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ON A SINGULAR DIRECTION OF A MEROMORPHIC FUNCTION

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Abstract. By using Ahlfors' theory of a covering surface, we establish the existence of a new singular direction for a meromorphic function f, namely a T direction for f, for which the Nevanlinna characteristic function $T(r,f)$ is used as a comparison function. Then we prove that every T direction is a Borel direction for meromorphic function with finite and positive regular growth order.

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1. Introduction and the Results

Let $f(z)$ be a transcendental meromorphic function defined on the whole complex plane. The singular direction for f is one of the main objects studied in the theory of value distribution of meromorphic function. Several types of singular directions have been introduced in the literature. Their existence has also been established.

In 1919, Julia introduced the concept of a Julia direction for a meromorphic function f in [1], that is, a ray $\arg z = \theta$ having the following property: for any $0 < \varepsilon < \pi$, and for all a (with at most two exceptions) on the Riemann sphere $\overline{C}$,

$$\lim_{r\to\infty} n(r,\theta,\varepsilon,a) = +\infty,$$

where $n(r,\theta,\varepsilon,a)$ is the number of solutions of $f(z) = a$ in $\{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\} \cap \{|z| < r\}$, multiple solutions being counted as many times as their multiplicity is. In the same paper Julia showed that every transcendental meromorphic function has at least one Julia direction and this is a refinement of the Picard theorem. In order to have a similar refinement for the Borel theorem, a more refined notion of Borel directions was introduced by Valiron in 1928. A ray $\arg z = \theta$ is called a Borel direction of order ρ for f if for every $0 < \varepsilon < \pi$,

$$\limsup_{r\to\infty} \frac{\log n(r,\theta,\varepsilon,a)}{\log r} \geq \rho,$$

for all a in $\overline{C}$ with at most two exceptions. Note that the definition is only meaningful in the case where $0 < \rho < \infty$. In this case it is well known that f must have at least one Borel direction [3]. When the order $\rho = 0$ or ∞, it is not good to use the order to characterize the growth of f. In this case, the

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Nevanlinna characteristic function $T(r,f)$ is certainly a more appropriate object to consider. Hence by using $T(r,f)$ instead of $\log r$ as a comparison function, Zheng [4] introduced a new singular direction, namely, the T direction for f.

Definition 1. A ray $\arg z=\theta$ is called a T direction for a meromorphic function $f(z)$ if for every $0<\varepsilon<\pi$,

$$\limsup_{r\to\infty}\frac{N(r,\theta,\varepsilon,a)}{T(r,f)}>0 \tag{1}$$

for all a in $\overline{C}$ with at most two exceptions. And a ray $\arg z=\theta$ is called a precise T direction if $N(r,\theta,\varepsilon,a)$ in (1) is replaced by $\overline{N}(r,\theta,\varepsilon,a)$.

We believe that it is more natural to use the Nevanlinna characteristic function $T(r,f)$ as a comparison function because in general $T(r,f)$ is the most basic function that one uses to describe the growth of meromorphic functions.

Suppose $w=f(z)$ is meromorphic in $|z|<R(\leq\infty)$, for any $0\leq r<R$. Let $A(r)$ be the area of $|z|\leq r$ on the w-sphere surface. Let

$$S(r,f)=\frac{A(r)}{\pi}=\frac{1}{\pi}\iint\limits_{|z|\leq r}\left(\frac{|f'(z)|}{(1+|f(z)|^2)}\right)^2 t\,d\theta\,dt,\qquad z=te^{i\theta},$$

$$T_0(r,f)=\int_0^r\frac{S(t,f)}{t}dt.$$

The latter is an increasing convex function of $\log r$ and $T_0(r,f)$ is called the Ahlfors–Shimizu characteristic function.

The main purpose of this paper is to prove the following theorems.

Theorem 1. *If $f(z)$ is a meromorphic function defined on the whole complex plane and*

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty, \tag{2}$$

then $f(z)$ has at least one T direction.

Theorem 2. *Let $f(z)$ be meromorphic on the whole complex plane. If*

$$\limsup_{r\to\infty}\frac{T(2r,f)}{T(r,f)}>1, \tag{3}$$

then $f(z)$ has at least one T direction.

Theorem 3. *Let $f(z)$ be meromorphic on the whole complex plane with finite and positive regular growth order ρ, then $w(z)$ has at least one T direction and every T direction is a Borel direction of order ρ.*

Remark. From Theorem 1.4 in [5], we have

$$|T(r,f)-T_0(r,f)-\log|f(0)||\leq\frac{1}{2}\log 2. \tag{4}$$

Hence it is sufficient to show that Theorem 1 holds if we can prove that $f(z)$ has a singular direction using $T_0(r,f)$ instead of $T(r,f)$ in Definition 1.

