choose numbers $y_0 < y_1 < \cdots < y_l$ such that $y_0 = -A$, $y_l = A$ and $y_{i+1} - y_i < \varepsilon/2$ for $i = 0, \ldots, l-1$. Choose numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_l$ such that $\alpha_0 = -A$, $\alpha_l = A$ and $\alpha_{i+1} - \alpha_i < \varepsilon/2$ for $i = 0, \ldots, l-1$. Then

$$\begin{aligned}\{w =&\{n\alpha + h_n\}_{n\in Z} \in Y_\omega \mid \alpha \in \{\alpha_0, \alpha_1, \ldots, \alpha_{l-1}\},\\ &h_j \in \{y_0, y_1, \ldots, y_{l-1}\},\ j = 1, \ldots, \omega - 1\,, h_0 = 0\}\end{aligned}$$

is a finite ε-net of Λ. □

Lemma 9. *Let L , P and Q be defined by* (4), (7) *and* (8) *respectively. Denote the inverse of the mapping $L\ |_{\text{Dom } L\cap \text{Ker } P}$: $(I-P)\,X_\omega \to \text{Im } L$ by K_P. Then K_P is continuous on* Im L.

Proof. The case of $k = 1$ is easy to see. Assume $k \geqslant 2$. For any $y = \{y_n\}_{n\in Z} \in$ Im L and $x = \{x_n\}_{n\in Z} \in$ Dom $L \cap$ Ker P such that $K_P y = x$, then

$$y_n = \Delta^{k-1} x_n - \Delta^{k-1} x_0, \quad n \in Z. \tag{12}$$

Since $x = \{x_n\}_{n\in Z} \in$ Dom $L \cap$ Ker P, $x_0 = 0$. In view of (12),

$$\Delta^{k-2} x_n - \Delta^{k-2} x_0 = \bigoplus_{i=0}^{n-1} y_i + \bigoplus_{i=0}^{n-1} \Delta^{k-1} x_0. \tag{13}$$

Since $\{\Delta^{k-2} x_n\}_{n\in Z}$ is ω-periodic, it follows that

$$\Delta^{k-1} x_0 = \frac{-1}{\omega} \bigoplus_{j_1=0}^{\omega-1} y_i. \tag{14}$$

From (13) and (14), we have

$$\Delta^{k-2} x_n - \Delta^{k-2} x_0 = \bigoplus_{i=0}^{n-1} y_i - \frac{1}{\omega} \bigoplus_{i=0}^{n-1} \bigoplus_{j_1=0}^{\omega-1} y_{j_1}. \tag{15}$$

Let

$$g_n^{(0)} = y_n, \tag{16}$$

and

$$g_n^{(j)} = \bigoplus_{i=0}^{n-1} g_i^{(j-1)} - \frac{1}{\omega} \bigoplus_{i=0}^{n-1} \bigoplus_{j_1=0}^{\omega-1} g_{j_1}^{(j-1)}, \quad j = 1, \ldots, k-1. \tag{17}$$

By induction (and the fact that $x_0 = 0$),

$$x_n = g_n^{(k-1)}, \quad n \in Z. \tag{18}$$

From (16) and (17), we see that

$$\left|g_n^{(j)}\right| \leq (2\omega)^{j-1} \|y\|_2\,, \quad j = 1, \ldots, k-1. \tag{19}$$

By (18) and (19), we have

$$\|K_P y\|_1 = \|x\|_1 \leq (2\omega)^{k-1} \|y\|_2 . \tag{20}$$

from which we see that K_P in bounded. The proof now follows from the linearity of K_p. □

Lemma 10. *Let L and N be defined by* (4) *and* (5) *respectively. Suppose Ω is an open and bounded subset of X_ω. Then N is L-compact on $\overline{\Omega}$.*

Proof. Since $\overline{\Omega}$ is bounded in X_ω, in view of Lemma 7, we know it is relatively compact. Note that N and Q are continuous mappings, hence $N(\overline{\Omega})$ and $QN(\overline{\Omega})$ are relatively compact in Y_ω and by Lemma 8, $N(\overline{\Omega})$ and $QN(\overline{\Omega})$ are bounded in Y_ω. By Lemma 9, K_P is continuous, so $K_P(I-Q)$is continuous. Note that $N(\overline{\Omega})$ is relatively compact in Y_ω, thus $\overline{K_P}(I-Q)N(\overline{\Omega})$ is relatively compact in X_ω and hence N is L-compact on $\overline{\Omega}$. □

Now, we consider the equation

$$\Delta^{k-1}x_n - \Delta^{k-1}x_0 = \lambda \bigoplus_{i=0}^{n-1} f\left(i, x_0, x_1, \ldots, x_l\right), \quad n \in Z, \tag{21}$$

where $\lambda \in (0,1)$.