2. Notation and Lemmas

Let $f(z)$ be meromorphic in an angular domain $\triangle(\theta,\alpha_0)=\{z:|\arg z-\theta|\le\alpha_0\}$, and $\triangle(\theta,\alpha)=\{z:|\arg z-\theta|\le\alpha\}$ be an angular domain contained in $\triangle(\theta,\alpha_0)$, where $\theta\in[0,2\pi)$ and $\alpha\le\alpha_0$. Let $\triangle_0(r)$, $\triangle(r)$ be the parts of $\triangle(\theta,\alpha_0)$, $\triangle(\theta,\alpha)$, respectively, which is contained in $|z|\le r$ and put

$$S(r,\triangle(\theta,\alpha))=\frac{A(r)}{\pi}=\frac{1}{\pi}\iint\limits_{\triangle(r)}\left(\frac{|f'(z)|}{(1+|f(z)|^2)}\right)^2 t\,d\theta\,dt,\quad z=te^{i\theta},$$

$$T_0(r,\triangle(\theta,\alpha))=\int_0^r\frac{S(t,\triangle(\theta,\alpha))}{t}dt.$$

Let $n(r,\theta,\alpha,a)$ be the number of zero points of $f(z)-a$ contained in $\triangle(r)$, multiple zeros being counted as many times as their multiplicity is, and put

$$N(r,\theta,\alpha,a)=\int_0^r\frac{n(t,\theta,\alpha,a)}{t}\,dt.$$

Note that $\overline{n}(r,\theta,\alpha,a)$ is the number of zero points of $f(z)-a$ contained in $\triangle(r)$, multiple zeros being counted only once, and put

$$\overline{N}(t,\theta,\alpha,a)=\int_0^r\frac{\overline{n}(t,\theta,\alpha,a)}{t}dt.$$

Then we introduce the following lemmas.

Lemma 1 ([6]). *Let $f(z)$ be meromorphic in complex plane, then*

$$S(r,\triangle(\theta,\alpha))\le 3\sum_{i=1}^{3}n(2r,\theta,\alpha_0,a_i)+O(\log r),$$
$$T_0(r,\triangle(\theta,\alpha))\le 3\sum_{i=1}^{3}N(2r,\theta,\alpha_0,a_i)+O(\log^2 r),$$

where a_1,a_2,a_3 are arbitrary three points in $\overline{C}$.

Lemma 2 ([7]). *Let $S(r)$ be a positive continuous non-decreasing function of r in $[0,+\infty)$. Suppose that*

$$\liminf_{r\to\infty}\frac{\log S(r)}{\log r}=\mu<+\infty,$$
$$\limsup_{r\to\infty}\frac{S(r)}{\log^2 r}=+\infty.$$

Then for any $h > 0$, *there exist sequences* $\{r_n\}$ *and* $\{R_n\}$, $R_n^{1-o(1)} \le r_n \le R_n (n \to \infty)$, *satisfying*

$$\lim_{n\to\infty} \frac{S(r_n)}{\log^2 r_n} = +\infty, \quad S(e^h R_n) \le e^{h\mu} S(R_n)(1+o(1)) \quad (n \to \infty).$$

Lemma 3 ([8]). *Let* $B(r)$ *be a positive and continuous function in* $[0, +\infty)$ *which satisfies* $\limsup_{r\to\infty} \frac{\log B(r)}{\log r} = \infty$, *then there exist continuously differentiable functions* $\rho(r)$ *and* $U(r)$, *which satisfy the following conditions:*

(i) $\rho(r) \downarrow 0$ *and* $\rho^{'}(r)$ *are monotonically increasing.*
(ii) $\lim_{r\to\infty} r\rho^{'}(r) \log r \log\log r = 0$.
(iii) *for a sufficiently large* r, *we have* $B(r) \ll U(r) = r^{\exp(\frac{1}{\rho(r)})}$.
(iv) $U(R) \le (1 + o(1))U(r)$ $(r \to \infty)$, *where* $R = r + \frac{r \log r}{\log U(r) \log^2 \log U(r)}$.

Proof. The detailed proof of Lemma 3 can be found in [8] and therefore we give only an out-line of the proof. When $B(r) \ge r^2 \ge 2$, we let $\mu(r) = [\log \frac{\log B(r)}{\log r}]^{-1}$, otherwise $\mu(r) = (\log 2)^{-1}$. By [8], we can find continuously differentiable functions $\rho(r)$ such that $0 < \rho(r) \ll \mu(r)$. Let $U(r) = r^{\exp(\frac{1}{\rho(r)})}$. It is easy to verify that $\rho(r)$ and $U(r)$ satisfy (i),(ii) and (iii). Now we prove (iv).

Put $H(r) = \exp(\frac{1}{\rho(r)})$; $x = \frac{r\rho^2(r)}{H(r)}$. By the mean value theorem we have

$$H(r+x) - H(r) = H^{'}(r+\theta x) \quad (0 < \theta < 1), \quad \text{and} \quad \rho(r+x) \ge \rho(r) + \rho^{'}(r)x.$$

Therefore

$$\begin{aligned}
H(r+x) - H(r) &= \exp\left(\frac{1}{\rho(r+\theta_1 x)}\right) \frac{-\rho^{'}(r+\theta_1 x)}{\rho^2(r+\theta_1 x)} \frac{r\rho^2(r)}{H(r)} \\
&\le \exp\left(\frac{1}{\rho(r+x)} - \frac{1}{\rho(r)}\right) \frac{-\rho^{'}(r) r\rho^2(r)}{\rho^2(r+x)} \\
&\le \exp\left(\frac{-\rho^{'}(r+\theta_2 x) r\rho^2(r)}{\rho^2(r+\theta_2 x)H(r)}\right) \frac{-\rho^{'}(r) r\rho^2(r)}{[\rho(r) + \rho^{'}(r)x]^2} \\
&\le \exp\left(\frac{-\rho^{'}(r) r\rho^2(r)}{\rho^2(r+x)H(r)}\right) \frac{-\rho^{'}(r) r}{[1 + \frac{\rho^{'}(r) r\rho(r)}{\exp(\frac{1}{\rho(r)})}]^2} \\
&\le \exp\left(\frac{-\rho^{'}(r) r\rho^2(r)}{[\rho(r) + \rho^{'}(r)x]^2 H(r)}\right) \frac{-\rho^{'}(r) r \log r \log\log r}{[1 + \frac{\rho^{'}(r) r\rho(r)}{\exp(\frac{1}{\rho(r)})}]^2 \log r \log\log r} \\
&= e^{o(1)} \frac{o(1)}{\log r \log\log r} = \frac{o(1)}{\log r \log\log r},
\end{aligned}$$

where $0 < \theta_1, \theta_2 < 1$. Because $U(r) = r^{H(r)}$, we have

$$U\left(r + \frac{r\log r}{\log U(r)\log^2\log U(r)}\right) = U\left(r + \frac{r\log r}{\log U(r)(\log H(r) + \log\log r)^2}\right)$$
$$\leq U\left(r + \frac{r}{H(r)\log^2 H(r)}\right) = \left(r + \frac{r\rho^2(r)}{H(r)}\right)^{H\left(r+\frac{r\rho^2(r)}{H(r)}\right)}$$
$$= r^{H(r)} r^{\frac{o(1)}{\log r\log\log r}}\left(1 + \frac{\rho^2(r)}{H(r)}\right)^{H(r)+o(1)} \leq (1+o(1))U(r). \qquad \square$$

Lemma 4 ([8])**.** *Let $f(z)$ be meromorphic in the sector $\Omega(\psi_1, \psi_2) = \{z : \psi_1 < \arg z < \psi_2\}$ $(\psi_1 < \psi_2)$. Suppose continuously differentiable functions $\rho(r)$ and $U(r)$ satisfy conditions* (i), (ii), (iv) *stated in Lemma* 3. *Let $T(r, \Omega(\psi_1, \psi_2), w) \leq U(r) = r^{\exp(\frac{1}{\rho(r)})}$ and $\{a_1, a_2, \ldots, a_q\}$ be $q(> 2)$ distinct points, then for arbitrary $\psi, \delta', \delta (0 < \delta' < \delta, \psi_1 < \psi - \delta < \psi - \delta' < \psi_2)$ we have*

$$(q-2)T(r, \Omega(\psi - \delta', \psi + \delta')) \leq \sum_{j=1}^{q} N(R, \Omega(\psi - \delta, \psi + \delta), a_j) + o(U(r)),$$

where R is the same as in Lemma 3.

3. The proof of the Theorems

Proof of Theorem 1. We need to consider two different cases.

I) $\liminf\limits_{r\to\infty} \frac{\log T(r,f)}{\log r} = \mu < +\infty$.