Lemma 11. *Suppose* (a_1), (b_1) *and* (c_1) *are satisfied. Then there is a positive constant D_0 such that for any ω-periodic solution $x = \{x_n\}_{n\in Z}$ of* (21),

$$\|x\|_1 = \max_{0\leq i\leq \omega-1} |x_i| \leq D_0. \tag{22}$$

Proof. Let $x = \{x_n\}_{n\in Z}$ be an ω-periodic solution of (21). Then

$$\bigoplus_{i=0}^{\omega-1} f\left(i, x_0, x_1, \ldots, x_l\right) = 0. \tag{23}$$

If we write

$$G_n^+ = \max\left\{f(n, x_0, x_1, \ldots, x_l), 0\right\}, \quad n \in Z, \tag{24}$$

and

$$G_n^- = \max\left\{-f(n, x_0, x_1, \ldots, x_l), 0\right\}, \quad n \in Z. \tag{25}$$

Then $\{G_n^+\}_{n\in Z}$ and $\{G_n^-\}_{n\in Z}$ are nonnegative real sequences and

$$f\left(n, x_0, x_1, \ldots, x_l\right) = G_n^+ - G_n^-, \quad n \in Z, \tag{26}$$

as well as

$$\left|f(n, x_0, x_1, \ldots, x_l)\right| = G_n^+ + G_n^-, \quad n \in Z. \tag{27}$$

In view of (c_1) and (25), we have

$$\left|G_n^-\right| = G_n^- \leq M, \quad n \in Z. \tag{28}$$

Thus

$$\bigoplus_{i=0}^{\omega-1} G_i^- \leq \omega M, \tag{29}$$

and in view of (23), (26) and (29),

$$\bigoplus_{i=0}^{\omega-1} G_i^+ = \bigoplus_{i=0}^{\omega-1} G_i^- \leq \omega M. \tag{30}$$

By (27) and (30), we know that

$$\bigoplus_{i=0}^{\omega-1} \left| f\left(i, x_0, x_1, \ldots, x_l\right) \right| \leq 2\omega M. \tag{31}$$

Let $x_\phi = \max_{0\leq i\leq \omega-1} x_i$ and $x_\psi = \min_{0\leq i\leq \omega-1} x_i$, where $0 \leq \phi, \psi \leq \omega - 1$. Set $\phi_1 = \max\{\phi, \psi\}$ and $\psi_1 = \min\{\phi, \psi\}$. Let

$$\Delta^{k-1} x_\theta = \max_{0\leq i\leq \omega-1} \Delta^{k-1} x_i,$$

and

$$\Delta^{k-1} x_\iota = \min_{0\leq i\leq \omega-1} \Delta^{k-1} x_i,$$

where $0 \leq \theta, \iota \leq \omega - 1$. There are two cases:

Case 1. When $k \geqslant 2$, by (21) and (31), we have

$$\begin{aligned}\Delta^{k-1} x_\theta - \Delta^{k-1} x_\iota &= \lambda \left| \bigoplus_{i=0}^{\theta-1} f\left(i, x_0, x_1, \ldots, x_l\right) - \bigoplus_{i=0}^{\iota-1} f\left(i, x_0, x_1, \ldots, x_l\right) \right| \\ &\leq 2 \bigoplus_{i=0}^{\omega-1} \left| f\left(i, x_0, x_1, \ldots, x_l\right) \right| \leq 4\omega M. \end{aligned} \tag{32}$$

Noting that $\{\Delta^{k-2} x_n\}_{n\in Z}$ is ω-periodic, by Lemma 1, we know that $\Delta^{k-1} x_\theta \geqslant 0$ and $\Delta^{k-1} x_\iota \leq 0$. Thus,

$$\Delta^{k-1} x_\theta \leq \Delta^{k-1}\, x_\iota + 4\omega M \leq 4\omega M,$$

and

$$\Delta^{k-1} x_\iota \geqslant \Delta^{k-1} x_\theta - 4\omega M \geqslant -4\omega M.$$