By the hypothesis and the relation of $T(r, f)$ and $T_0(r, f)$, we have

$$\liminf_{r\to\infty} \frac{\log T_0(r, f)}{\log r} = \mu < +\infty,$$
$$\limsup_{r\to\infty} \frac{T_0(r, f)}{\log^2 r} = +\infty.$$

Applying Lemma 2, we have

$$\lim_{n\to\infty} \frac{T_0(r_n, f)}{\log^2 r_n} = +\infty,$$
$$T_0(2R_n, f) = T_0(e^{\log 2}R_n, f) \leq e^{\mu\log 2}T_0(R_n, f)(1 + o(1))(n \to \infty).$$

where $R_n^{1-o(1)} \leq r_n \leq R_n$ $(n \to \infty)$.

Hence we have

$$\lim_{n\to\infty} \frac{T_0(R_n, f)}{\log^2 R_n} = +\infty, \tag{5}$$

and

$$\lim_{n\to\infty} \frac{T_0(R_n, f)}{T_0(2R_n, f)} > 0. \tag{6}$$

Expression (6) implies that there is a ray $\arg z = \theta_0$ $(0 \le \theta_0 \le 2\pi)$ such that for any $\varepsilon(0 < \varepsilon < \pi)$

$$\lim_{n\to\infty} \frac{T_0(R_n, \triangle(\theta_0, \varepsilon))}{T_0(2R_n, f)} > 0. \tag{7}$$

Now we are in the position to prove that the ray $\arg z = \theta_0$ is the T direction.

For arbitrary $\delta \in (0, \pi)$ and any three distinct points a_1, a_2, a_3 in $\overline{C}$, by using Lemma 1 for $\triangle(\theta_0, \delta))$ and $\triangle(\theta_0, \delta/2))$, we have

$$T_0(R_n, \triangle(\theta_0, \delta/2))) \le 3\sum_{i=1}^{3} N(2R_n, \theta_0, \delta, a_i) + O(\log^2 R_n).$$

Both sides of the above expression divided by $T_0(2R_n, f)$. Applying (5) and (7) shows that

$$\lim_{n\to\infty} \frac{\sum\limits_{i=1}^{3} N(2R_n, \theta_0, \delta, a_i)}{T_0(2R_n, f)} > 0,$$

that is to say, for any $a \in \overline{C}$ with at most two exceptions, we have

$$\lim_{n\to\infty} \frac{N(2R_n, \theta_0, \delta, a)}{T_0(2R_n, f)} > 0.$$

Hence

$$\limsup_{r\to\infty} \frac{N(r, \theta_0, \delta, a)}{T_0(r, f)} > 0.$$

Therefore

$$\limsup_{r\to\infty} \frac{N(r, \theta_0, \delta, a)}{T(r, f)} > 0.$$

II) $\liminf\limits_{r\to\infty} \frac{\log T(r,f)}{\log r} = \infty$, that is to say $f(z)$ is an infinite order function.

By Lemma 3, there exists $U(r)$ satisfying conditions (i), (ii), (iii), (iv) stated in Lemma 3. Let m $(m \ge 4)$ be a positive integer, $\theta_0 = 0, \theta_1 = \frac{2\pi}{m}, \cdots, \theta_{m-1} = \frac{(m-1)2\pi}{m}$, $\theta_m = \theta_0$ and $\triangle(\theta_i) = \{z || \arg z - \theta_i| < \frac{2\pi}{m}\}$, $i = 0, 1, \ldots, m-1$; $\triangle(\theta_m) = \triangle(\theta_0)$. We can assert that among these m angular domains $\{\triangle(\theta_i)\}$, there is at least one angular domain $\triangle(\theta_i)$ such that the relative expression

$$\limsup_{r\to\infty} \frac{N(r, \triangle(\theta_i), a)}{U(r)} > 0, \tag{8}$$

holds for all $a \in \overline{C}$ with at most two exceptions.

In fact, if this is not case, then for any angular domain $\triangle(\theta_i)$ $(1 \le i \le m)$, we have three distinct points a_i^j $(j = 1, 2, 3)$ in $\overline{C}$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{3} N(r, \triangle(\theta_i), a_i^j) = o(U(r)). \tag{9}$$

Put the angular domain

$$\overline{\triangle}(\theta_i) = \left\{ z \,\Big|\, \frac{\theta_{i-1} + \theta_i}{2} \le \arg z \le \frac{\theta_i + \theta_{i+1}}{2} \right\},$$