It follows that

$$-4\omega M \leq \Delta^{k-1} x_\iota \leq \Delta^{k-1} x_n \leq \Delta^{k-1} x_\theta \leq +4\omega M, \quad 0 \leq n \leq \omega - 1,$$

that is

$$\left|\Delta^{k-1} x_n\right| \leq D_{k-1}, \quad 0 \leq n \leq \omega - 1, \tag{33}$$

where $D_{k-1} = 4\omega M$. In view of Lemma 2, there are positive constants $D_1, \ldots, D_{k-1}$ such that

$$\max_{0\leq n\leq \omega-1} \left|\Delta^j x_n\right| \leq D_j, \quad j = 1, \ldots, k - 1. \tag{34}$$

If there is some x_s, $0 \leq s \leq \omega - 1$, such that $|x_s| < D$, then in view of (34), for any $n \in \{0, 1, \ldots, \omega - 1\}$, we have

$$|\, x_n| = |\, x_s| + |x_n - \, x_s| \leq D + \left| \bigoplus_{i=0}^{n-1} \Delta x_i - \bigoplus_{i=0}^{s-1} \Delta x_i \right|$$

$$\leq D + 2 \bigoplus_{i=0}^{\omega-1} |\Delta x_i| \leq D + 2\omega D_1.$$

It not, by (a_1), (b_1) and (23), $x_\phi \geqslant D$ and $x_\psi \leq -D$. From (34), we have

$$x_\phi - \, x_\psi = \left| \bigoplus_{i=\psi_1}^{\phi_1 - 1} \Delta x_i \right| \leq \omega D_1. \tag{35}$$

From (35), we have

$$x_\phi \leq \, x_\psi + \omega D_1 \leq -D + \omega D_1,$$

and

$$x_\psi \geqslant x_\phi - \omega D_1 \geqslant D - \omega D_1.$$

It follows that

$$D - \omega D_1 \leq x_\psi \leq x_n \leq x_\phi \leq -D + \omega D_1, \quad \leq n \leq \omega - 1,$$

or,

$$|x_n| \leq D + \omega D_1, \quad 0 \leq n \leq \omega - 1. \tag{36}$$

Case 2. When $k = 1$, by (21) and (31), we have

$$x_\phi - \, x_\psi =| \, x_\phi - \, x_\psi \, |$$

$$= \lambda \left| \bigoplus_{i=0}^{\phi-1} f\left(i, x_0, x_{_1}, \ldots, x_l\right) - \bigoplus_{i=0}^{\psi-1} f\left(i, x_0, x_{_1}, \ldots, x_l\right) \right|$$

$$\leq 2 \bigoplus_{i=0}^{\omega-1} |f\left(i, , x_i, x_{i-1}, \ldots, x_l\right)| \leq 4\omega M. \tag{37}$$

If there is some x_μ, where $0 \leq \mu \leq \omega - 1$, such that $|x_\mu| < D$, then in view of (21) and (31), for any $n \in \{0, 1, \ldots, \omega - 1\}$, we have

$$|\, x_n| \leq |\, x_\mu| + |x_n - \, x_\mu|$$

$$\leq D + \left| \bigoplus_{i=0}^{n-1} f\left(i, x_0, x_{_1}, \ldots, x_l\right) - \bigoplus_{i=0}^{\mu-1} f\left(i, x_0, x_{_1}, \ldots, x_l\right) \right|$$

$$\leq D + 2 \bigoplus_{i=0}^{\omega-1} \left| f\left(i, x_0, x_{_1}, \ldots, x_l\right) \right| \leq D + 4\omega M. \tag{38}$$

If not, then by (a_1), (b_1) and (23), $x_\phi \geqslant D$ and $x_\psi \leq -D$. From (37), we have

$$x_\phi \leq \, x_\psi + 4\omega M \leq -D + 4\omega M,$$

and

$$x_\psi \geqslant x_\phi - 4\omega M \geqslant D - 4\omega M.$$

It follows that

$$D - 4\omega M \le x_\psi \le x_n \le x_\phi \le -D + 4\omega M, \quad 0 \le n \le \omega - 1,$$

or,

$$|x_n| \le D + 4\omega M, \quad 0 \le n \le \omega - 1. \tag{39}$$

The proof is complete. □

Proof of Theorem 1. Let L, N, P and Q be defined by (4), (5), (7) and (8) respectively. By Lemma 11, there is a positive constant D_0 such that (22) holds for any ω-periodic solution $x = \{x_n\}_{n\in Z}$ of (21). By Lemma 11, there is positive constant D_0 such that for any ω-periodic solution $x = \{x_n\}_{n\in Z}$ of (21) such that (22) holds. Set

$$\Omega = \left\{x \in X_\omega | \quad \|x\|_1 < \overline{D}\right\},$$

where $\overline{D}$ is a fixed number which satisfies $\overline{D} > D_0$. It is easy to see that Ω is an open and bounded subset of X_ω. Furthermore, in view of Lemma 5 and Lemma 10, L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. By Lemma 11, for each $\lambda \in (0,1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Next note that a sequence $x = \{x_n\}_{n\in Z} \in \partial\Omega \cap \text{Ker } L$ must be constant : $\{x_n\}_{n\in Z} = \{\overline{D}\}_{n\in Z}$ or $\{x_n\}_{n\in Z} = \{-\overline{D}\}_{n\in Z}$. Hence by (a_1), (b_1), (5) and (8),

$$(QNx)_n = \frac{n}{\omega} \bigoplus_{i=0}^{\omega-1} f(i, x_0, \ldots, x_0), \quad n \in Z,$$

so

$$QNx \neq \theta_2.$$

The isomorphism $J : \text{Im } Q \to \text{Ker } L$ is defined by $(J(n\alpha))_n = \alpha$ for $\alpha \in R$, $n \in Z$. Then

$$(JQNx)_n = \frac{1}{\omega} \bigoplus_{i=0}^{\omega-1} f(i, x_0, \ldots, x_0) \neq 0, \quad n \in Z, \tag{40}$$

In particular, we see that if $\{x_n\}_{n\in Z} = \{\overline{D}\}_{n\in Z}$, then

$$(JQNx)_n = \frac{1}{\omega} \bigoplus_{i=0}^{\omega-1} f\left(i, \overline{D}, \ldots, \overline{D}\right) > 0, \quad n \in Z, \tag{41}$$

and if $\{x_n\}_{n\in Z} = \{-\overline{D}\}_{n\in Z}$, then

$$(JQNx)_n = \frac{1}{\omega} \bigoplus_{i=0}^{\omega-1} f\left(i, -\overline{D}, \ldots, -\overline{D}\right) < 0, \quad n \in Z. \tag{42}$$

Consider the mapping

$$H(x, s) = sx + (1 - s) JQNx, \quad 0 \le s \le 1. \tag{43}$$

From (41) and (43), for each $s \in [0,1]$ and $\{x_n\}_{n\in Z} = \{\overline{D}\}_{n\in Z}$, we have

$$(H(x, s))_n = s\overline{D} + (1 - s)\frac{1}{\omega} \bigoplus_{i=0}^{\omega-1} f\left(i, \overline{D}, \ldots, \overline{D}\right) > 0, \quad n \in Z. \tag{44}$$

Similarly, from (42) and (43), for each $s \in [0,1]$ and $\{x_n\}_{n\in Z} = \{-\overline{D}\}_{n\in Z}$, we have

$$(H(x,s))_n = -s\overline{D} + (1-s)\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i, -\overline{D}, \dots, -\overline{D}\right) < 0, \quad n \in Z. \tag{45}$$

By (44) and (45), $H(x,s)$ is a homotopy. This shows that

$$\deg\left(JQNx, \Omega \cap \operatorname{Ker} L, \theta_1\right) = \deg\left(-x, \Omega \cap \operatorname{Ker} L, \theta_1\right) \neq 0.$$

By Theorem A, we see that the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$. In other words, (1) has an ω-periodic solution $x = \{x_n\}_{n\in Z}$. □

3. Example

Consider the difference equation

$$\begin{aligned}\Delta^3 x_n = &\left(x_n + x_{n-1} + x_{n-2} - 2 - \sin\frac{\pi n}{2}\right)\\ &\times \exp\left(-x_n - x_{n-1} - x_{n-2} + \sin\frac{\pi n}{2}\right),\end{aligned} \tag{46}$$

we can prove that (46) has a 4-periodic nontrivial solution. Indeed, take $k = 3$ and

$$f(t, x_0, x_1, x_2) = \left(x_0 + x_1 + x_2 - 2 - \sin\frac{\pi t}{2}\right)\exp\left(-x_0 - x_1 - x_2 + \sin\frac{\pi t}{2}\right),$$

and let $D = 4$, $M = 9$. Then the conditions in (iii) of Theorem 1 are satisfied. Therefore (46) has a 4-periodic solution. Furthermore, this solution is nontrivial since $f(t,0,0,0)$ is not identically zero.

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