Applying Lemma 4 to $\overline{\triangle}(\theta_i) \subset \triangle(\theta_i)$ and a_i^j $(j = 1, 2, 3)$, we have

$$T(r, \overline{\triangle}(\theta_i)) \leq \sum_{j=1}^{3} N(R, \triangle(\theta_i), a_i^j) + o(U(r)).$$

Denoting $T(r, f) = \sum_{i=1}^{m} T(r, \overline{\triangle}(\theta_i))$ and summing both sides of the above inequality over $i = 1, \dots, m$ we obtain

$$T(r, f) \leq \sum_{i=1}^{m} \sum_{j=1}^{3} N(R, \triangle(\theta_i), a_j) + o(U(r)).$$

Dividing both sides of the above expression by $U(r)$, and using (i),(iv) stated in Lemma 3, by (9), we have

$$\begin{aligned} 1 &\leq \limsup_{r\to\infty} \frac{\sum_{i=1}^{m}\sum_{j=1}^{3} N(R, \triangle(\theta_i), a_i^j)}{U(r)} = \limsup_{r\to\infty} \frac{\sum_{i=1}^{m}\sum_{j=1}^{3} N(R, \triangle(\theta_i), a_i^j)}{U(R)} \frac{U(R)}{U(r)} \\ &\leq \limsup_{r\to\infty} \frac{\sum_{i=1}^{m}\sum_{j=1}^{3} N(2r, \triangle(\theta_i), a_i^j)}{U(2r)} \limsup_{r\to\infty} \frac{U(R)}{U(r)} = 0. \end{aligned}$$

This contradiction means that (8) is true.

By (8) and the condition (iii) of Lemma 3, we have

$$\begin{aligned} \limsup_{r\to\infty} \frac{N(r, \triangle(\theta_i), a)}{T(r, f)} &= \limsup_{r\to\infty} \frac{N(r, \triangle(\theta_i), a)}{U(r)} \frac{U(r)}{T(r, f)} \\ &\geq \limsup_{r\to\infty} \frac{N(r, \triangle(\theta_i), a)}{U(r)} \liminf_{r\to\infty} \frac{U(r)}{T(r, f)} > 0. \end{aligned}$$

So for an arbitrary positive integer m, there exists an angular domain $\triangle(\theta_m) = \{z || \arg z - \theta_m| < \frac{2\pi}{m}\}$ such that for any a, we have

$$\limsup_{r\to\infty} \frac{N(r, \triangle(\theta_m), a)}{T(r, f)} > 0 \tag{10}$$

with two exceptions at most. Choosing a subsequence of $\{\theta_m\}$ and denoting it also by $\{\theta_m\}$, we assume that $\theta_m \to \theta_0$. Put $T : \arg z = \theta_0$, then T is the T direction of Theorem 1.

In fact, for any $\delta (0 < \delta < \pi/2)$, when m is sufficiently large, we have $\triangle(\theta_m) \subset \triangle(\theta_0, \delta)$. By (10), we obtain

$$\limsup_{r\to\infty} \frac{N(r, \theta_0, \delta, a)}{T(r, f)} \geq \limsup_{r\to\infty} \frac{N(r, \triangle(\theta_m), a)}{T(r, f)} > 0$$

with at most two exceptions for a. Hence Theorem 1 holds in this case too. □

Proof of Theorem 2. Since

$$T(r, f) = O(\log^2 r)$$

implies

$$T(2r, f) \sim T(r, f),$$

by (3) we have

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = +\infty.$$

Then we deduce Theorem 2 from Theorem 1 directly. □

Proof of Theorem 3. It is easy to verify the existence of a T direction for $f(z)$ by Theorem 1. Now we prove the second assertion.

Let $L : \arg z = \theta_0$ be a T direction of $f(z)$, then for any $\delta(0 < \delta < \pi/2)$ and each a (with at most two exceptions), we have

$$\limsup_{r\to\infty} \frac{N(r,\theta_0,\delta,a)}{T(r,f)} > \varepsilon > 0.$$

Then there exists $\{r_n\}$ such that for any sufficiently large n we have $N(r_n,\theta_0,\delta,a) > \frac{\varepsilon}{2}T(r_n,f)$. In this case, we have

$$\limsup_{r\to\infty} \frac{\log N(r,\theta_0,\delta,a)}{\log r} \geq \rho$$

for all $a \in \overline{C}$ with at most two exceptions.

By [9],

$$\limsup_{r\to\infty} \frac{\log N(r,\theta_0,\delta,a)}{\log r} = \limsup_{r\to\infty} \frac{\log n(r,\theta_0,\delta,a)}{\log r}.$$

Hence the T direction is a Borel direction. □

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